

Twistor and ambitwistor string theories

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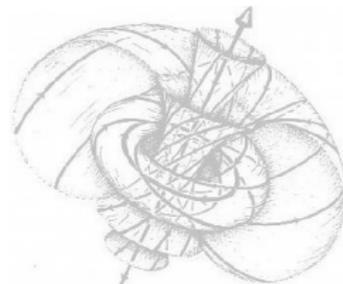
Amplitudes, Motives & Beyond; Mainz

With David Skinner. [arxiv:1311.2564](https://arxiv.org/abs/1311.2564), and collaborations with Tim Adamo, Eduardo Casali, Yvonne Geyer, Arthur Lipstein, Ricardo Monteiro & Kai Roehrig, Piotr Tourkine, [1312.3828](https://arxiv.org/abs/1312.3828), [1404.6219](https://arxiv.org/abs/1404.6219), [1405.5122](https://arxiv.org/abs/1405.5122), [1406.1462](https://arxiv.org/abs/1406.1462), etc..

[Cf. also Cachazo, He, Yuan [arxiv:1306.2962](https://arxiv.org/abs/1306.2962), [1306.6575](https://arxiv.org/abs/1306.6575), [1307.2199](https://arxiv.org/abs/1307.2199), [1309.0885](https://arxiv.org/abs/1309.0885), [1412.3479](https://arxiv.org/abs/1412.3479)]

Ambitwistor spaces: spaces of complex null geodesics.

- Extends Penrose/Ward's gravity/Yang-Mills twistor constructions to non-self-dual fields.
- Yang-Mills Witten and Isenberg, et. al. 1978, 1985.
- Conformal and Einstein gravity LeBrun [1983,1991]
Baston & M. [1987] .



Ambitwistor Strings:

- Tree S-Matrices in all dimensions for gravity, YM etc. [CHY]
- From strings in ambitwistor space [M. & Skinner 1311.2564]
- Models for Einstein-YM, DBI, BI, NLS, etc. CGMMRS 150??.?
- Related to \mathcal{I} , null geodesic scattering and the BMS group
- New form of maximal supergravity loop integrand.

Provide string theories at $\alpha' = 0$ for field theory amplitudes.

The scattering equations

Take n null momenta $k_i \in \mathbb{R}^d$, $i = 1, \dots, n$, $k_i^2 = 0$, $\sum_i k_i = 0$,

- define $P : \mathbb{CP}^1 \rightarrow \mathbb{C}^d$

$$P(\sigma) := \sum_{i=1}^n \frac{k_i}{\sigma - \sigma_i}, \quad \sigma, \sigma_i \in \mathbb{CP}^1.$$

- Solve for $\sigma_i \in \mathbb{CP}^1$ with the n scattering equations

$$k_i \cdot P(\sigma_i) = \text{Res}_{\sigma_i} P(\sigma) \cdot P(\sigma) = \sum_{j=1}^n \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0.$$

$$\Rightarrow P(\sigma) \cdot P(\sigma) = 0 \quad \forall \sigma.$$

- For Möbius invariance $\Rightarrow P \in \mathbb{C}^d \otimes K$, $K = \Omega^{1,0} \mathbb{CP}^1$
- There are $(n-3)!$ solutions.

Arise in large α' strings [Gross-Mende 1988] & twistor-strings [Witten 2004].

Amplitude formulae for massless theories.

Theorem (Cachazo, He, Yuan 2013,2014)

Tree-level massless amplitudes in d-dims are integrals/sums

$$\mathcal{M}(1, \dots, n) = \delta^d \left(\sum_i k_i \right) \int_{(\mathbb{CP}^1)^n} \frac{l^l l^r \prod_i \bar{\delta}(k_i \cdot P(\sigma_i))}{\text{Vol SL}(2, \mathbb{C}) \times \mathbb{C}^3}$$

where $l^l/r = l^l/r(\epsilon_i^l/r, k_i, \sigma_i)$ depend on the theory.

