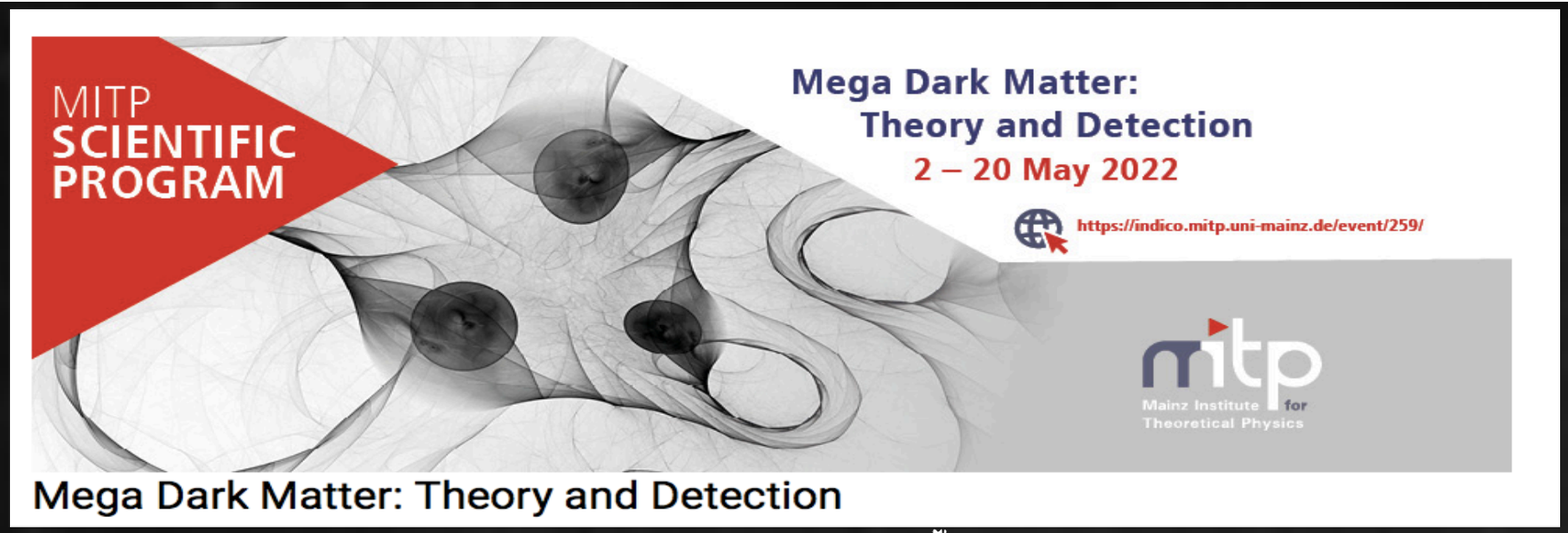
Vector Misalignment

Fatemeh Elahi

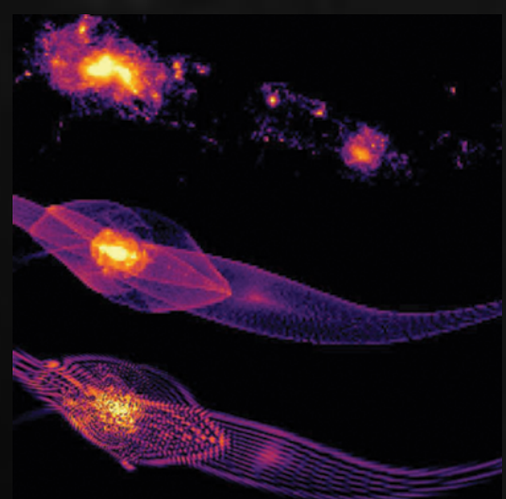
JGU Mainz

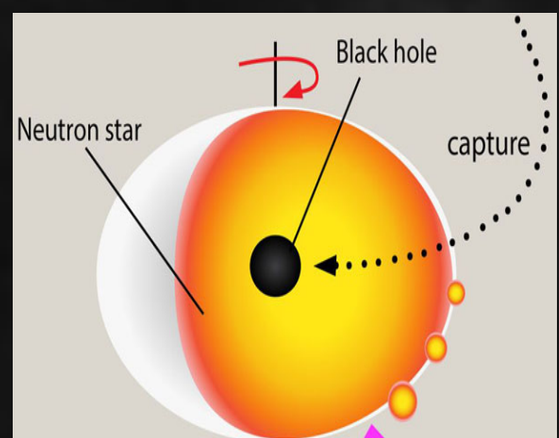
inspired by a work with

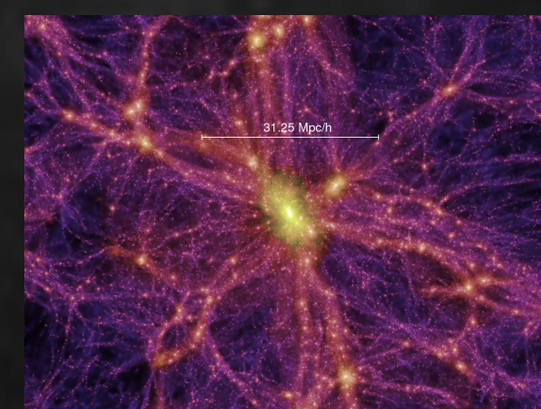
Sara Khatibi [arXiv:2204.04012]

