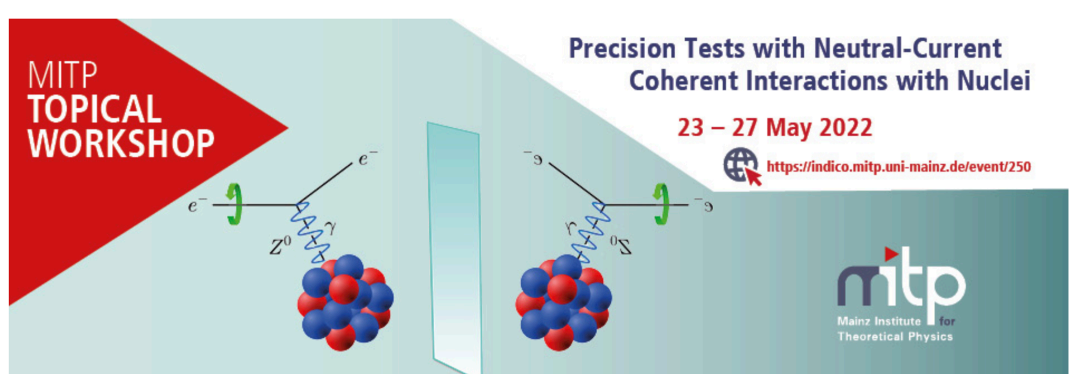
Phenomenology of Quadratic approximants
The Pion Form Factor Case

Pere Masjuan
(masjuan@ifae.es)

Preliminary and ongoing work done in collaboration with
Cristian Alarcón, Sergi González-Solís and Pablo Sanchez-Puertas



Precision Tests with Neutral-Current Coherent Interactions with Nuclei

IFAE
EXCELENCIA SEVERO OCHOA
BIST Barcelona Institute of Science and Technology
UAB
Universitat Autònoma de Barcelona

The Problem

Think about your calculations, your result.

Did you use a model?

- Vector Meson Dominance
- Breit Wigner
- Omnès Representation
- Z-parameterization

Does the model have?

- Statistical error coming from fitting data?
- Parametric uncertainty coming from input errors?
- An uncertainty accounting for the modeling used?

The Problem

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Does the model have?

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The Solution

What then?

Use a Rational Approximants!

- Robust
- Model independent
- Economic
- Provides an error
- Set of powerful theorems of convergence exist!

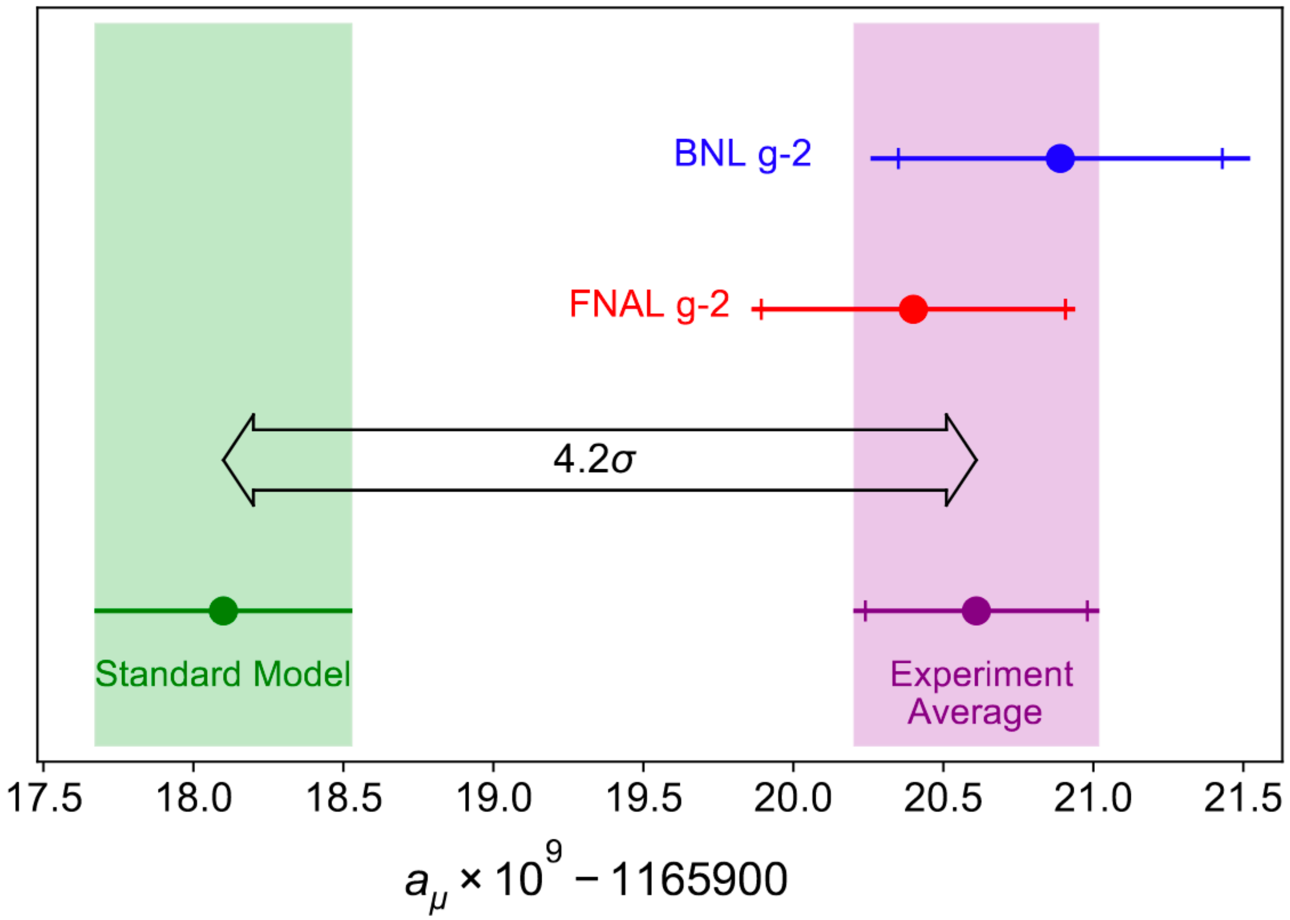
Outline

- Motivation: New Physics in muon g-2?
- Why Rational Approximants?
- Introduction to Padé and Quadratic approximants
    - Pion Form Factor case
- Outlook and Conclusions

The anomalous magnetic moment of the muon

FermiLab release

2104.03281



(6.4)

$$a_\mu^{\text{BNL}} = 11\,659\,209.1(5.4)(3.3) \times 10^{-10}$$

$$a_\mu^{\text{FNL}} = 11\,659\,204.0(5.1)(1.9) \times 10^{-10}$$

(5.4)

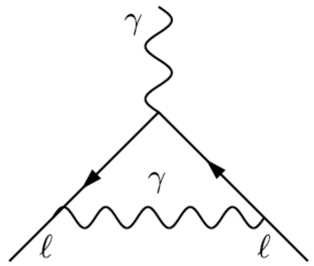
$$a_\mu^{\text{exp}} = 11\,659\,206.1(4.1) \times 10^{-10}$$

$$a_\mu^{\text{th}} = 11\,659\,181.0(4.3) \times 10^{-10}$$

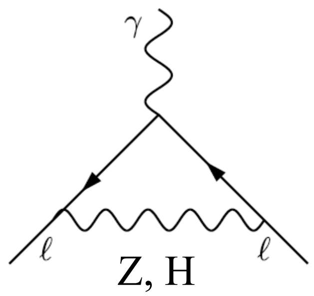
$$\Delta a_\mu = (25.1 \pm 5.9) \times 10^{-10}$$

The anomalous magnetic moment of the muon

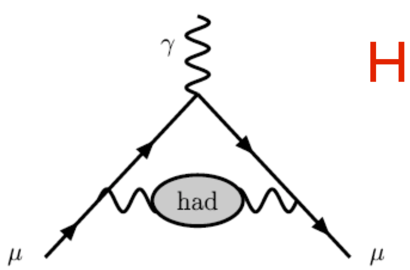
$$a_\mu^{th} = a_\mu^{QED} + a_\mu^{weak} + a_\mu^{had}$$



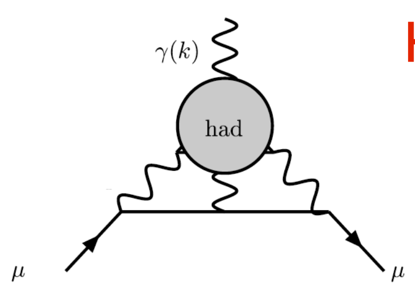
QED: $11658471.8931(0.0104) \times 10^{-10}$