- polarizations ϵ_i^l for spin 1, $\epsilon_i^l \otimes \epsilon_i^r$ for spin-2 ($k_i \cdot \epsilon_i = 0 \dots$).
- Introduce skew $2n \times 2n$ matrices $M = \begin{pmatrix} A & C \\ -C^t & B \end{pmatrix}$,

$$A_{ij} = \frac{k_i \cdot k_j}{\sigma_i - \sigma_j}, \quad B_{ij} = \frac{\epsilon_i \cdot \epsilon_j}{\sigma_i - \sigma_j}, \quad C_{ij} = \frac{k_i \cdot \epsilon_j}{\sigma_i - \sigma_j}, \quad \text{for } i \neq j$$

and $A_{ii} = B_{ii} = 0$, $C_{ii} = \epsilon_i \cdot P(\sigma_i)$.

- For YM, $l^l = Pf^l(M)$, $l^r = \prod_i \frac{1}{\sigma_i - \sigma_{i-1}}$.
- For GR $l^l = Pf^l(M^l)$, $l^r = Pf^r(M^r)$, & many more.

Bosonic ambitwistor string action:

- Σ Riemann surface, coordinate $\sigma \in \mathbb{C}$
- Complexify space-time (M, g) , coords $X \in \mathbb{C}^d$, g hol.
- $(X, P) : \Sigma \rightarrow T^*M$, $P \in K$, holomorphic 1-forms on Σ .

$$S_B = \int_{\Sigma} P_{\mu} \bar{\partial} X^{\mu} - e P^2 / 2.$$

Underlying geometry:

- e enforces $P^2 = 0$,
- P^2 generates gauge freedom: $\delta(X, P, e) = (\alpha P, 0, 2\bar{\partial}\alpha)$.

So target is

$$\mathbb{A} = T^*M|_{P^2=0} / \{\text{gauge}\}.$$

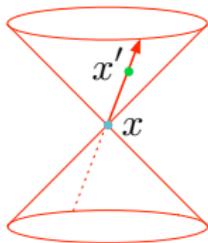
This is *Ambitwistor space*, space of complexified light rays.

The geometry of the space of light rays

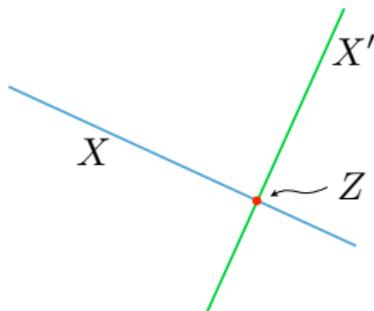
Ambitwistor space \mathbb{A} is space of complexified light rays.

- Light rays primary, events determined by lightcones $X \subset \mathbb{A}$ of light rays incident with x .
- Space-time $M =$ space of such $X \subset \mathbb{A}$.

Space-time



Twistor Space



Space-time geometry is encoded in complex structure of \mathbb{A} .

Theorem (LeBrun 1983 following Penrose 1976)

Complex structure of \mathbb{A} determines M and conformal metric g . Correspondence is stable under deformations of the complex structure of $P\mathbb{A}$ that preserve existence of symplectic potential $\theta = p_\mu dx^\mu$.

Idea of proof:

- Use Kodaira theory.
- Reconstruct M as moduli space of projective lightcones (quadrics) $PX \subset P\mathbb{A}$.
- Incidence determines null geodesics.
- Existence of symplectic potential \leftrightarrow torsion-free connection for geodesics.

Quantize bosonic ambitwistor string:

- $(X, P) : \Sigma \rightarrow T^*M,$

$$S_B = \int_{\Sigma} P_{\mu}(\bar{\partial} + \tilde{e}\partial)X^{\mu} - e P^2/2.$$

- Gauge fix $\tilde{e} = e = 0, \rightsquigarrow$ ghosts & BRST Q
- Introduce vertex operators $V_i \leftrightarrow$ field perturbations.

Amplitudes are computed as correlators of vertex ops

$$\mathcal{M}(1, \dots, n) = \langle V_1 \dots V_n \rangle$$

For gravity add type II worldsheet susy $S_{\Psi_1} + S_{\Psi_2}$ where

$$S_{\Psi} = \int_{\Sigma} \Psi_{\mu} \bar{\partial} \Psi^{\mu} + \chi P \cdot \Psi.$$

From deformations of \mathbb{A} to the scattering equations

Gravitons \leftrightarrow vertex operators $V_i = \text{def'm of action } \delta S = \int_{\Sigma} \delta\theta.$

- θ determines complex structure on $P\mathbb{A}$ via $\theta \wedge d\theta^{d-2}$. So:
- Deformations of complex structure $\leftrightarrow [\delta\theta] \in H^1_{\bar{\partial}}(P\mathbb{A}, L)$.