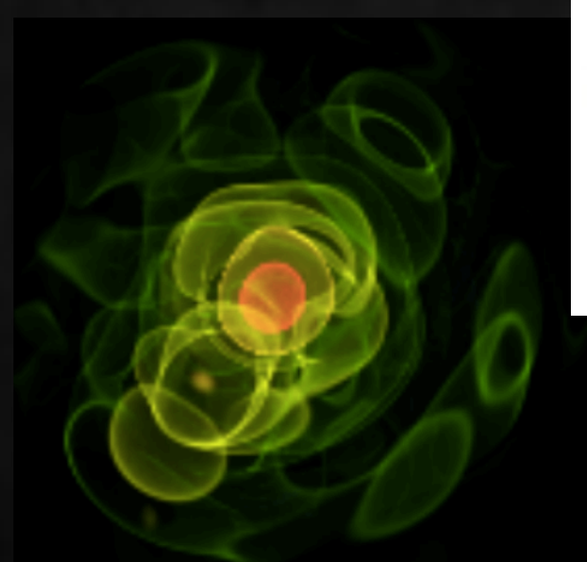


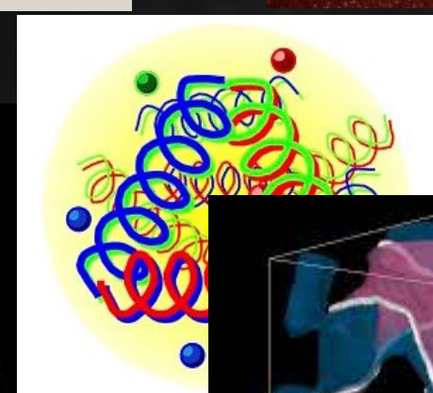
M_{PL} $M_\odot$ $10^n M_\odot$

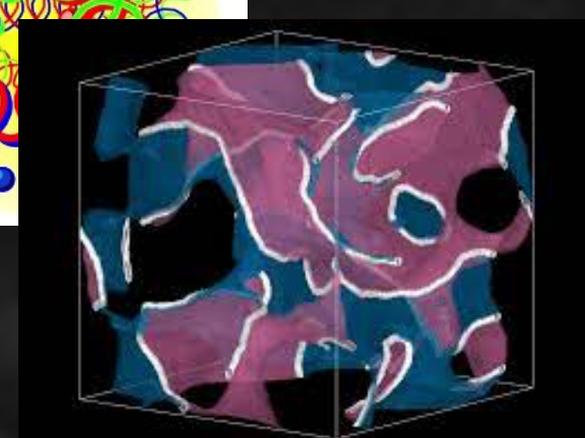


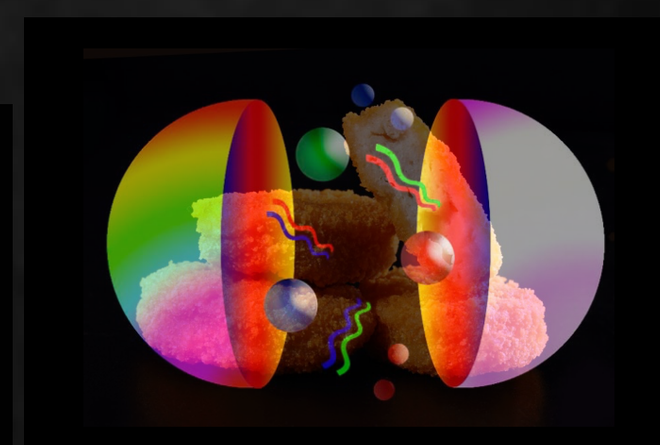


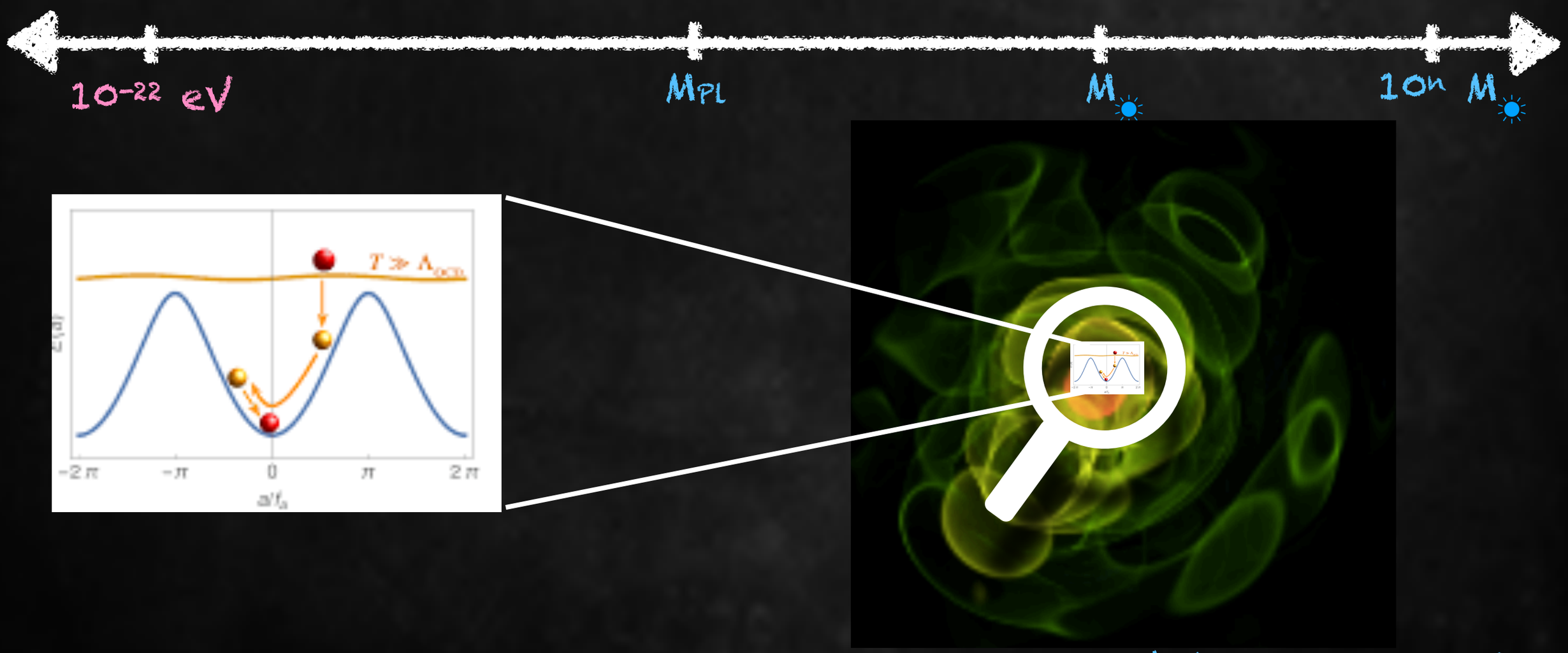












Josh Eby's talk tomorrow!

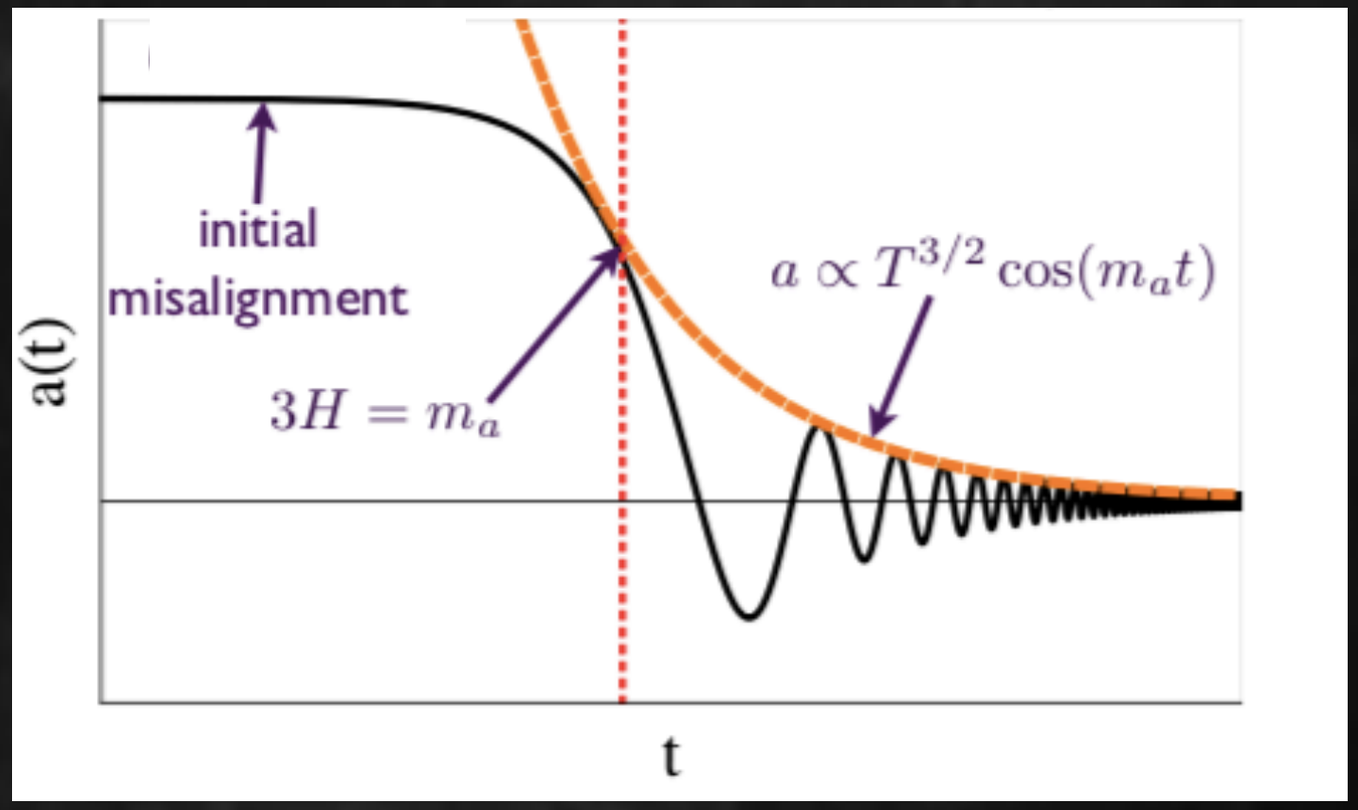
Axion Misalignment

Early time

a: constant

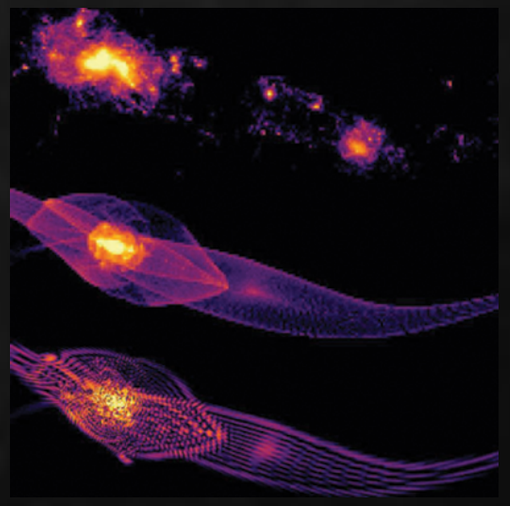
Late time

a: redshifts like matter



How about other bosons?

How about vectors?



Mustafa Amin's talk!

Why even consider vector DM?

SM — Portal — DS

Scalar portal	$\lvert H\rvert^2\lvert S\rvert^2$	exotic scalar
Vector portal	$F_{\mu\nu}F^{'\mu\nu}$	Dark Photon
Neutrino portal	LHN	sterile neutrino

Dark Photon

Suppose there is an additional $U(1)_D$ in nature that is broken.

$$\mathcal{L} \supset -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} - \frac{\epsilon}{2}F_{\mu\nu}F'^{\mu\nu} + \frac{1}{2}m_{A'}^2 A'_\mu A'^\mu$$

Going to the mass basis

$$\mathcal{L} \supset -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \frac{1}{2}m_{A'}^2 A'_\mu A'^\mu + J_\mu^{\mathrm{EM}}(A_\mu - \epsilon A'_\mu)$$

Dark photon interacts with EM current.