EW: $15.36(0.10) \times 10^{-10}$



Hadronic VP: $684.5(4.0) \times 10^{-10}$



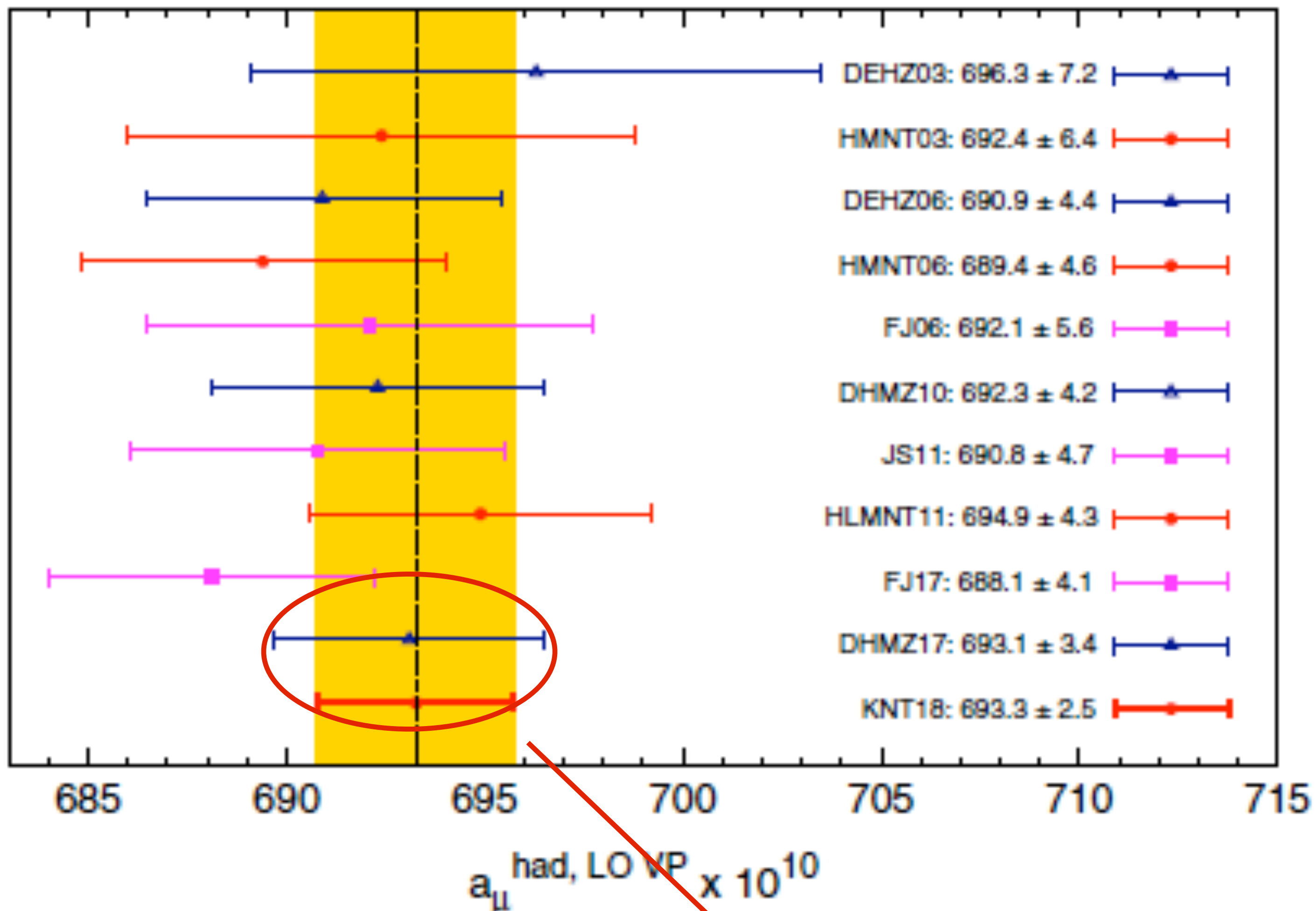
Hadronic LBL: $9.2(1.8) \times 10^{-10}$

[White Paper: 2006.04822]

SM: $11659181.0(4.3) \times 10^{-10}$	$\Delta a_\mu = 27.9(7.6) \times 10^{-10} (3.7)\sigma$

Hadronic Vacuum Polarization

White Paper Result



[Keshavarzi et al 18]

$$a_\mu^{\text{had, LO VP}} = 693.26(2.46) \times 10^{-10}$$

$$a_\mu^{\text{had, NLO VP}} = -9.82(0.04) \times 10^{-10}$$

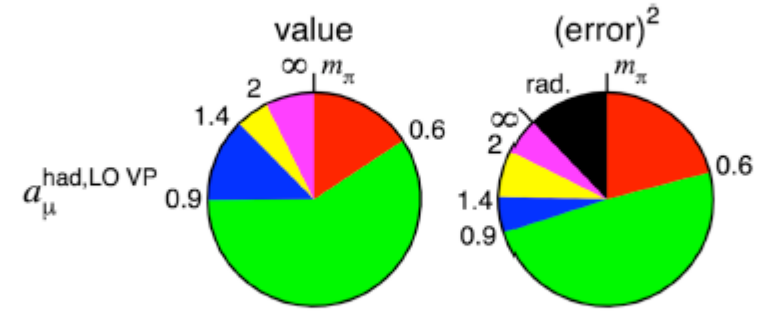
$$a_\mu^{\text{had, NNLO VP}} = 1.24(0.01) \times 10^{-10}$$

$$a_\mu^{\text{HVP}} = 684.5(4.0) \times 10^{-10}$$

[White Paper: 2006.04822]

Hadronic Vacuum Polarization

[Keshavarzi et al 18]



Channel	Energy range [GeV]	$a_\mu^{\mathrm{had,LO\,VP}} \times 10^{10}$
Chiral perturbation theory (ChPT) threshold contributions		
$\pi^0\gamma$	$m_\pi \le \sqrt{s} \le 0.600$	0.12 ± 0.01
$\pi^+\pi^-$	$2m_\pi \le \sqrt{s} \le 0.305$	0.87 ± 0.02
$\pi^+\pi^-\pi^0$	$3m_\pi \le \sqrt{s} \le 0.660$	0.01 ± 0.00
$\eta\gamma$	$m_\eta \le \sqrt{s} \le 0.660$	0.00 ± 0.00
Data based channels ($\sqrt{s} \le 1.937$ GeV)		
$\pi^0\gamma$	$0.600 \le \sqrt{s} \le 1.350$	4.46 ± 0.10
$\pi^+\pi^-$	$0.305 \le \sqrt{s} \le 1.937$	502.97 ± 1.97
$\pi^+\pi^-\pi^0$	$0.660 \le \sqrt{s} \le 1.937$	47.79 ± 0.89
$\pi^+\pi^-\pi^+\pi^-$	$0.613 \le \sqrt{s} \le 1.937$	14.87 ± 0.20
$\pi^+\pi^-\pi^0\pi^0$	$0.850 \le \sqrt{s} \le 1.937$	19.39 ± 0.78
$(2\pi^+2\pi^-\pi^0)_{\mathrm{no}\eta}$	$1.013 \le \sqrt{s} \le 1.937$	0.99 ± 0.09
$3\pi^+3\pi^-$	$1.313 \le \sqrt{s} \le 1.937$	0.23 ± 0.01
$(2\pi^+2\pi^-2\pi^0)_{\mathrm{no}\eta\omega}$	$1.322 \le \sqrt{s} \le 1.937$	1.35 ± 0.17
K^+K^-	$0.988 \le \sqrt{s} \le 1.937$	23.03 ± 0.22
$K_S^0K_L^0$	$1.004 \le \sqrt{s} \le 1.937$	13.04 ± 0.19
$KK\pi$	$1.260 \le \sqrt{s} \le 1.937$	2.71 ± 0.12
$KK2\pi$	$1.350 \le \sqrt{s} \le 1.937$	1.93 ± 0.08
$\eta\gamma$	$0.660 \le \sqrt{s} \le 1.760$	0.70 ± 0.02
$\eta\pi^+\pi^-$	$1.091 \le \sqrt{s} \le 1.937$	1.29 ± 0.06
$(\eta\pi^+\pi^-\pi^0)_{\mathrm{no}\omega}$	$1.333 \le \sqrt{s} \le 1.937$	0.60 ± 0.15
$\eta 2\pi^+2\pi^-$	$1.338 \le \sqrt{s} \le 1.937$	0.08 ± 0.01
$\eta\omega$	$1.333 \le \sqrt{s} \le 1.937$	0.31 ± 0.03
$\omega(\to \pi^0\gamma)\pi^0$	$0.920 \le \sqrt{s} \le 1.937$	0.88 ± 0.02
$\eta\phi$	$1.569 \le \sqrt{s} \le 1.937$	0.42 ± 0.03
$\phi \to$ unaccounted	$0.988 \le \sqrt{s} \le 1.029$	0.04 ± 0.04
$\eta\omega\pi^0$	$1.550 \le \sqrt{s} \le 1.937$	0.35 ± 0.09
$\eta(\to \mathrm{npp})K\bar{K}_{\mathrm{no}\phi\to K\bar{K}}$	$1.569 \le \sqrt{s} \le 1.937$	0.01 ± 0.02
$p\bar{p}$	$1.890 \le \sqrt{s} \le 1.937$	0.03 ± 0.00
$n\bar{n}$	$1.912 \le \sqrt{s} \le 1.937$	0.03 ± 0.01
Estimated contributions ($\sqrt{s} \le 1.937$ GeV)		
$(\pi^+\pi^-3\pi^0)_{\mathrm{no}\eta}$	$1.013 \le \sqrt{s} \le 1.937$	0.50 ± 0.04
$(\pi^+\pi^-4\pi^0)_{\mathrm{no}\eta}$	$1.313 \le \sqrt{s} \le 1.937$	0.21 ± 0.21
$KK3\pi$	$1.569 \le \sqrt{s} \le 1.937$	0.03 ± 0.02
$\omega(\to \mathrm{npp})2\pi$	$1.285 \le \sqrt{s} \le 1.937$	0.10 ± 0.02
$\omega(\to \mathrm{npp})3\pi$	$1.322 \le \sqrt{s} \le 1.937$	0.17 ± 0.03
$\omega(\to \mathrm{npp})KK$	$1.569 \le \sqrt{s} \le 1.937$	0.00 ± 0.00
$\eta\pi^+\pi^-2\pi^0$	$1.338 \le \sqrt{s} \le 1.937$	0.08 ± 0.04
Other contributions ($\sqrt{s} > 1.937$ GeV)		
Inclusive channel	$1.937 \le \sqrt{s} \le 11.199$	43.67 ± 0.67
J/ψ	...	6.26 ± 0.19
ψ'	...	1.58 ± 0.04
$\Upsilon(1S-4S)$	...	0.09 ± 0.00
pQCD	$11.199 \le \sqrt{s} \le \infty$	2.07 ± 0.00
Total	$m_\pi \le \sqrt{s} \le \infty$	693.26 ± 2.46