Proposition

For perturbation $\delta g_{\mu\nu} = e^{ik \cdot x} \epsilon_{\mu} \epsilon_{\nu}$ of flat space-time

$$\delta\theta = \bar{\partial}h = \bar{\delta}(k \cdot P) e^{ik \cdot X} (\epsilon \cdot P)^2, \quad h = \frac{e^{ik \cdot x} (\epsilon \cdot P)^2}{k \cdot P}.$$

Ambitwistor repn $\Rightarrow \bar{\delta}(k \cdot P) \Rightarrow$ scattering equs.

Proposition

CHY formulae for massless tree amplitudes e.g. YM & gravity arise from appropriate choices of worldsheet matter.

- Take $e^{ik_j \cdot X(\sigma_j)}$ factors into action to give

$$S = \frac{1}{2\pi} \int_{\Sigma} P \cdot \bar{\partial} X + 2\pi \sum_i ik \cdot X(\sigma_i).$$

- Gives field equations $\bar{\partial} X = 0$ and,

$$\bar{\partial} P = 2\pi \sum_i ik \delta^2(\sigma - \sigma_i).$$

- Solutions $X(\sigma) = X = \text{const.}$, $P(\sigma) = \sum_i \frac{k_j}{\sigma - \sigma_j} d\sigma$.

Thus path-integral reduces to

$$\mathcal{M}(1, \dots, n) = \delta^d \left(\sum_i k_i \right) \int_{(\mathbb{CP}^1)^{n-3}} \frac{\prod_i \delta(k_i \cdot P) (\epsilon_i \cdot P(\sigma_i))^2}{\text{Vol } G}$$

We see $P(\sigma)$ appearing and scattering equations.

Unfortunately: amplitudes for $S \sim \int_M R + R^3$

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We see $P(\sigma)$ appearing and scattering equations.

Unfortunately: amplitudes for $S \sim \int_M R + R^3$.

- Decorate null geodesics with spin vectors, vectors for internal degrees of freedom & other holomorphic CFTs.
- Take

$$S = S_B + S^l + S^r$$

where S^l, S^r are some worksheet matter CFTs.

- Total vertex operators given by

$$v^l v^r \bar{\delta}(k \cdot P) e^{ik \cdot X}$$

with v^l, v^r worksheet currents from S^l, S^r resp..

- Amplitudes become

$$\mathcal{M}(1, \dots, n) = \delta^d(\sum_i k_i) \int_{(\mathbb{CP}^1)^n} \frac{l^l l^r \prod_i \bar{\delta}(k_i \cdot P)}{\text{Vol Gauge}}$$

where l^l, l^r are worksheet correlators of v^l 's, v^r 's resp..

- In good situations, Q -invariance and discrete symmetries (GSO) rule out unwanted vertex operators.