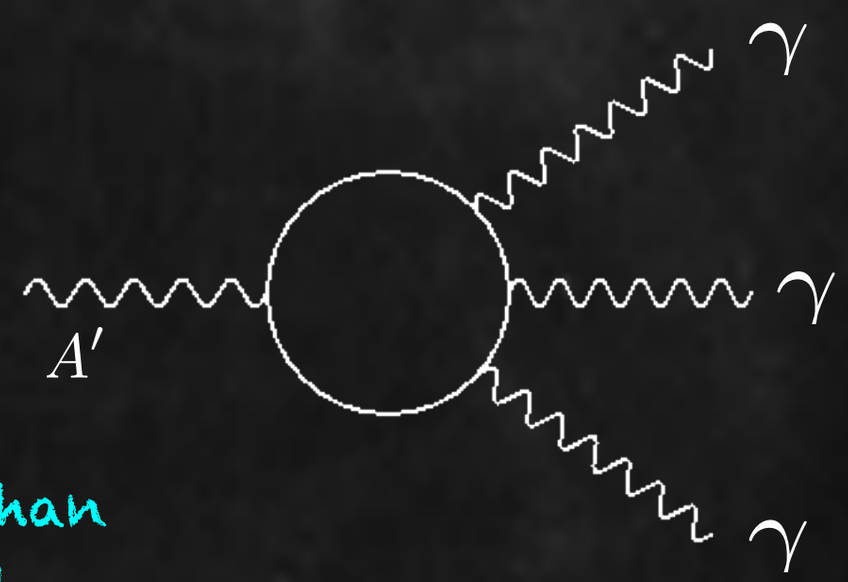
Holdom '86

Can Dark Photon be a DM candidate?

Stability?

for $m_{A'} < 2\, m_e$

$$\Gamma_{A' \to 3\gamma} = \frac{17 \epsilon^2 \alpha^4}{2^7 3^6 5^3 \pi^3} \frac{m_{A'}^9}{m_e^8}$$



For a dark photon lighter than eV, it is easily long lived

Can Dark Photon be a DM candidate?

Can it produce the right relic abundance?

Dark Photon Dark Matter Produced by Axion Oscillations

Raymond T. Co,[1] Aaron Pierce,[1] Zhengkang Zhang,[1,2,3] and Yue Zhao[1,4]

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[4] *Department of Physics and Astronomy, University of Utah, Salt Lake City, UT 84112, USA*

(Dated: October 18, 2018)

on dark matter may be produced. A coherently
r its energy density to a dark photon field via a
subsequently depleted via couplings to the visible
tter. We ensure the cosmologies of both the axion
straints. We find that the mechanism works for a
se of interest for ongoing experiments and proposed

Relic Abundance of Dark Photon Dark Matter

Prateek Agrawal,[1] Naoya Kitajima,[2] Matthew Reece,[1] Toyokazu Sekiguchi,[3] and Fuminobu Takahashi

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[2] *Department of Physics and Astrophysics, Nagoya University, Chikusa, Nagoya 464-8602, Japan*
[3] *Research Center for the Early Universe (RESCEU), Graduate School of Science, the University of Tokyo, Tokyo 113-0033, Japan*
[4] *Department of Physics, Tohoku University, Sendai, Miyagi 980-8578, Japan*
[5] *Kavli Institute for the Physics and Mathematics of the Universe (Kavli IPMU), UTIAS, WPI, The University of Tokyo, Kashiwa, Chiba 277-8568, Japan*
[6] *Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA*

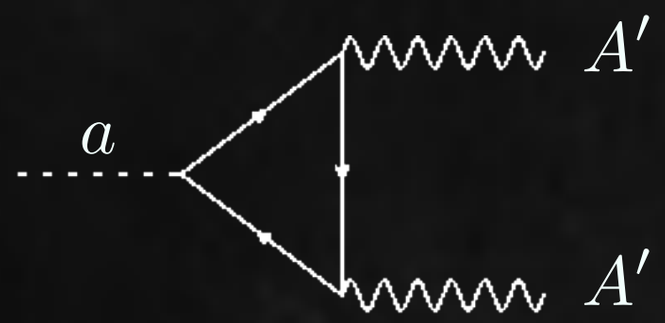
(Dated: October 18, 2018)

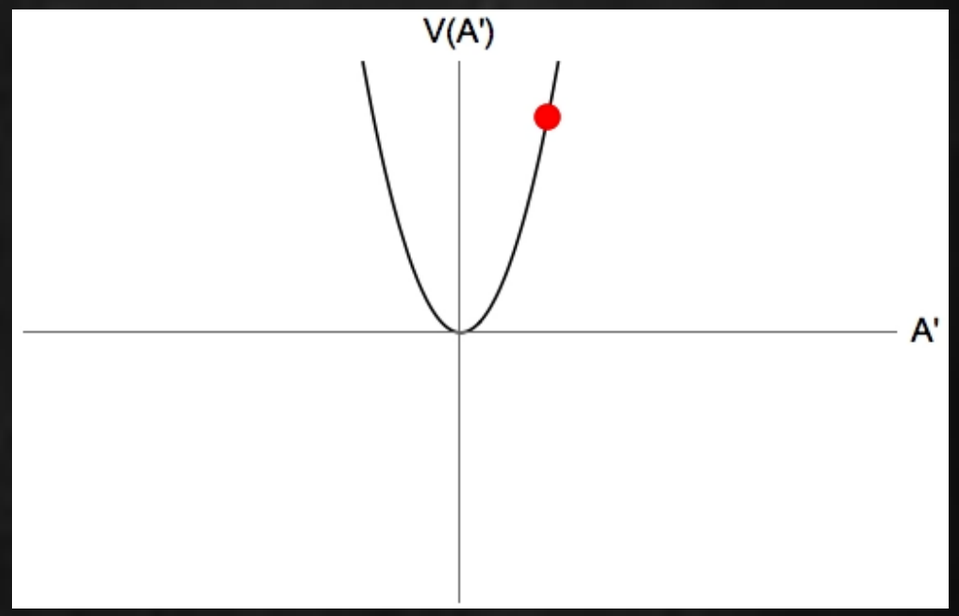
We present a new mechanism for producing the correct relic abundance of dark photon dark matter over a wide range of its mass, extending down to 10^{-20} eV. The dark matter abundance is initially stored in an axion which is misaligned from its minimum. When the axion starts oscillating, it efficiently transfers its energy into dark photons via a tachyonic instability. If the dark photon mass is within a few orders of magnitude of the axion mass, $m_{\gamma'}/m_a = \mathcal{O}(10^{-3} - 1)$, then dark photons make up the dominant form of dark matter today. We present a numerical lattice simulation for a benchmark model that explicitly realizes our mechanism. This mechanism firms up the motivation for a number of experiments searching for dark photon dark matter.

Can Dark Photon be a DM candidate?

Can it produce the right relic abundance?

$$\mathcal{L} \supset \frac{\alpha_D}{8\pi}\frac{a}{f_a}F'_{\mu\nu}\tilde{F}'^{\mu\nu}$$





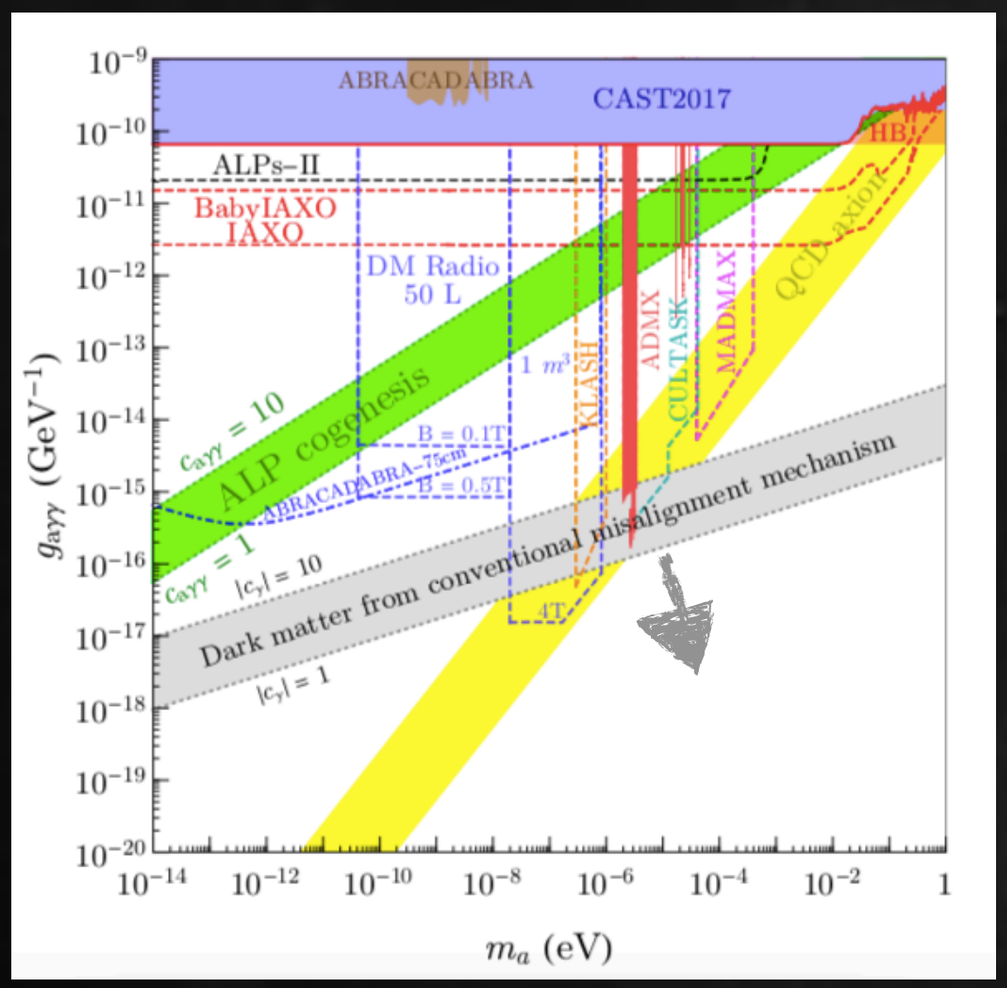
Raymond Co (U. Michigan)