Hadronic Vacuum Polarization

[Keshavarzi et al 18]

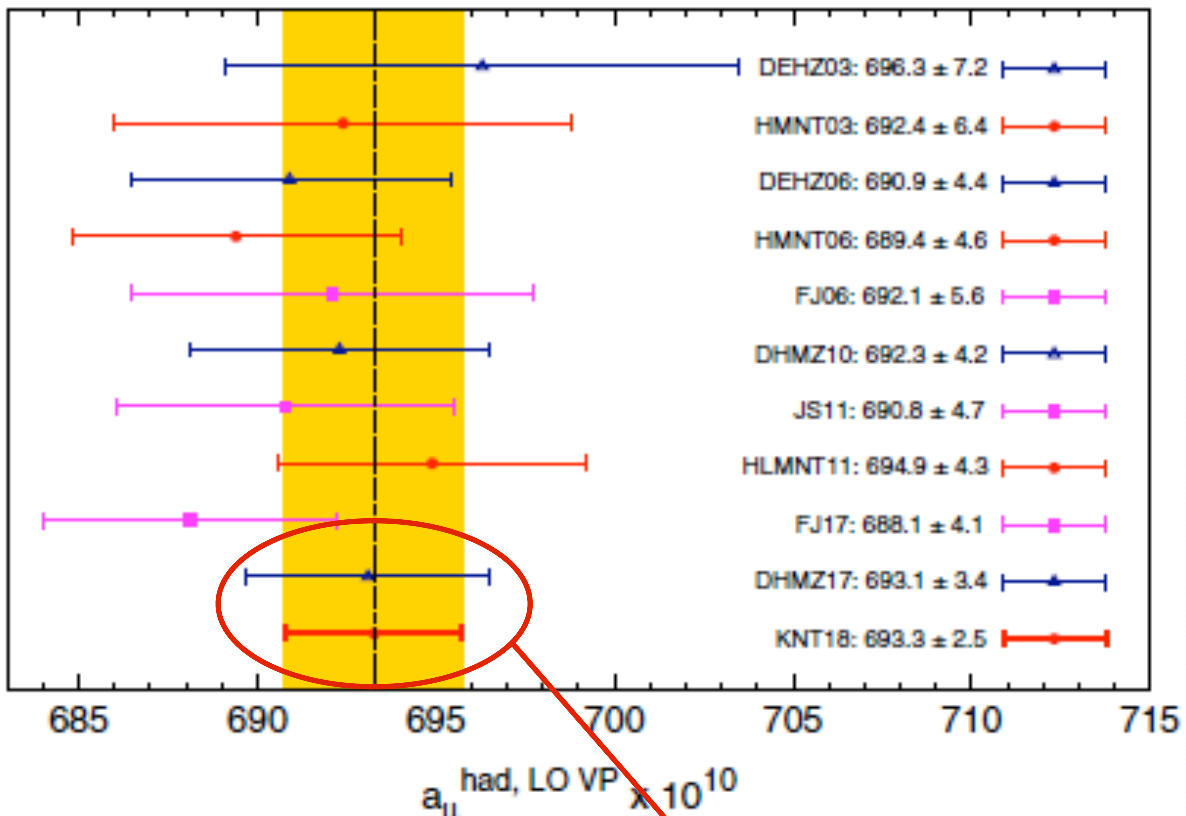
Channel	This work (KNT18)	DHMZ17 [78]	Difference
	Data based channels ($\sqrt{s} \leq 1.8$ GeV)		
$\pi^0\gamma$ (data + ChPT)	4.58 ± 0.10	4.29 ± 0.10	0.29
$\pi^+\pi^-$ (data + ChPT)	503.74 ± 1.96	507.14 ± 2.58	−3.40
$\pi^+\pi^-\pi^0$ (data + ChPT)	47.70 ± 0.89	46.20 ± 1.45	1.50
$\pi^+\pi^-\pi^+\pi^-$	13.99 ± 0.19	13.68 ± 0.31	0.31
$\pi^+\pi^-\pi^0\pi^0$	18.15 ± 0.74	18.03 ± 0.54	0.12
$(2\pi^+2\pi^-\pi^0)_{\mathrm{no}\eta}$	0.79 ± 0.08	0.69 ± 0.08	0.10
$3\pi^+3\pi^-$	0.10 ± 0.01	0.11 ± 0.01	−0.01
$(2\pi^+2\pi^-2\pi^0)_{\mathrm{no}\eta\omega}$	0.77 ± 0.11	0.72 ± 0.17	0.05
K^+K^-	23.00 ± 0.22	22.81 ± 0.41	0.19
$K^0_SK^0_L$	13.04 ± 0.19	12.82 ± 0.24	0.22
$KK\pi$	2.44 ± 0.11	2.45 ± 0.15	−0.01
$KK2\pi$	0.86 ± 0.05	0.85 ± 0.05	0.01
$\eta\gamma$ (data + ChPT)	0.70 ± 0.02	0.65 ± 0.02	0.05
$\eta\pi^+\pi^-$	1.18 ± 0.05	1.18 ± 0.07	0.00
$(\eta\pi^+\pi^-\pi^0)_{\mathrm{no}\omega}$	0.48 ± 0.12	0.39 ± 0.12	0.09
$\eta 2\pi^+2\pi^-$	0.03 ± 0.01	0.03 ± 0.01	0.00
$\eta\omega$	0.29 ± 0.02	0.32 ± 0.03	−0.03
$\omega(\to \pi^0\gamma)\pi^0$	0.87 ± 0.02	0.94 ± 0.03	−0.07
$\eta\phi$	0.33 ± 0.03	0.36 ± 0.03	−0.03
$\phi \to$ unaccounted	0.04 ± 0.04	0.05 ± 0.00	−0.01
$\eta\omega\pi^0$	0.10 ± 0.05	0.06 ± 0.04	0.04
$\eta(\to \mathrm{npp})K\bar{K}_{\mathrm{no}\phi\to K\bar{K}}$	0.00 ± 0.01	0.01 ± 0.01	−0.01 [a]

Combined in quadrature: 4.0×10^{-10}

Combined linearly: 6.5×10^{-10}

Hadronic Vacuum Polarization

White Paper Result



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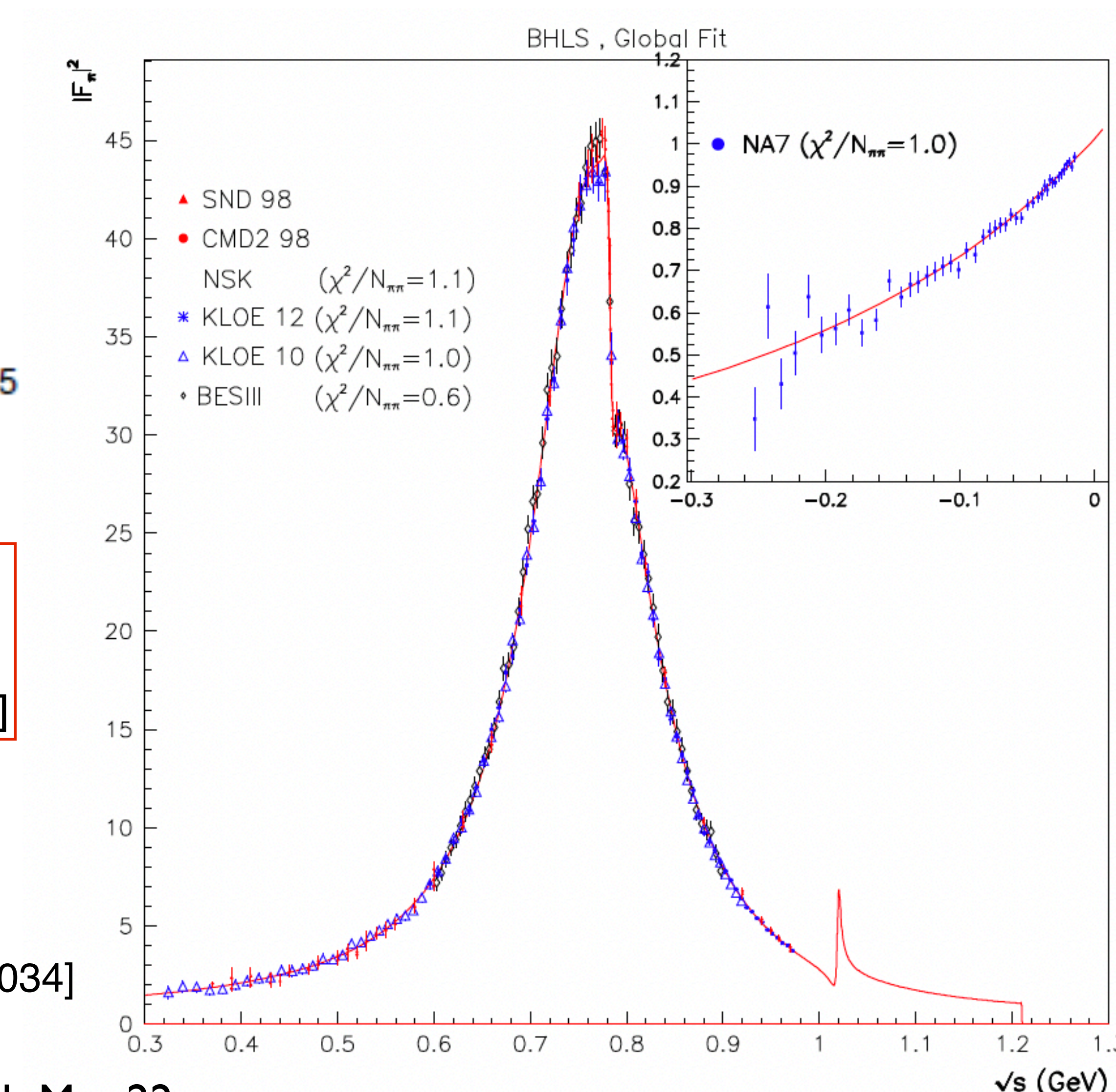
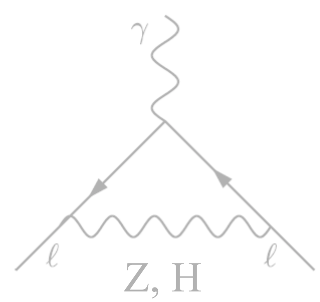