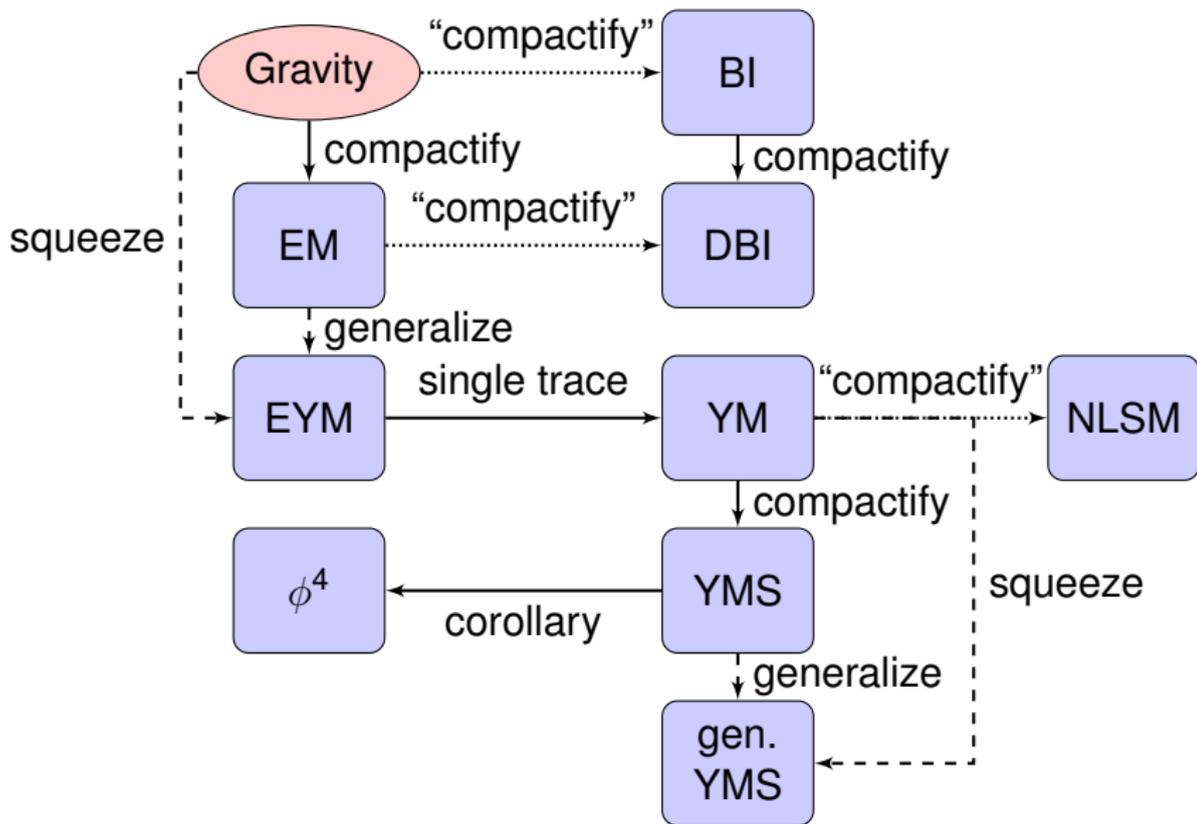


Figure: Theories studied by CHY and operations relating them.

Ambitwistor strings with combinations of matter

CGMMRS 150?

$S' \backslash S'$	S_Ψ	S_{Ψ_1, Ψ_2}	$S_{\rho, \Psi}^{(\tilde{m})}$	$S_{CS, \Psi}^{(\tilde{N})}$	$S_{CS}^{(\tilde{N})}$
S_Ψ	E				
S_{Ψ_1, Ψ_2}	BI	Galileon			
$S_{\rho, \Psi}^{(m)}$	EM $U(1)^m$	DBI	EMS $U(1)^m \times U(1)^{\tilde{m}}$		
$S_{CS, \Psi}^{(N)}$	EYM	ext. DBI	$EYMS$ $SU(N) \times U(1)^{\tilde{m}}$	$EYMS$ $SU(N) \times SU(\tilde{N})$	
$S_{CS}^{(N)}$	YM	Nonlinear σ	$EYMS$ $SU(N) \times U(1)^{\tilde{m}}$	<i>gen. YMS</i> $SU(N) \times SU(\tilde{N})$	<i>Biadjoint Scalar</i> $SU(N) \times SU(\tilde{N})$

Table: Theories arising from the different choices of matter models.

Spinning light rays and worldsheet SUSY, S_Ψ

Let $\Psi^\mu \in K^{1/2}$, spin 1/2 fermions on Σ ,

$$S_\Psi = \int g_{\mu\nu} \Psi^\mu \bar{\partial} \Psi^\nu - \chi P_\mu \Psi^\mu$$

$\chi \rightsquigarrow$ constraints $P \cdot \Psi = 0$ and gauge field for worldline susy

$$D = \Psi \cdot \frac{\partial}{\partial X} + P \cdot \frac{\partial}{\partial \Psi}, \quad \{D, D\} = P \cdot \frac{\partial}{\partial X}.$$

- Gauge fix $\chi = 0 \rightsquigarrow$ bosonic ghosts $(\beta, \gamma) \in (K^{3/2}, T^{1/2})$.
- Extend BRST operator with $Q_\Psi = \int \gamma P \cdot \Psi$.
- For v^l or v^r , must replace $\epsilon \cdot P$ by

$$u = \delta(\gamma) \epsilon \cdot \Psi \quad (\text{fixed}), \quad v = \epsilon \cdot P + \epsilon \cdot \Psi k \cdot \Psi.$$

- Worldsheet correlator $\langle u_1 u_2 v_3 \dots v_n \rangle = Pf'(M)$ for l^l or l^r .
- GSO symmetry $(\Psi, \gamma, \beta) \rightarrow (-\Psi, -\gamma, -\beta) \Rightarrow$ need u or v .