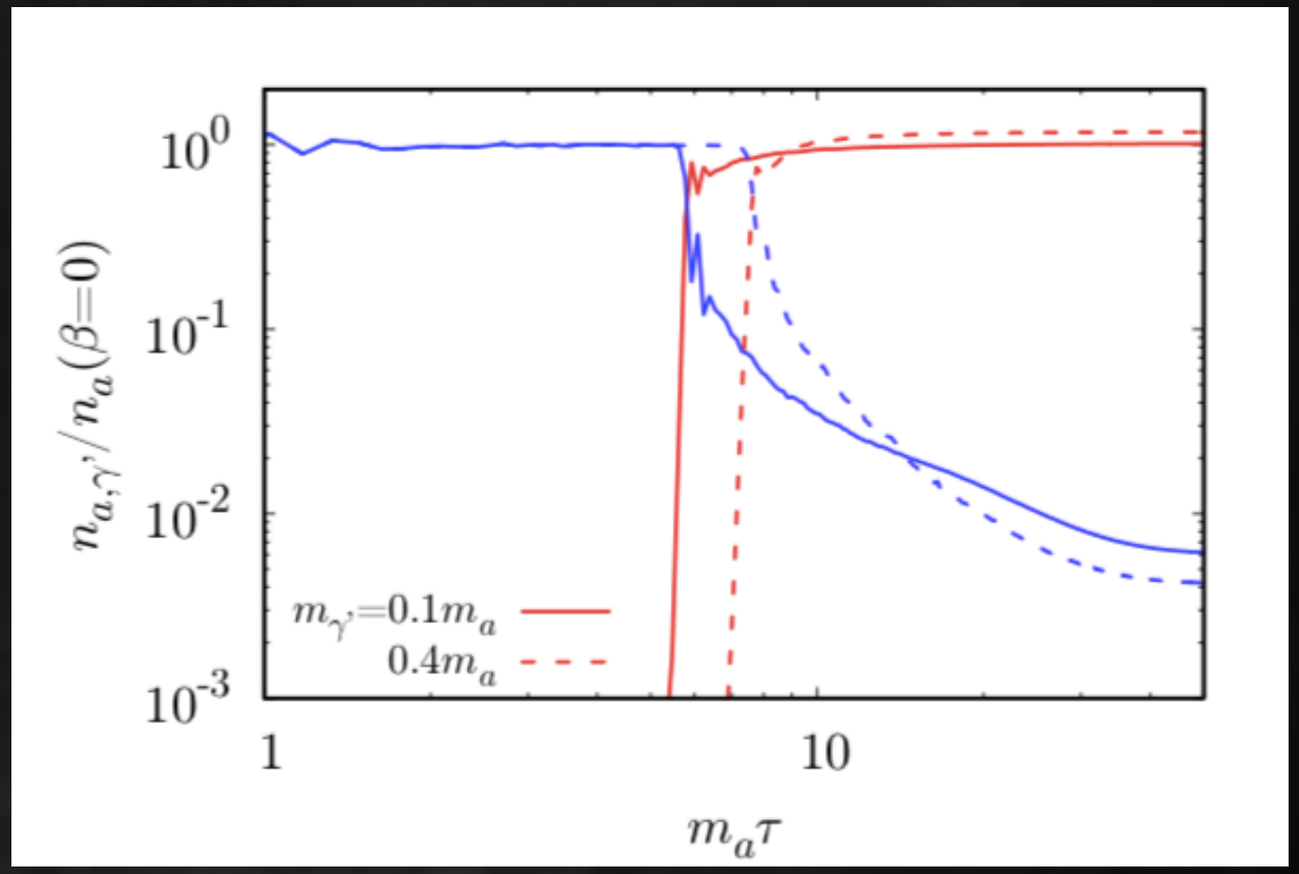
$$\frac{\partial^2 A'_\pm}{\partial\eta^2} + \left(m_{A'}^2 + k_{A'}^2 \pm \frac{\alpha_D k_{A'}}{2\pi f_a}\frac{\partial a}{\partial\eta}\right)A'_\pm = 0$$

$$m_{A'}^2 + k_{A'}^2 \pm \frac{\alpha_D k_{A'}}{2\pi f_a}\frac{\partial a}{\partial\eta} < 0$$

efficient energy transfer from axion to dark photon

Advantages?





Increasing the viable parameter space

Producing dark photon relic

Vector Misalignment

Misalignment : Scalar Field

Action for the scalar field :

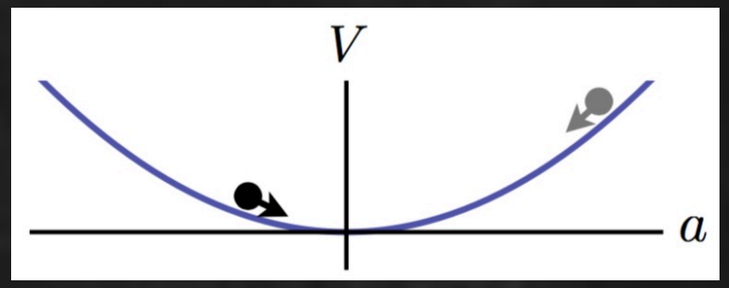
$$S = \int d^4x \sqrt{-g} \left(\frac{M_p^2}{2} R - \frac{1}{4} g^{MN} \, \partial_M \phi \, \partial_N \phi - \frac{1}{2} m^2 \phi^2 \right)$$

EOM spatially homogeneous but time-dependent complex scalar field :

$$\ddot{\phi} + 3H\dot{\phi} + m_\phi^2 \phi = 0$$

Misalignment : Scalar Field

$$\ddot{\phi} + 3H\dot{\phi} + m_\phi^2\phi = 0$$



Early time : $H \gg m_\phi$

Hubble friction dominates

$\phi =$ **constant**

$\rho_\phi = m_\phi^2\phi^2$

$\rho_\phi =$ **constant**

Late time : $H \ll m_\phi$

Oscillations begin

$\phi \propto a^{-3/2}$

$\rho_\phi \propto a^{-3}$

Misalignment : Abelian Field

Action for the abelian vector field :

$$S = \int d^4x \sqrt{-g} \left(\frac{M_p^2}{2} R - \frac{1}{4} g^{MN} g^{KL} \mathcal{F}_{MK} \mathcal{F}_{NL} - \frac{1}{2} m^2 g^{MN} A_M A_N \right)$$

EOM spatially homogeneous but time-dependent vector field :

$$\ddot{A}_i + H\dot{A}_i + m_A^2 A_i = 0$$

$$A_0 = 0$$

Misalignment : Abelian Field

$$\ddot{A}_i + H\dot{A}_i + m_A^2 A_i = 0$$

Early time : $H \gg m_A$

Hubble friction dominates

$A_i \simeq$ **constant**

$$\rho_A = T_{00} = \frac{1}{2a^2}\left(\dot{A}_i^2 + m_A^2 A_i^2\right)$$

$$\rho_{A_i} \propto a^{-2}$$

Redshifts too fast

Late time : $H \ll m_A$

Oscillations begin

$$A_i \propto a^{-1/2}$$

Misalignment : Abelian Field

- We should look at the physical field (since it is canonical in the physical coordinate) :

$$\bar{A}_i \equiv A_i / a$$

- because the kinetic term

$$S \supset \int d\tau d^3x \frac{1}{2} A_i'^2 = \int dt d^3X \frac{1}{2a^2} \dot{A}_i^2 = \int dt d^3X \left(\frac{1}{2} \dot{A}_i^2 - \frac{1}{2} \left(H^2 - \dot{H} \right) \bar{A}_i^2 \right)$$