[Benayoun, Delbuono, Jegerlehner, 1903.11034]

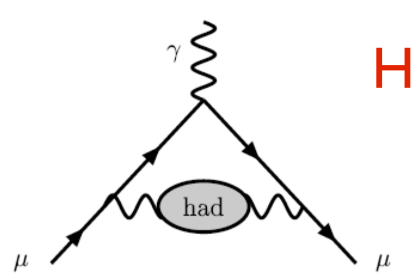
Hadronic Vacuum Polarization

$$a_\mu^{th} = a_\mu^{QED} + a_\mu^{weak} + a_\mu^{had}$$

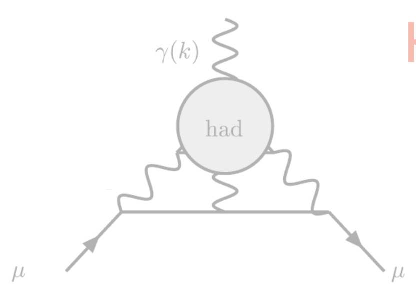
QED:



EW:



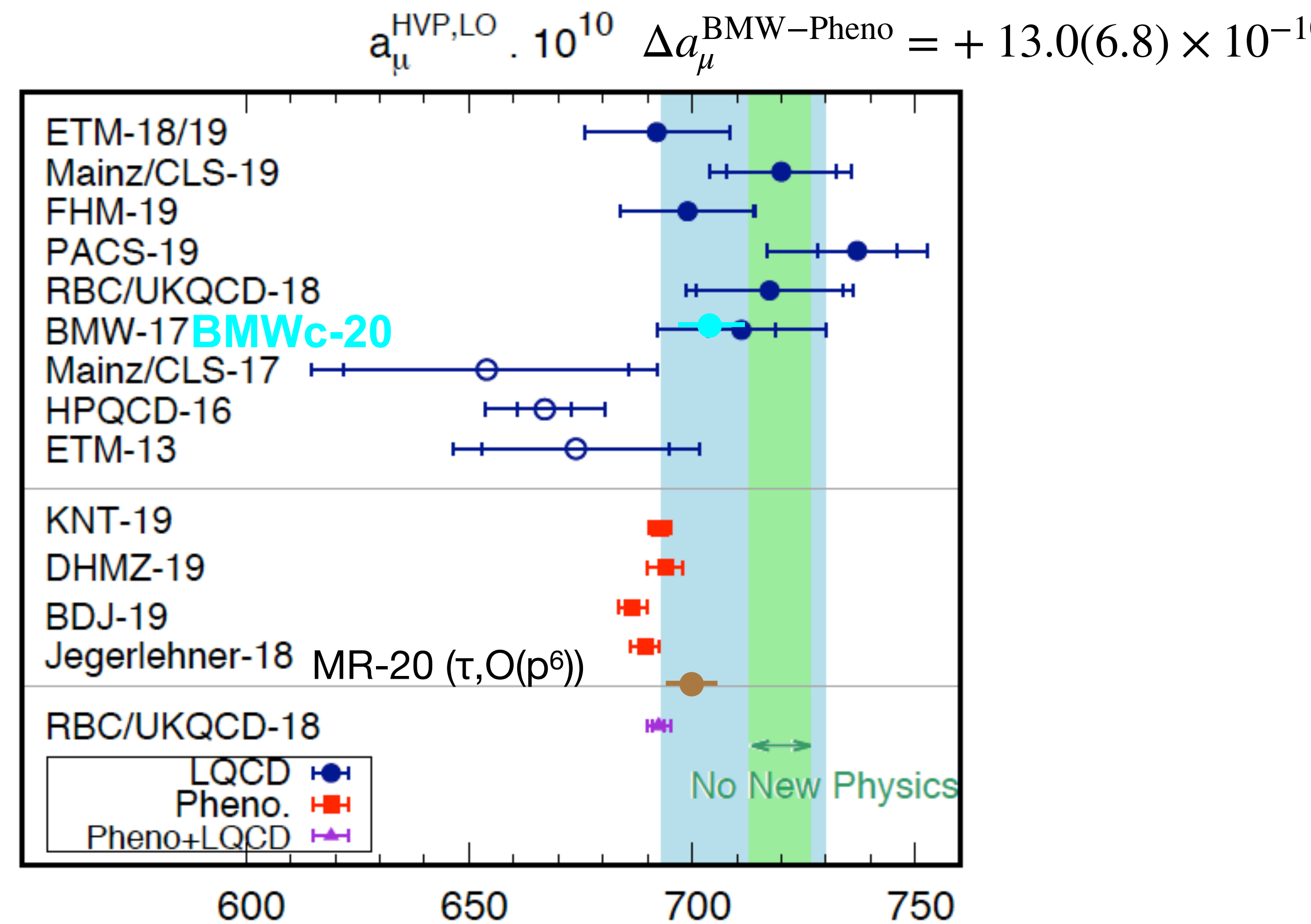
Hadronic VP:



Hadronic LBL:

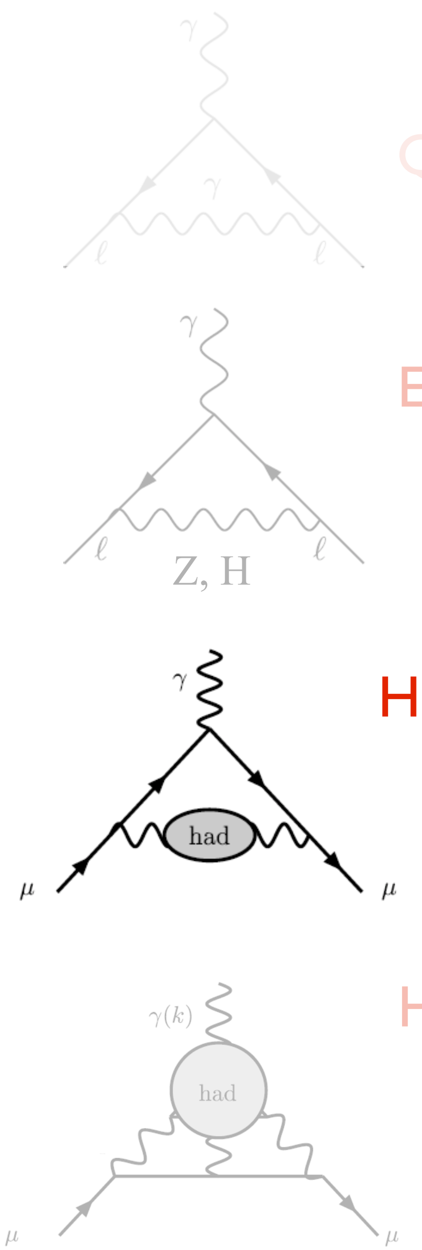
Progress on Hadronic Vacuum Polarization:

Lattice simulation!