Free Fermions and current algebras

- Free 'real' Fermions $\rho^a \in \mathbb{C}^m \otimes K^{1/2}$

$$S_\rho = \int_\Sigma \delta_{ab} \rho^a \bar{\partial} \rho^b, \quad a = 1, \dots, m,$$

- These generate current algebra e.g., if \mathfrak{g} is Lie algebra with structure constants f^{abc} , $m = \dim \mathfrak{g}$ then

$$j^a = f^{abc} \rho^b \rho^c, \quad j^a(\sigma) j^b(0) = \frac{k \delta^{ab}}{\sigma^2} + \frac{f^{abc} j^c}{\sigma} + \dots$$

Here level $k = C$.

- Current algebra gives Vertex ops

$$v = t \cdot j.$$

- Correlators give 'Parke-Taylor' factors + unwanted multi-trace terms

$$\langle v_1 \dots v_n \rangle = \frac{\text{tr}(t_1 \dots t_n)}{\sigma_{12} \sigma_{23} \dots \sigma_{n1}} + \dots$$

where $\sigma_{ij} = \sigma_i - \sigma_j$.

- To avoid multitrace terms, take current algebra level $k = 0$.
- But, then there is no trace $\langle v_1 \dots v_n \rangle = 0$.
- Gauge fermionic spin 3/2 current to give two fixed vertex operators to end chain of structure constants 'comb'.
- Use fermions $\tilde{\rho}^a, \rho^a \in \mathfrak{g} \otimes K^{1/2}$, bosons $q^a, y^a \in \mathfrak{g} \otimes K^{1/2}$

$$S_{CS} = \int_{\Sigma} \tilde{\rho}_a \bar{\partial} \rho^a + q_a \bar{\partial} y^a + \chi \operatorname{tr} \rho \left(\frac{[\tilde{\rho}, \rho]}{2} + [q, y] \right).$$

- Gauge fix $\chi = 0 \rightsquigarrow$ ghosts (β, γ)
- Gives vertex operators

$$u = \delta(\gamma) t \cdot \rho \quad \tilde{u} = \delta(\gamma) t \cdot \tilde{\rho}, \quad v = t \cdot [\rho, \rho], \quad \tilde{v} = [\tilde{\rho}, \rho] + [q, y].$$

- To be nontrivial, correlator must have just one untilded VO

$$\langle u_1 \tilde{u}_2 \tilde{v}_3 \dots \tilde{v}_n \rangle = \mathcal{C}(1, \dots, n) := \frac{\operatorname{tr}(t_1 [t_2, [t_3, \dots [t_{n-1}, t_n] \dots]])}{\sigma_{12} \dots \sigma_{n1}}.$$

The 2013 CHY formulae & ambitwistor models

Above lead essentially to original models & formulae:

- $(S', S^r) = (S_{\tilde{\psi}}, S_{\psi}) \rightsquigarrow$ type II gravity,
- $(S', S^r) = (S_{CS}, S_{\psi}) \rightsquigarrow$ heterotic with YM,
- $(S', S^r) = (S_{CS}, S_{CS}) \rightsquigarrow$ bi-adjoint scalar.

The latter two come with unphysical gravity.

S_{CS} improves on current algebras in avoiding multi-trace terms and all models critical in 10d.