$(\tau, \vec{x})$: **co-moving coordinate**

$(t, \vec{X})$: **physical coordinate**

$$' = \frac{\partial}{\partial \tau} = a \frac{\partial}{\partial t}$$

Misalignment : Abelian Field

Using $\bar{A}_i \equiv A_i/a$

- EOM of physical vector field :

$$\ddot{\bar{A}}_i + 3H\dot{\bar{A}}_i + \left(m_A^2 + 2H^2 + \dot{H}\right)\bar{A}_i = 0 \xrightarrow{R = 6\,(2H + \dot{H})} \ddot{\bar{A}}_i + 3H\dot{\bar{A}}_i + \left(m_A^2 + \frac{R}{6}\right)\bar{A}_i = 0$$

Early time : $H \gg m_A$

$$\bar{A}_i \propto e^{-Ht}$$

(The field is exponentially damped and vanish after inflation)

Late time : $H \ll m_A$

$$\bar{A}_i \propto a^{-3/2}$$

Misalignment : Abelian Field

- Action for the abelian vector field with non-minimal coupling to Ricci scalar :

$$S = \int d^4x\sqrt{-g}\left(\frac{M_p^2}{2}R - \frac{1}{4}g^{MN}g^{KL}\mathcal{F}_{MK}\mathcal{F}_{NL} - \frac{1}{2}g^{MN}A_M A_N\left(m^2 - \frac{\kappa}{6}R\right)\right)$$

- EOM physical vector field :

$$\ddot{\bar{A}}_i + 3H\dot{\bar{A}}_i + \left(m_A^2 + \left(\frac{1}{6} - \frac{\kappa}{6}\right)R\right)A_i = 0$$

- For $\kappa = 1$, the Hubble-induced mass term is vanished and the EOM becomes the same as a massive scalar field.

Paola Arias, et.al : 1201.5902
Gonzalo Alonso-Álvarez, et.al :1905.09836

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not gauge invariant

- EOM physical vector field :

$$\ddot{\bar{A}}_i + 3H\dot{\bar{A}}_i + \left(m_A^2 + \left(\frac{1}{6} - \frac{\kappa}{6}\right)R\right)A_i = 0$$

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Misalignment : Abelian Field

$$\ddot{\bar{A}}_i + 3H\dot{\bar{A}}_i + \left(m_A^2 + \left(\frac{1}{6} - \frac{\kappa}{6}\right)R\right)A_i = 0$$

Early time : $H \gg m_A$

Hubble friction dominates

$\bar{A}_i =$ **constant**

$$\rho_A = \frac{1}{2}\dot{\bar{A}}_i^2 + \frac{m^2}{2}\bar{A}_i^2 + (1-\kappa)\left(\frac{1}{2}H^2\bar{A}_i^2 + H\dot{\bar{A}}_i^2\bar{A}_i\right)$$

$\kappa = 1$

$\rho_{\bar{A}} =$ **constant**

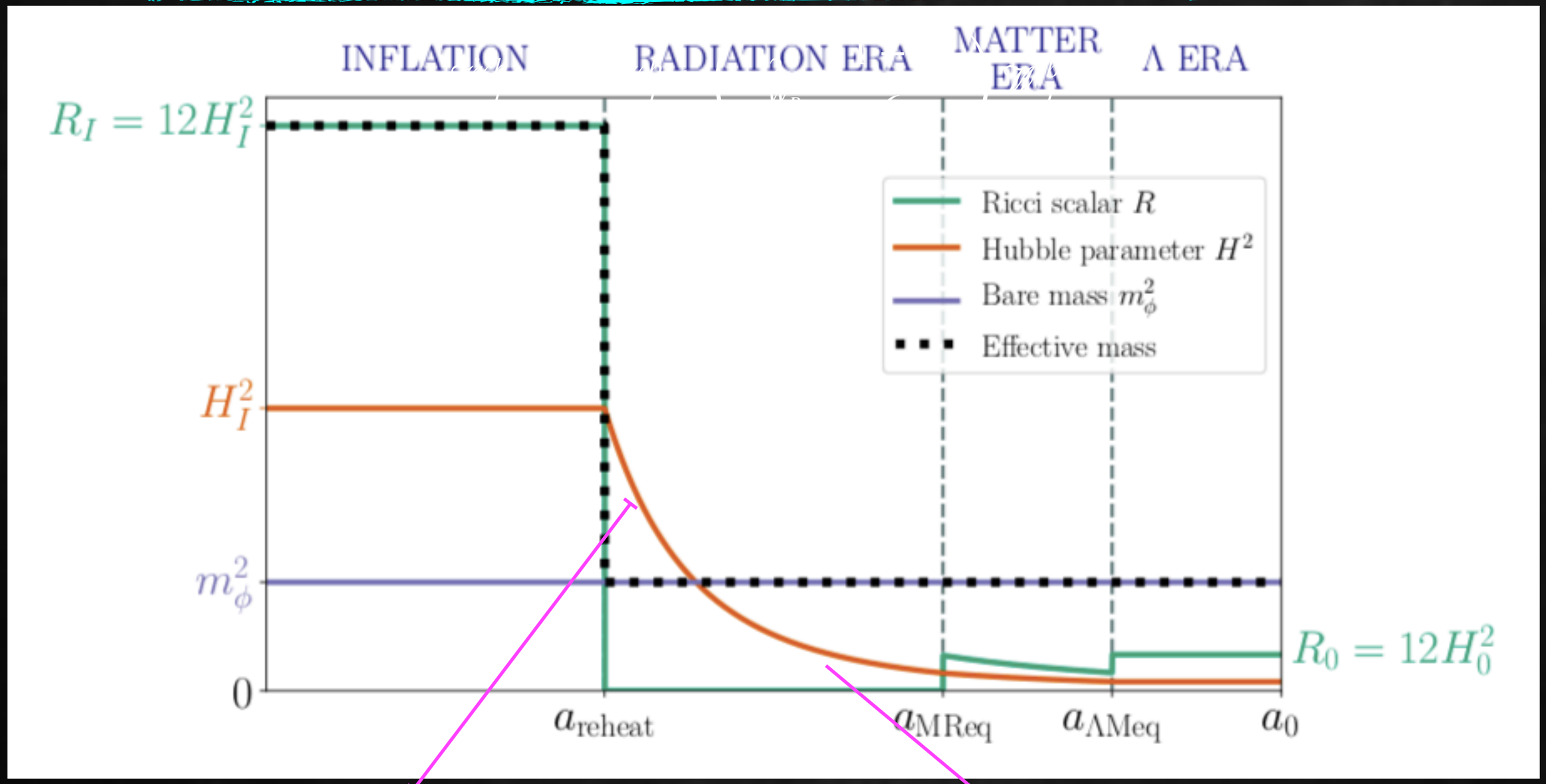
Late time : $H \ll m_A$

Oscillations begin

$\bar{A}_i \propto a^{-3/2}$

$\bar{A}_i \propto a^{-3}$

Misalignment : Abelian Field



$$\bar{A}^b_{i,e} \simeq \bar{A}^b_{i,s} \mathrm{e}^{-\frac{1}{2}(3-\sqrt{1+8\kappa})N_{tot}}$$

$$\bar{A}^b_i(t) = \bar{A}^b_{i,e}\left(\frac{a_\star}{a}\right)^{\frac{3}{2}} \cos\left(m_A\left(t-t_\star\right)\right)$$