Hadronic Vacuum Polarization

$$a_\mu^{th} = a_\mu^{QED} + a_\mu^{weak} + a_\mu^{had}$$



QED:

EW:

Hadronic VP:

Hadronic LBL

Progress on Hadronic Vacuum Polarization:

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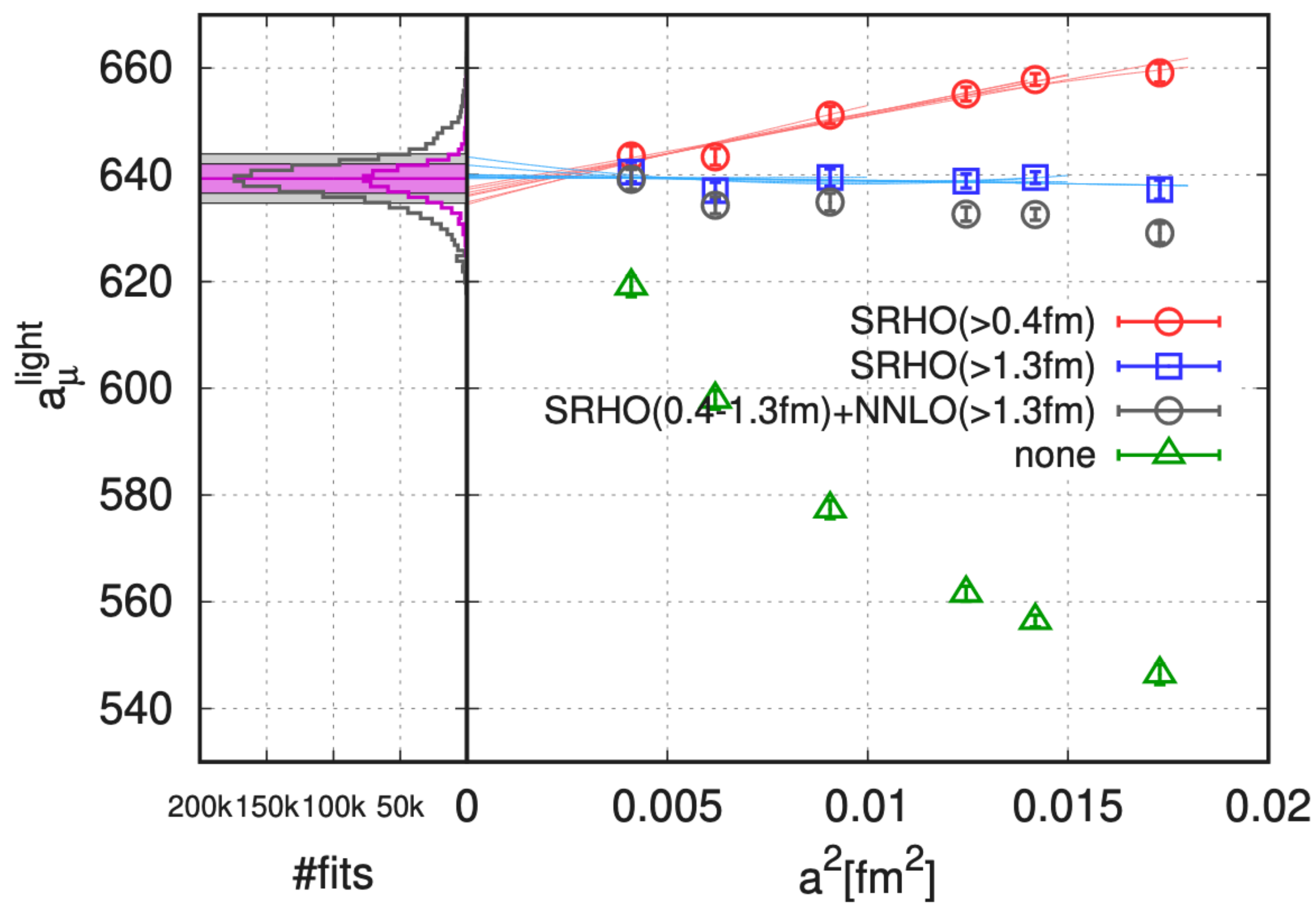

Why Rational Approximants?

- To THEORY:
  - our problem is "the general rational Hermite interpolation problem": solution is RAs
  - in many cases we need $\int ds\, f(s)$ not $f(s)$
  - it is proven: some RAs slower convergence than others
  - Use prefered model as "Leading Order"
- To DISPERSION Theory:
  - excellent interpolation tool: bring DRs (TL!) to ∞ (link with pQCD?)
    - even more: RA can be the function to be used for the DR (respects unitary)
    - or DR can provide LECs to feed in the RA
- To LATTICE:
  - used recently for lattice fitting of HVP: suggestion use it in its full glory!
- To EXPERIMENT:

  what energy and precision to measure: use RAs to identify!

  (high energies are as well important vs DRs)

Padé approximants

Padé approx: $Q(z)f(z) + R(z) = \mathcal{O}\left(z^{q+r+1}\right)$

$R(z), Q(z)$ are polynomials

Let f(z)

$$f(z) = \sum_{k=0} a_k z^k$$ then its PA $$P^N_M(z) = \frac{\sum_{n=0}^{N} r_n z^n}{\sum_{m=0}^{M} q_n z^n}$$

and the PA has a <u>contact</u> with f(z) or order N+M+1

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and the PA has a <u>contact</u> with f(z) or order N+M+1

$$\begin{cases} P_M^N(z) = r_0 + (r_1 - r_0 q_1)z + (r_2 - r_1 q_1 + r_0 q_1^2 - r_0 q_2)z^2 + \mathcal{O}(z^3) \\ f(z) = a_0 + a_1 z + a_2 z^2 + \mathcal{O}(z^3) \end{cases}$$

Examples: $$P_1^0(z) = \frac{a_0}{1 - \frac{a_1}{a_0} z}$$ $$P_1^1(z) = \frac{a_0 + \frac{a_1^2 - a_0 a_2}{a_1}}{1 - \frac{a_2}{a_1} z}$$

Padé approximants

Padé approx: $Q(z)f(z)+R(z)=\mathcal{O}\left(z^{q+r+1}\right)$

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Stieltjes theorem:

$$\lim_{N\to\infty} P^N_{N+1}(z) \leq f(z) \leq \lim_{N\to\infty} P^N_N(z)$$

(others: Montessus, Pommerenke, Nutall, Baker, Chisholm...)

Example: $f(z)=\frac{1}{z}\log(1-z)$

$$f(z)=-\sum_{k=0}\frac{z^k}{k+1}=-1-\frac{z}{2}-\frac{z^2}{3}-\frac{z^3}{4}-\frac{z^4}{5}+\mathcal{O}(z^6)$$

Padé approximants

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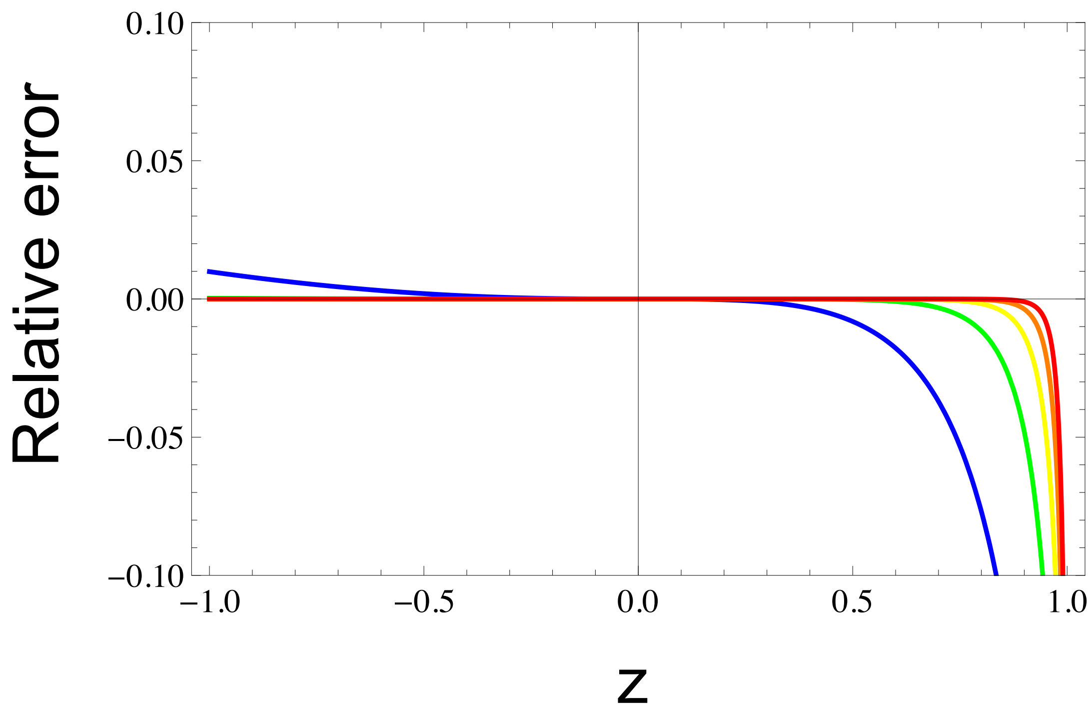
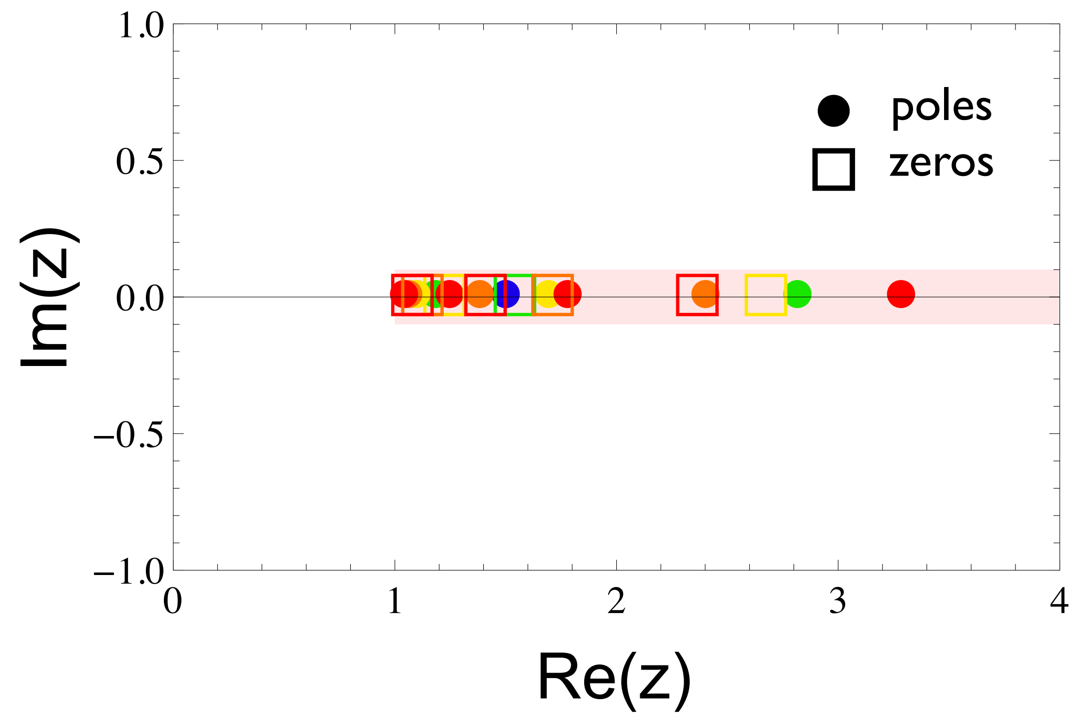
$P_N^N(z)$ for N=1,2,3,4,5