Combined matter systems

$S_{\Psi_1, \Psi_2} = S_{\Psi_1} + S_{\Psi_2}$ two worldsheet susy's for S^l or S^r . This is maximum. It gives VO currents

$$u = \delta(\gamma_1)k \cdot \Psi_2, \quad v = k \cdot \Psi_1 k \cdot \Psi_2.$$

$S_{\Psi, \rho} = S_{\Psi} + S_{\rho}$ combines 'real' Fermions with susy, \rightsquigarrow VO currents as usual for S_{Ψ} and

$$u_t = \delta(\gamma)t \cdot \rho, \quad v_t = k \cdot \Psi t \cdot \rho.$$

$$S_{\Psi, CS} = \int_{\Sigma} \Psi \cdot \bar{\partial} \Psi + \tilde{\rho}_a \bar{\partial} \rho^a + q_a \bar{\partial} y^a + \chi \left(P \cdot \Psi + \text{tr} \rho \left(\frac{[\tilde{\rho}, \rho]}{2} + [q, y] \right) \right).$$

With ghosts etc., the VO currents are those for S_{Ψ} and

$$\begin{aligned} \tilde{u}_t &= \delta(\gamma)t \cdot \tilde{\rho}, & u_t &= \delta(\gamma)t \cdot \rho, \\ \tilde{v}_t &= k \cdot \Psi t \cdot \tilde{\rho} + t \cdot ([\tilde{\rho}, \rho] + [q, y]), & v_t &= k \cdot \Psi t \cdot \rho + t \cdot [\rho, \rho]. \end{aligned}$$

GSO now reverses signs of all fields in matter system.

With $(S^l, S^r) = (S_{\Psi, CS}, S_{\tilde{\Psi}})$ we obtain Einstein- T^* YM.

- Graviton vertex operators now come from

$$u^l u^r = \delta(\gamma) \epsilon \cdot \Psi \delta(\tilde{\gamma}) \tilde{\epsilon} \cdot \tilde{\Psi}, \quad v^l v^r = \epsilon \cdot (P + \Psi k \Psi) \tilde{\epsilon} \cdot (P + \Psi k \cdot \Psi).$$

Worldsheet correlators give $Pf'(M)Pf'(\tilde{M})$.

- Two types of gluon VOs from $u_t^l u^r$, $v_t^l v^r$ and $\tilde{u}_t^l u^r$, $\tilde{v}_t^l v^r$.
- With gluons, worldsheet correlator gives sum of

$$\mathcal{C}(T_1) \dots \mathcal{C}(T_r) Pf'(\Pi) Pf'(\tilde{M})$$

where (T_1, \dots, T_r) is a partition of gluons, Π from CHY.

Theory is linear YM on full YM background i.e., T^* YM + gravity

$$\int_M R_{NS} \text{dvol} + \text{tr}(A \wedge D_{\tilde{A}}^* F_{\tilde{A}})$$

with $A \leftrightarrow u_t^l u^r, v_t^l v^r, \tilde{A} \leftrightarrow \tilde{u}_t^l u^r, \tilde{v}_t^l v^r$.

Can replace S_{CS} with anomalous S_{YM} with single gluon type.

Models from different geometric realizations of \mathbb{A}

We can start with other formulations of null superparticles

- Green-Schwarz version:

$$S = \int P \cdot \bar{\partial} X + P_\mu \gamma_{\alpha\beta}^\mu \theta^\alpha \bar{\partial} \theta^\beta .$$

- Pure spinor version (Berkovits) $S = \int P \cdot \bar{\partial} X + p_\alpha \bar{\partial} \theta^\alpha + \dots$
- $d = 4$, Twistor-strings of Witten, Berkovits & Skinner

$$\mathbb{A} = \{(Z, W) \in \mathbb{T} \times \mathbb{T}^* \mid Z \cdot W = 0\} / \{Z \cdot \partial_Z - W \cdot \partial_W\}$$

$$S = \int_\Sigma W \cdot \bar{\partial} Z + a Z \cdot W$$

- In 4d have full ambitwistor representation [Geyer, Lipstein, M. 1404.6219]