Gonzalo Alonso-Álvarez, et.al :1905.09836

Isocurvature

adiabatic density inhomogeneity

The same source of generation

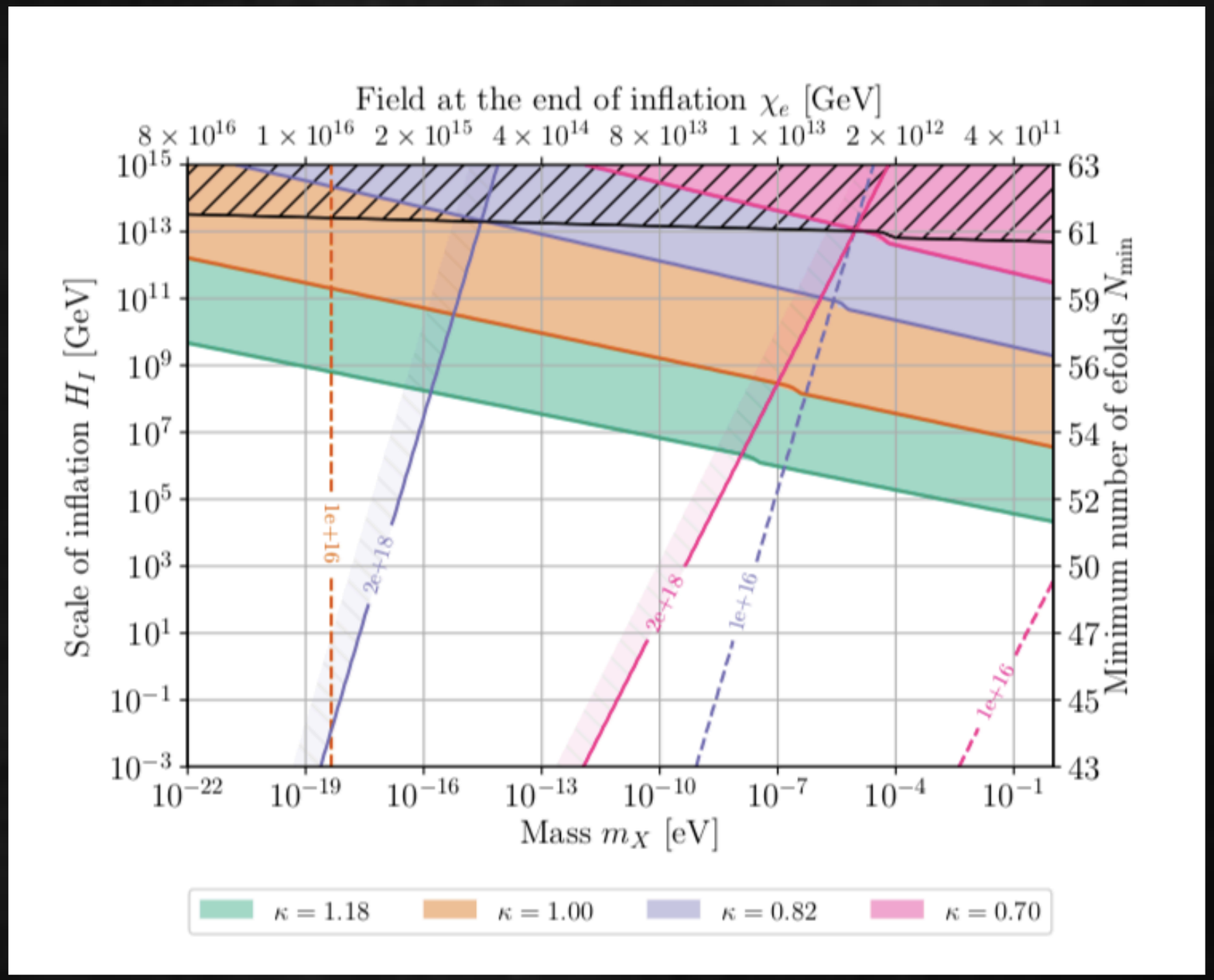
Isocurvature density inhomogeneity

different source of generation

Sweet Region of the parameter space

$$\frac{\Omega_{\bar{A}}}{\Omega_{\rm DM}} \simeq 5.2\mathscr{F}\left(T_\star\right)\sqrt{\frac{m_A}{\rm eV}}\left(\frac{\bar{A}_{i,e}}{10^{12}{\rm GeV}}\right)^2$$

$$\bar{A}_{i,e} \simeq \bar{A}_{i,s}{\rm e}^{-\frac{1}{2}(3-\sqrt{1+8\kappa})N_{tot}}$$



Gonzalo Alonso-Álvarez, et.al :1905.09836

Ghost instability

The action for the transverse and longitudinal modes :

$$S = S_T + S_L$$

$$S_T = \int \frac{d^3k d\tau}{(2\pi)^3} \frac{1}{2} \left(\left| \vec{A}'_T \right|^2 - \left(k^2 + a^2 m^2\right) \left| \vec{A}_T \right|^2 \right)$$

$$S_L = \int \frac{d^3k d\tau}{(2\pi)^3} \frac{1}{2} \left(\boxed{\frac{a^2 m^2}{k^2 + a^2 m^2}} \left| A'_L \right|^2 - a^2 m^2 \left| A_L \right|^2 \right)$$

- The non-minimal coupling scenario suffers from ghost instability.

After adding non-minimal coupling term $m^2 \to m^2 - \frac{\kappa}{6} R$

The kinetic term of the longitudinal mode
has a wrong sign in the short wavelength limit.

Before going to other solutions..
A few words about non-abelian symmetry

- The action of the non-abelian vector field, with non-minimal coupling to the Ricci scalar :

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2} \left(M_{\rm pl}^2 + \frac{\kappa}{6} W^a_\mu W^{a\mu} \right) R - \frac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} - \frac{1}{2} m_W^2 W^a_\mu W^{a\mu} \right)$$

- EOM spatially homogeneous but time-dependent Non-Abelian vector field :

$$\partial_\mu \left(\sqrt{-g} W^{b\mu\nu} \right) = \sqrt{-g} \left(\frac{\kappa}{6} R - m_W^2 \right) W^{b\nu} \xrightarrow{\sqrt{-g} = a^3} \partial_0 \left[a^3 W^b_{0\nu} \right] = a^3 \left(\frac{\kappa}{6} R - m_W^2 \right) W^{b\nu}$$

$$\left.\begin{aligned} 3H W^b_{00} + \partial_0 W^b_{00} &= - \left(\frac{\kappa}{6} R - m_W^2 \right) W^b \\ W^b_{00} = \partial_0 W^b_0 - \partial_0 W^b_0 + g_D f^b_{cd} W^c_0 W^d_0 &= 0 \end{aligned}\right\} \longrightarrow W^b_0 = 0$$

$$\left.\begin{aligned} \partial_0 W^b_{0i} + H W^b_{0i} + \left(m_W^2 - \frac{\kappa}{6} R \right) W^b_i &= 0 \\ W^b_0 &= 0 \end{aligned}\right\} \longrightarrow \ddot{W}^{bb}_R + H \dot{W}^b_R + \left(m^2_{W_R} - \frac{\kappa}{6} R \right) W^b_{Ri} = 0$$

$$\bar{W}^b_i \equiv W^b_{Ri}/a \longrightarrow \ddot{\bar{W}}^b_i + 3H \dot{\bar{W}}^b_i + \left(m_W^2 + \frac{1-\kappa}{6} R \right) \bar{W}^b_i = 0$$

FE, S. Khatibi, arXiv: 2204.04012

Alternative solution to Vector Misalignment

We can modify the kinetic term!

$$S = \int d^4x\sqrt{-g}\left(-\frac{1}{4}f^2(\phi)g^{MN}g^{KL}\mathcal{F}_{MK}\mathcal{F}_{NL} - \frac{1}{2}m^2 g^{MN}\mathcal{A}_M\mathcal{A}_N\right)$$

The energy density, in this case becomes:

$$\bar{A} \equiv \frac{f}{a}A$$

$$\rho_A = T_{00} = \frac{1}{2a^2}\left(f^2\dot{A}_i^2 + m^2 A_i^2\right) = \frac{1}{2}\left[\dot{\bar{A}}_i + \left(H - \frac{\dot{f}}{f}\right)\bar{A}_i\right]^2 + \frac{m^2}{2f^2}\bar{A}_i^2$$

Kazunori Nakayama, arXiv[1907.06243]

Misalignment – kinetic term modification

$$\ddot{\overline{A}}_i + 3H\dot{\overline{A}}_i + \left(\frac{m^2}{f^2} + 2H^2 + \dot{H} - H\frac{\dot{f}}{f} - \frac{\ddot{f}}{f}\right)\overline{A}_i = 0.$$

We have to specify $f(\phi)$: $f(\phi) \propto a^\alpha$ during inflation and $f(\phi) \to 1$ after inflation.

what should α be?

$$\ddot{\overline{A}}_i + 3H\dot{\overline{A}}_i + \left(\frac{m^2}{f^2} - \frac{(\alpha+4)(\alpha-2)}{4}H^2 + \frac{2-\alpha}{2}\dot{H}\right)\overline{A}_i = 0.$$

$$\boxed{\alpha = 2,\ \alpha = -4}$$

Kazunori Nakayama, arXiv[1907.06243]

Misalignment – kinetic term modification

$$\ddot{\bar{A}}_i + 3H\dot{\bar{A}}_i + \left(\frac{m^2}{f^2} - \frac{(\alpha+4)(\alpha-2)}{4}H^2 + \frac{2-\alpha}{2}\dot{H}\right)\bar{A}_i = 0.$$

$$\alpha = 2,\ \alpha = -4$$

Early time : $H \gg m_A/f$

Hubble friction dominates

$$\bar{A}_i = \text{constant}$$

$$\rho_A = \frac{1}{2}\left[\dot{\bar{A}}_i + (1-\alpha)H\bar{A}_i\right]^2 + \frac{m^2}{2f^2}\bar{A}_i^2$$

$$\rho_{\bar{A}} = \text{constant}$$

Late time : $H \ll m_A/f$

Oscillations begin

$$\bar{A}_i \propto a^{-3/2}$$

$$\bar{A}_i \propto a^{-3}$$

Kazunori Nakayama, arXiv[1907.06243]

How can $f(\phi) \propto a^{\alpha}$?