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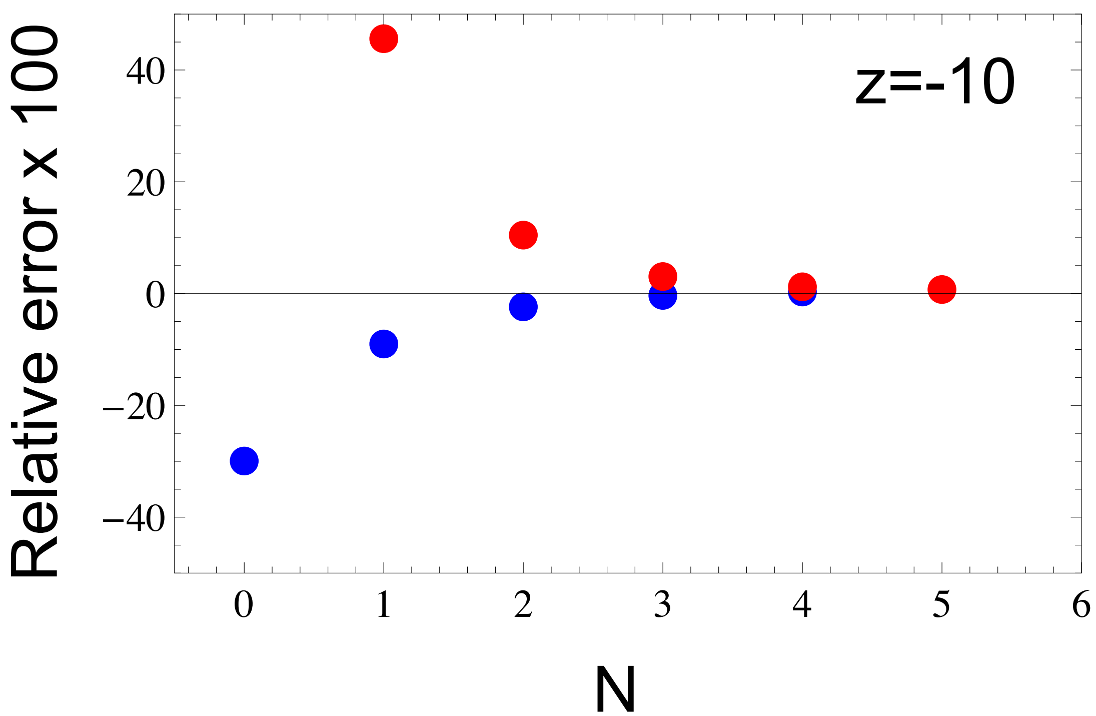

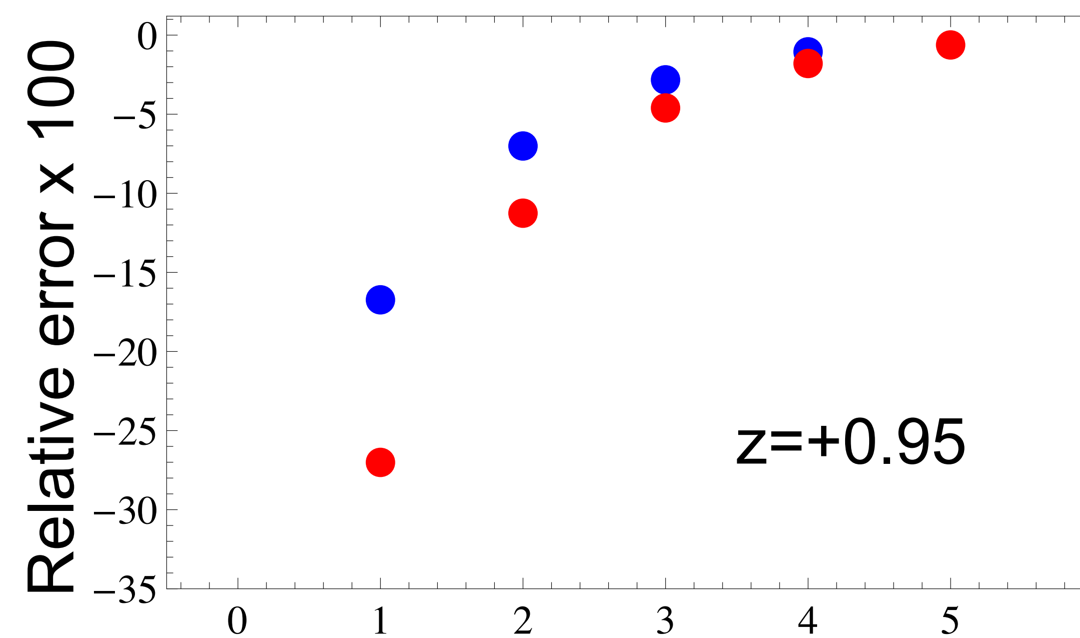

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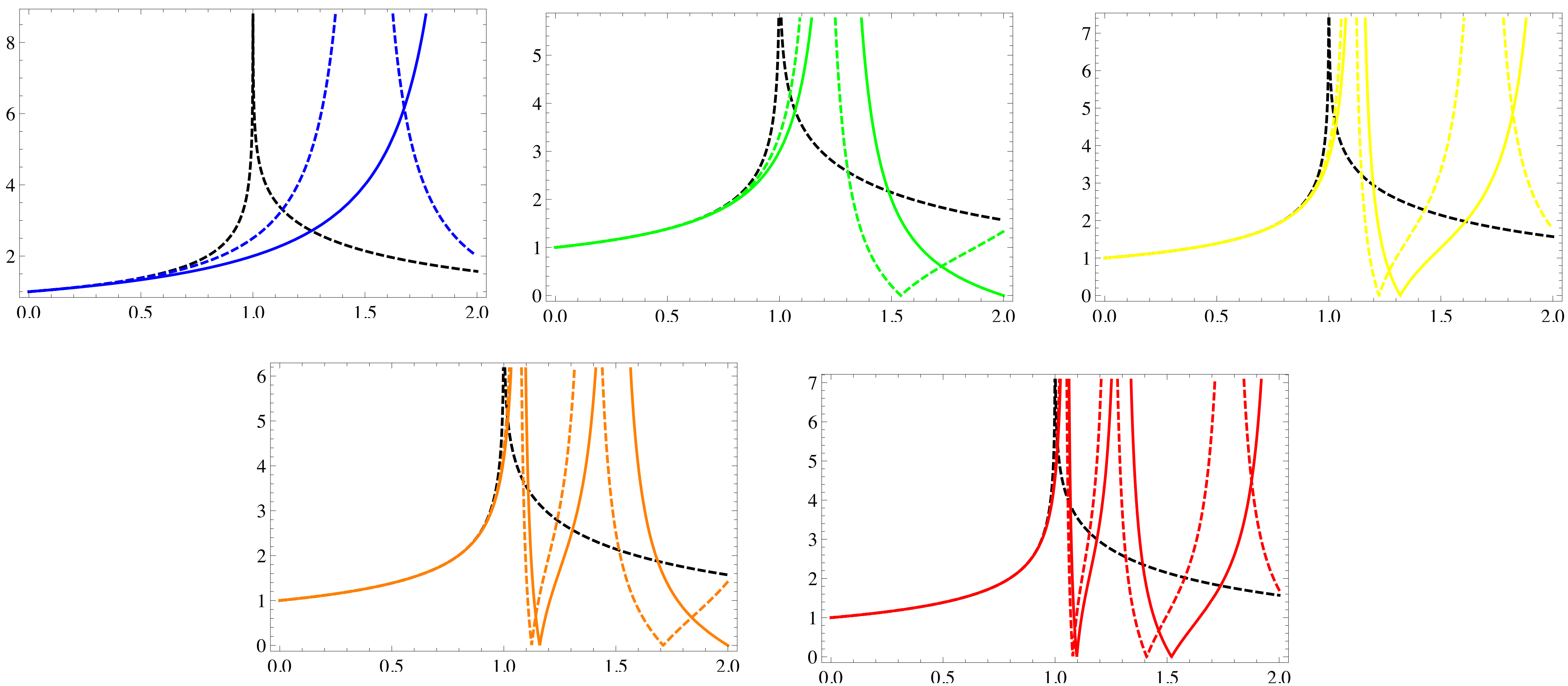