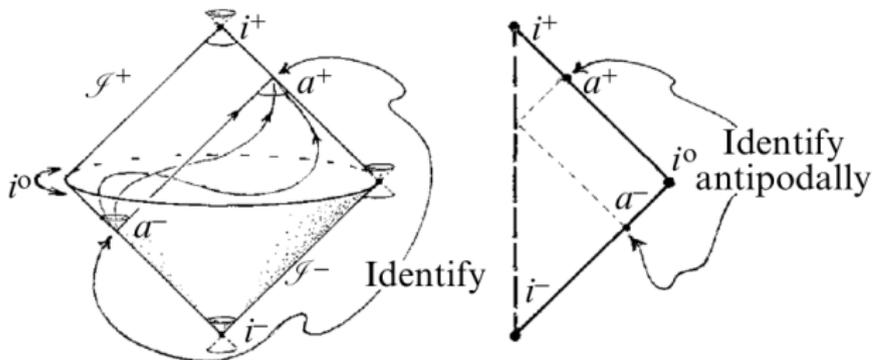
$$S = \int_\Sigma Z \cdot \bar{\partial} W - W \cdot \bar{\partial} Z + a Z \cdot W$$

Not twistor string: $(Z, W) \in K^{1/2}$ gives simpler 4d formulae with no moduli. Nonchiral, working at $N = 0$.

Relation to null infinity, BMS and soft gravitons

Geyer, Lipstein & M. 1406.1462 (cf. Adamo, Casali & Skinner 1405.5122, 1503.02304).

Take space-time asymptotically simple:



Real light rays intersect \mathcal{I}^+ and \mathcal{I}^- , $\mathbb{A}_{\mathbb{R}} = T^*\mathcal{I}^+ = T^*\mathcal{I}^-$.

- **Flat space-time:** identification is identity and global.
- **Curved space-time:** identification only for real light rays:

$$\mathbb{A} = T^*\mathcal{I}_{\mathbb{C}}^+ \cup T^*\mathcal{I}_{\mathbb{C}}^- \quad \text{glued over } \mathbb{A}_{\mathbb{R}}.$$

Infinitesimally glued by Hamiltonian h for light ray scattering

$$\text{Vertex op} = \delta\theta = \bar{\partial}h.$$

BMS $^\pm$ group acts on \mathcal{I}^\pm hence on $T^*\mathcal{I}^\pm$ with Hamiltonians h .

- worldsheet generators have same form as Vertex Ops:

$$\oint_{\Sigma} h = \int_{\Sigma} \bar{\partial} h.$$

- **Soft gravitons:** $k \rightarrow 0$ then $h \rightsquigarrow$ supertranslation.
- Subleading term generates ‘superrotation’.
- Diagonal subgroup $\subset \text{BMS}^+ \times \text{BMS}^-$ are symmetries.

\rightsquigarrow versions of (subleading) soft graviton thm as Ward identity.

Theorem (Lysov/Strominger/Weinberg)

Weinberg soft theorem: as $k_n \rightarrow 0$

$$\mathcal{M}(1, \dots, n) \rightarrow \mathcal{M}(1, \dots, n-1) \sum_{i=1}^{n-1} \frac{(\epsilon_n \cdot k_i)^2}{k_n \cdot k_i},$$

follows from supertranslation equivariance.

The $d = 10$ supergravity loop integrand

[Adamo, Casali, Skinner 2013, Casali Tourkine 2015 Geyer, M., Monteiro, Tourkine. . .]

10d type II gravity model is critical so extends to higher genus:

- At genus g , P is a 1-form and acquires dg zero-modes.
- These are the loop momenta for g -loops.
- Standard string technology can be adapted at all g .
- E.g., at 1-loop, $n = 4$, obtain modular invariant sum over spin structures of

$$\mathcal{M}_4^{(1)}(\alpha; \beta) = \delta^{10} \left(\sum_i k_i \right) \int d^{10} p \wedge d\tau \wedge \bar{\delta} \left(P^2(\sigma_1; \tau) \right) \\ \prod_{j=2}^4 d\sigma_j \bar{\delta}(k_j \cdot P(\sigma_j)) \frac{\vartheta_\alpha(\tau)^4 \vartheta_\beta(\tau)^4}{\eta(\tau)^{24}} \text{Pf}(M_\alpha) \text{Pf}(\tilde{M}_\beta)$$

Now checked in many ways and related to standard integrand.

Chiral $\alpha' = 0$ ambitwistor strings use LeBrun's correspondence to give theories underlying CHY formulae old & new.

- Incorporates colour/kinematics Yang-Mills/gravity ideas. Any insight into geometry of kinematic factors?
- Quantization ties scattering of null geodesics into that for gravitational waves.
- Critical models extend to loops .
- Does new representation give new insights into loop integrands?

Thank You