One way of parameterizing this function is the following:

$$f^2(\phi) = e^{c\phi^2/M_{\rm P}^2} \simeq \left(\frac{a_{end}}{a\left(N_e\right)}\right)^{2cn}$$

$$c \equiv -\alpha/(2n)$$

Monomial inflaton potential is disfavored by the observation of the cosmic microwave background!

Kazunori Nakayama, arXiv[1907.06243]

Are these values of α safe?

For $\alpha = -4$, the kinetic function is exponentially decreasing during inflation.

f is exponentially large as time goes back.
→ One might worry about the back-reaction to the scalar field dynamic.
To avoid this effect, we need

$$\left|\left(\partial_\phi f^2\right)\mathcal{F}\mathcal{F}\right| \lesssim \left|\partial_\phi V(\phi)\right|$$

which leads to

$$\left(\frac{\overline{A^{(\mathrm{in})}}}{M_{\mathrm{P}}}\right)^2 \lesssim \frac{M_{\mathrm{P}}^2}{2}\left(\frac{\partial_\phi V(\phi)}{V(\phi)}\right)^2 = \frac{r}{16} \sim 10^{-2}\left(\frac{H_{\mathrm{inf}}}{10^{14}\mathrm{GeV}}\right)^2$$

Kazunori Nakayama, arXiv[1907.06243]

Are these values of α safe?

For $\alpha = 2$, the kinetic function is exponentially increasing during inflation.

So, f is exponentially small in the beginning.
$\rightarrow$ theory is in a strongly coupled regime since f is roughly an inverse of the gauge coupling

(Problem for non-abelian symmetries, but not for abelian symmetries)

On the other hand, since we need $m/f \ll H$ during inflation, we get

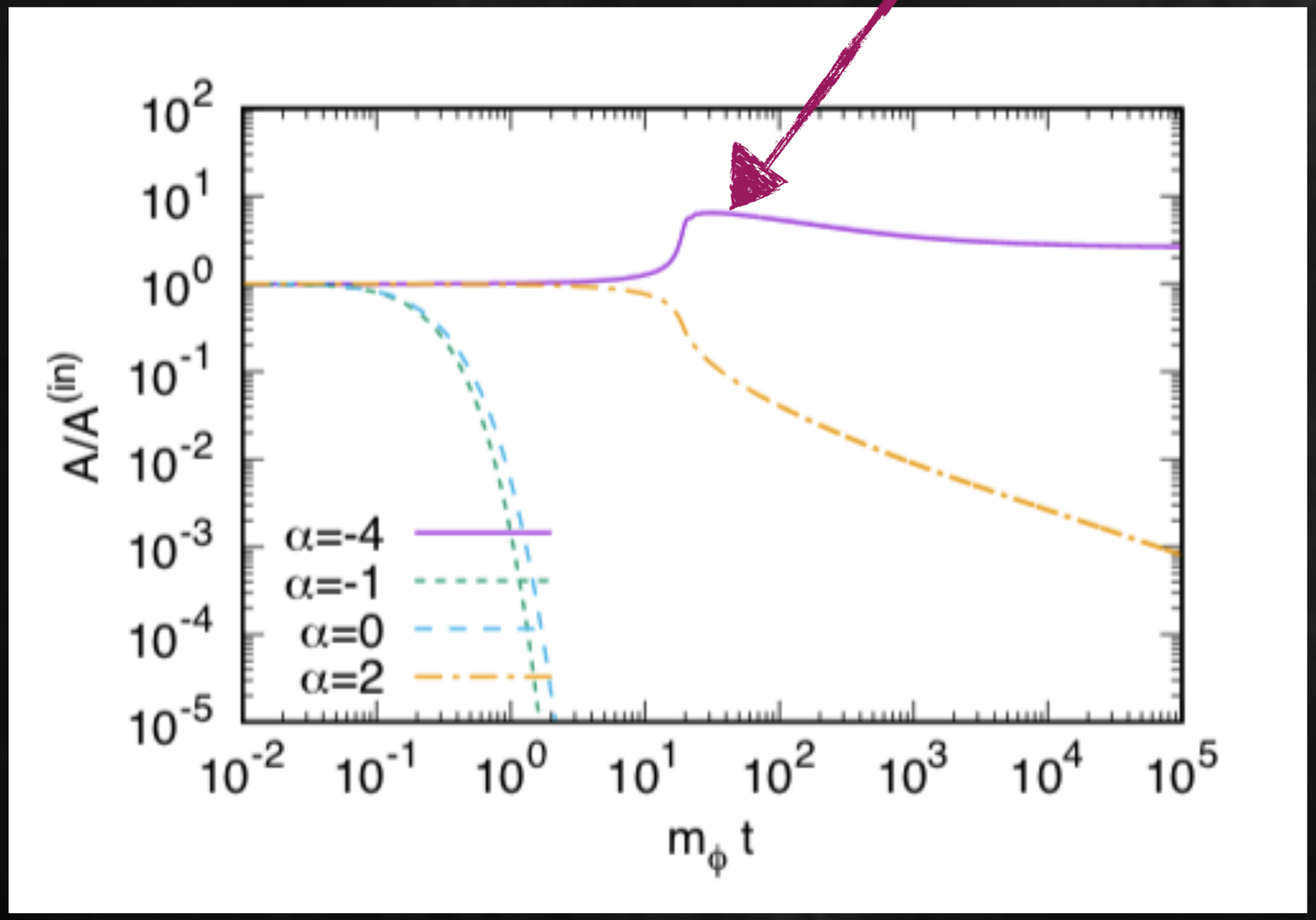
$$m \ll e^{-50} H_{\rm inf} \sim 10^{-22} H_{\rm inf} = 10 \text{ eV} \left(\frac{H_{\rm inf}}{10^{14} \text{GeV}} \right)$$

Works best for very light masses

Kazunori Nakayama, arXiv[1907.06243]

What happens outside of these values of α?

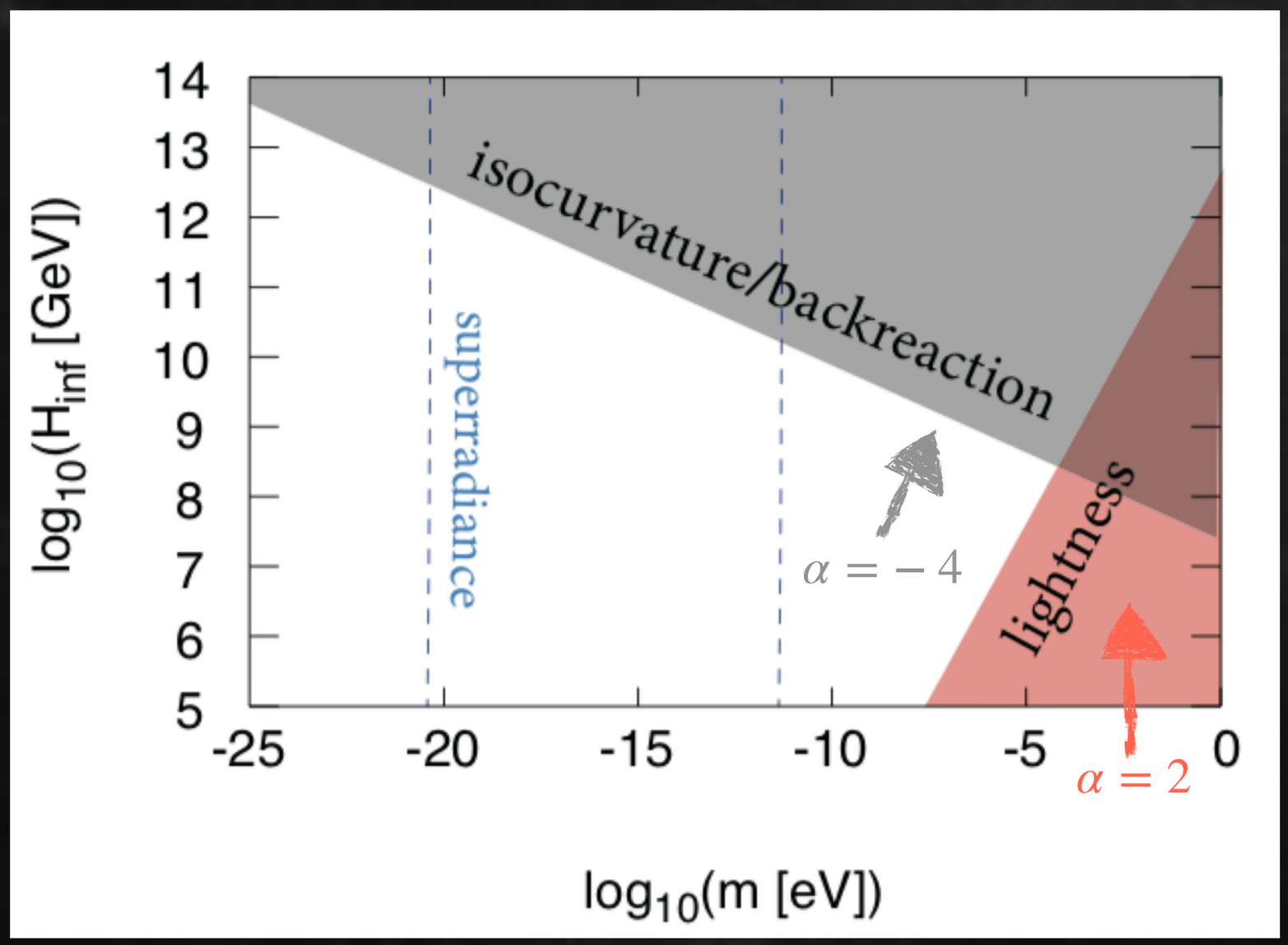
$$\ddot{\overline{A}}_i + 3H\dot{\overline{A}}_i + \left(\frac{m^2}{f^2} - \frac{(\alpha+4)(\alpha-2)}{4}H^2 + \boxed{\frac{2-\alpha}{2}\dot{H}}\right)\overline{A}_i = 0.$$