Padé approximants

Padé approx: $Q(z)f(z) + R(z) = \mathcal{O}\left(z^{q+r+1}\right)$

$R(z), Q(z)$ are polynomials

First conclusion: PA OK in Space-like region but have difficulties to explore the branch cut

Reason: strong cut

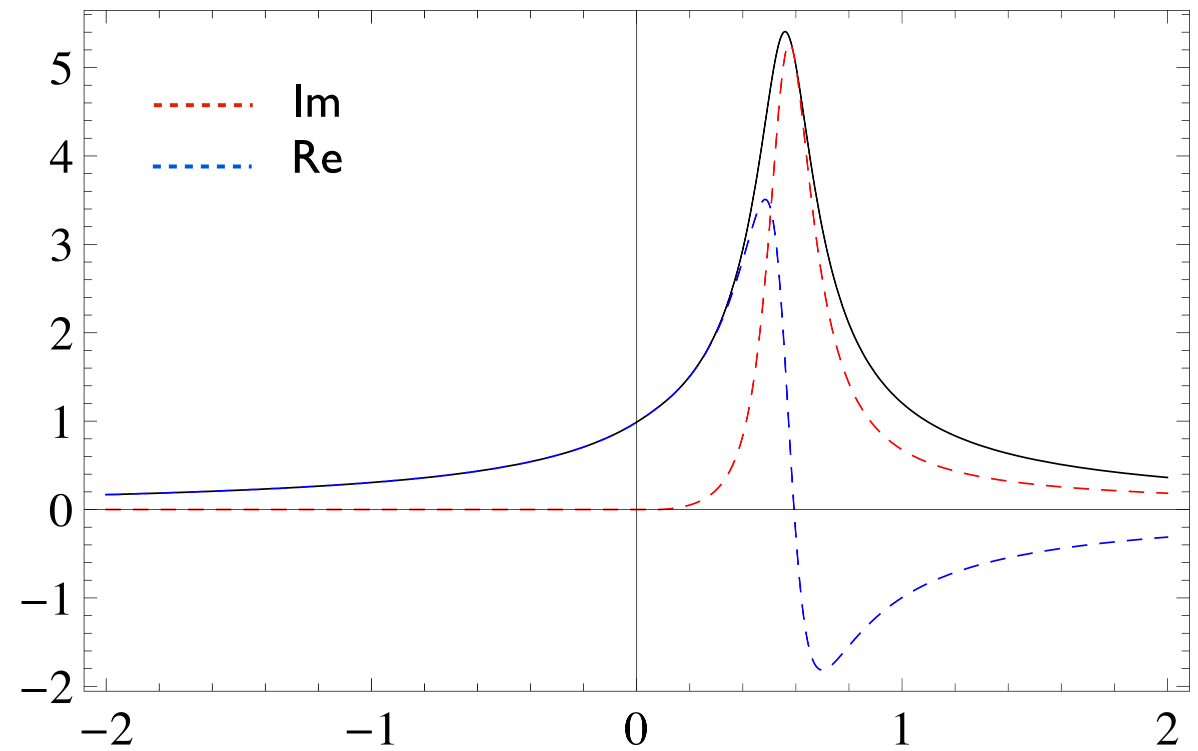
Padé approximants

Padé approx: $Q(z)f(z) + R(z) = \mathcal{O}\left(z^{q+r+1}\right)$

$R(z), Q(z)$ are polynomials

Last example: vector FF (mild cut)

$$F_V(s) = \frac{M_\rho^2}{M_\rho^2 - s - i\frac{\Gamma_\rho s}{M_\rho}\frac{\beta(s)^3}{\beta(M_\rho^2)^3}} \qquad \beta(s) = \sqrt{1 - \frac{4m^2}{s}}$$



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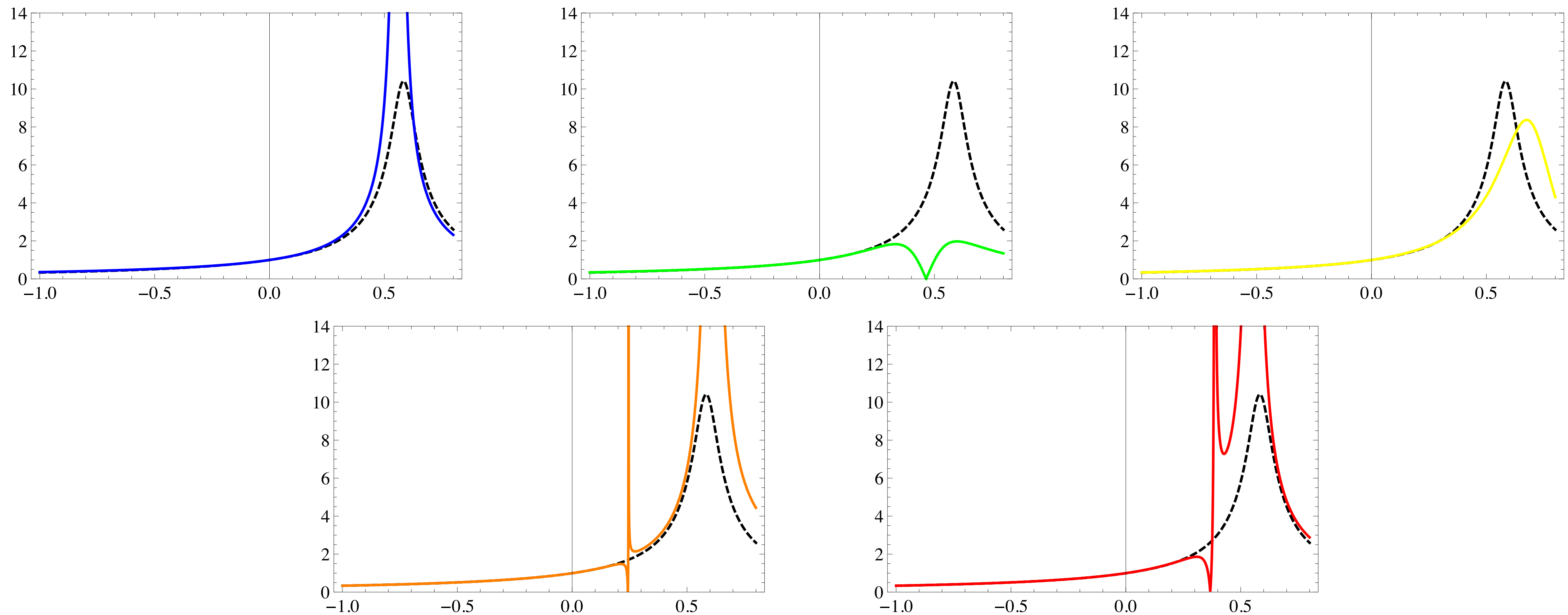
$P_N^N(z)$

for N=1,2,3,4,5



Padé approximants

Padé approx: $Q(z)f(z) + R(z) = \mathcal{O}\left(z^{q+r+1}\right)$

$R(z), Q(z)$ are polynomials

Second <u>conclusion</u>:

the strength of the cut matters, still a good description of space-like region is possible

Quadratic Approximants

Quadratic Approximants

Padé approx: $Q(z)f(z)+R(z)=\mathcal{O}\left(z^{q+r+1}\right)$

Quadratic approx: $Q(z)f^2(z)+2R(z)f(z)+S(z)=\mathcal{O}\left(z^{q+r+s+2}\right)$

$R(z), S(z), Q(z)$ are polynomials

$$\mathbb{Q}_q^{r,s}(z)=\frac{-R(z)\pm\sqrt{[R(z)]^2-Q(z)S(z)}}{Q(z)}$$
$$=\frac{-S(z)}{R(z)\pm\sqrt{[R(z)]^2-Q(z)S(z)}}$$

- When S(z)=0, $\mathbb{Q}_q^{r,s}(s)\ \to P_q^r(z)$
- Lowest order ~ Breit-Wigner param.
- If info about poles, threshold, LEPs is known, easy to implement
- Satisfy Disp. Rel. (is a solution to Omnès)
- Relation with z-param. used in D,B decays

Quadratic Approximants

Quadratic approx: $$Q(z)f^2(z)+2R(z)f(z)+S(z)=\mathcal{O}\left(z^{q+r+s+2}\right)$$

$R(z), S(z), Q(z)$ are polynomials

General form for the Q[1,1,1]:

$$\mathbb{Q}_1^{1,1}(z)=\frac{-(R_0+R_1z)\pm\sqrt{[R_0+R_1z]^2-(1+Q_1z)(S_0+S_1z)}}{(1+Q_1z)}$$

We need to solve the equation:

$$(1+Q_1z)[f(z)]^2+2(R_0+R_1z)f(z)+(S_0+S_1z)=\mathcal{O}(z^5)$$

Quadratic Approximants

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Example: $$f(z)=\frac{1}{z}\log(1-z)$$