Kazunori Nakayama, arXiv[1907.06243]

Relic Abundance

$$\frac{\rho_A}{s} = \simeq 1\mathrm{GeV}\left(\frac{m}{10^{-9}\mathrm{eV}}\right)^{1/2}\left(\frac{\overline{A^{(\mathrm{in})}}}{M_{\mathrm{P}}}\right)^2$$



Kazunori Nakayama, arXiv[1907.06243]

Conclusion

$$\ddot{A}_i + H\dot{A}_i + m_A^2 A_i = 0$$

$$\bar{A}_i \equiv A_i/a$$

$$\ddot{\bar{A}}_i + 3\dot{H}\dot{\bar{A}}_i + \left(m_{\bar{A}}^2 + \frac{R}{6}\right)\bar{A}_i = 0$$

$$\mathscr{L} \supset \frac{R}{6}\bar{A}_\mu \bar{A}^\mu R$$

$$\mathscr{L} \supset f^2(\phi)\mathscr{F}\mathscr{F}$$

$$\ddot{\bar{A}}_i + 3H\dot{\bar{A}}_i + \left(m_A^2 + \left(\frac{1}{6} - \frac{\kappa}{6}\right)R\right)\bar{A}_i = 0$$

$$\ddot{\overline{A}}_i + 3H\dot{\overline{A}}_i + \left(\frac{m^2}{f^2} - \frac{(\alpha+4)(\alpha-2)}{4}H^2 + \frac{2-\alpha}{2}\dot{H}\right)\overline{A}_i = 0.$$

$$\alpha = 2,\ \alpha = -4$$

Problem with ghost instability and gauge invariance!

Can easily be generalized to non-abelian symmetries!

No ghost instability!

The dynamic of inflation?
generalization to non-abelian symmetries?

Conclusion

$$\ddot{A}_i + H\dot{A}_i + m^2 A_i = 0$$

Thank you for your attention!

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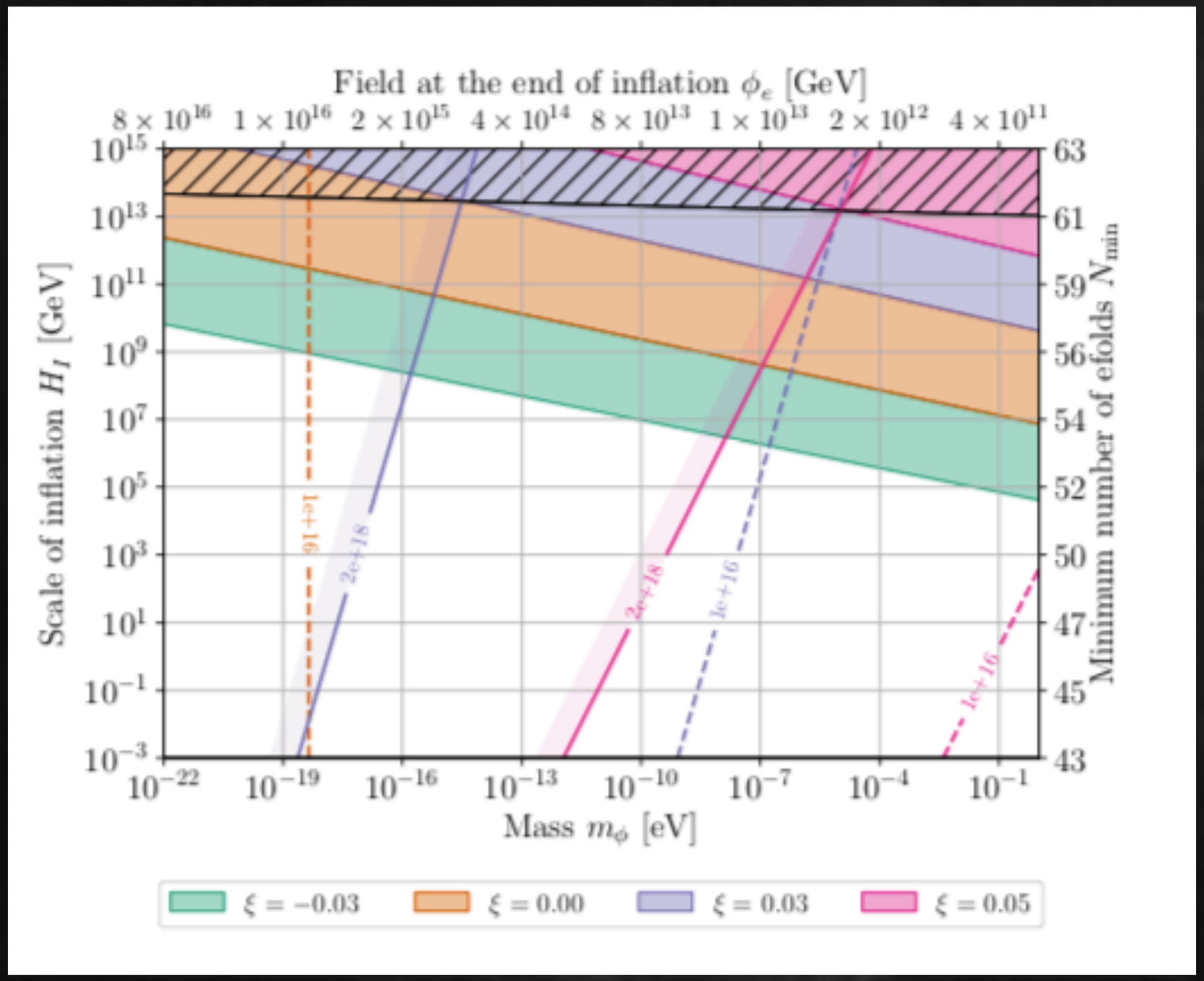
Back up



Gonzalo Alonso-Álvarez, et.al :1905.09836

$$\partial_0 \left[a^3 X_{0\nu} \underbrace{g^{00}}_{-1} g^{\nu\beta} \right] = a^3 \left(-m_\chi^2 + \frac{kR}{6}\right) X_\nu g^{\nu\beta} \qquad \boxed{\partial_i X_\mu = 0}$$

$$\boxed{\beta = 0} \longrightarrow X_0 = 0$$

$$\boxed{\beta = i} \quad \partial_0 \left(-a^3 X_{0i} \underbrace{g^{ii}}_{-\frac{1}{a^2}}\right) = a^3 \left(-m_\chi^2 + \frac{kR}{6}\right) X_i \underbrace{g^{ii}}_{\frac{1}{a^2}}$$

$$-\left(\dot{a} X_{0i} + a \dot{X}_{0i}\right) = a \left(-m_\chi^2 + \frac{kR}{6}\right) X_i$$

$$-\left(H X_{0i} + \dot{X}_{0i}\right) = \left(-m_\chi^2 + \frac{kR}{6}\right) X_i$$

$$\partial_0 \left(\partial_0 X_i - \cancel{\partial_i X_0}^{\,0}\right) + H \left(\partial_0 X_i - \cancel{\partial_i X_0}^{\,0}\right) + \left(m_\chi^2 - \frac{kR}{6}\right) X_i$$

$$\ddot{X}_i + H \dot{X}_i + \left(m_\chi^2 - \frac{k}{6} R\right) X_i = 0 \quad \checkmark$$