We need to solve the equation:

$$(1+Q_1 z)[f(z)]^2+2(R_0+R_1 z)f(z)+(S_0+S_1 z)=\mathcal{O}(z^5)$$

Two solutions:

$$\mathbb{Q}_1^{1,1}(z)=\frac{(\frac{7}{8}-\frac{5}{12}z)\pm\sqrt{[-\frac{7}{8}+\frac{5}{12}z]^2-(1-\frac{13}{12}z)(-\frac{11}{4}+\frac{1}{24}z)}}{(1-\frac{13}{12}z)}$$

Quadratic Approximants

Quadratic approx: $Q(z)f^2(z) + 2R(z)f(z) + S(z) = \mathcal{O}\left(z^{q+r+s+2}\right)$

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Example: $f(z) = \frac{1}{z}\log(1-z)$

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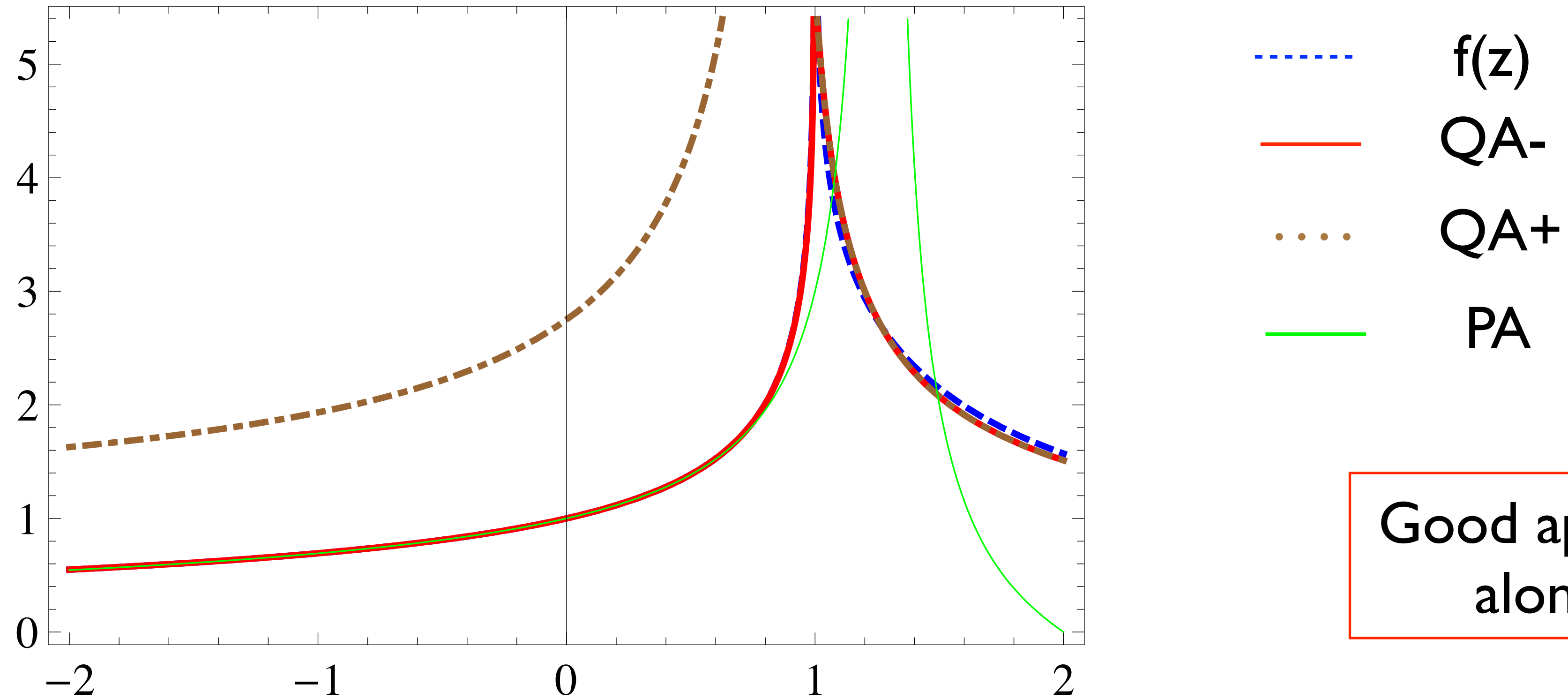


Good approach even along the cut

Quadratic Approximants

Quadratic approx: $Q(z)f^2(z) + 2R(z)f(z) + S(z) = \mathcal{O}\left(z^{q+r+s+2}\right)$

Example: $f(z) = \frac{1}{z}\log(1-z)$

$R(z), S(z), Q(z)$ are polynomials

Convergence theorems? Unkown so far!

- We have *proof* for some particular cases if Stieltjes:

$$QA_n^{n,n} \qquad QA_n^{n,0} \qquad QA_{n+1}^{0,n-1} \qquad QA_{n-k}^{n-k,1+2(k-1)}$$

- For PAs: PA_n^{n-1}, PA_n^n
- Impose high-energy constraints, threshold constraints, etc, not yet demonstration

Quadratic Approximants

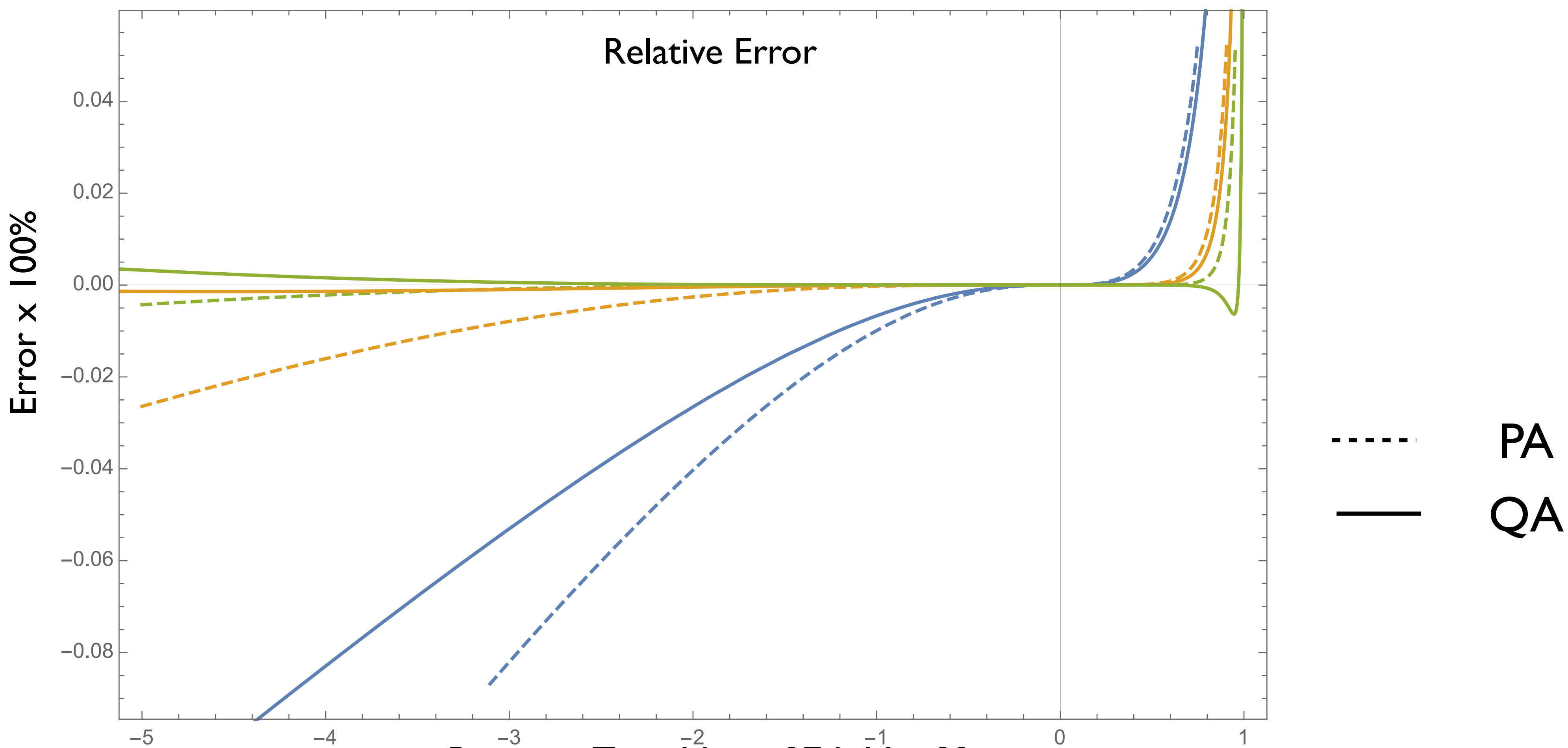
Quadratic approx: $Q(z)f^2(z) + 2R(z)f(z) + S(z) = \mathcal{O}\left(z^{q+r+s+2}\right)$

$R(z), S(z), Q(z)$ are polynomials

Example: $f(z) = \frac{1}{z}\log(1-z)$

Convergence theorems? Unkown so far!

- We have *proof* of faster convergence than PAs!



Quadratic Approximants

Quadratic approx: $$Q(z)f^2(z)+2R(z)f(z)+S(z)=\mathcal{O}\left(z^{q+r+s+2}\right)$$

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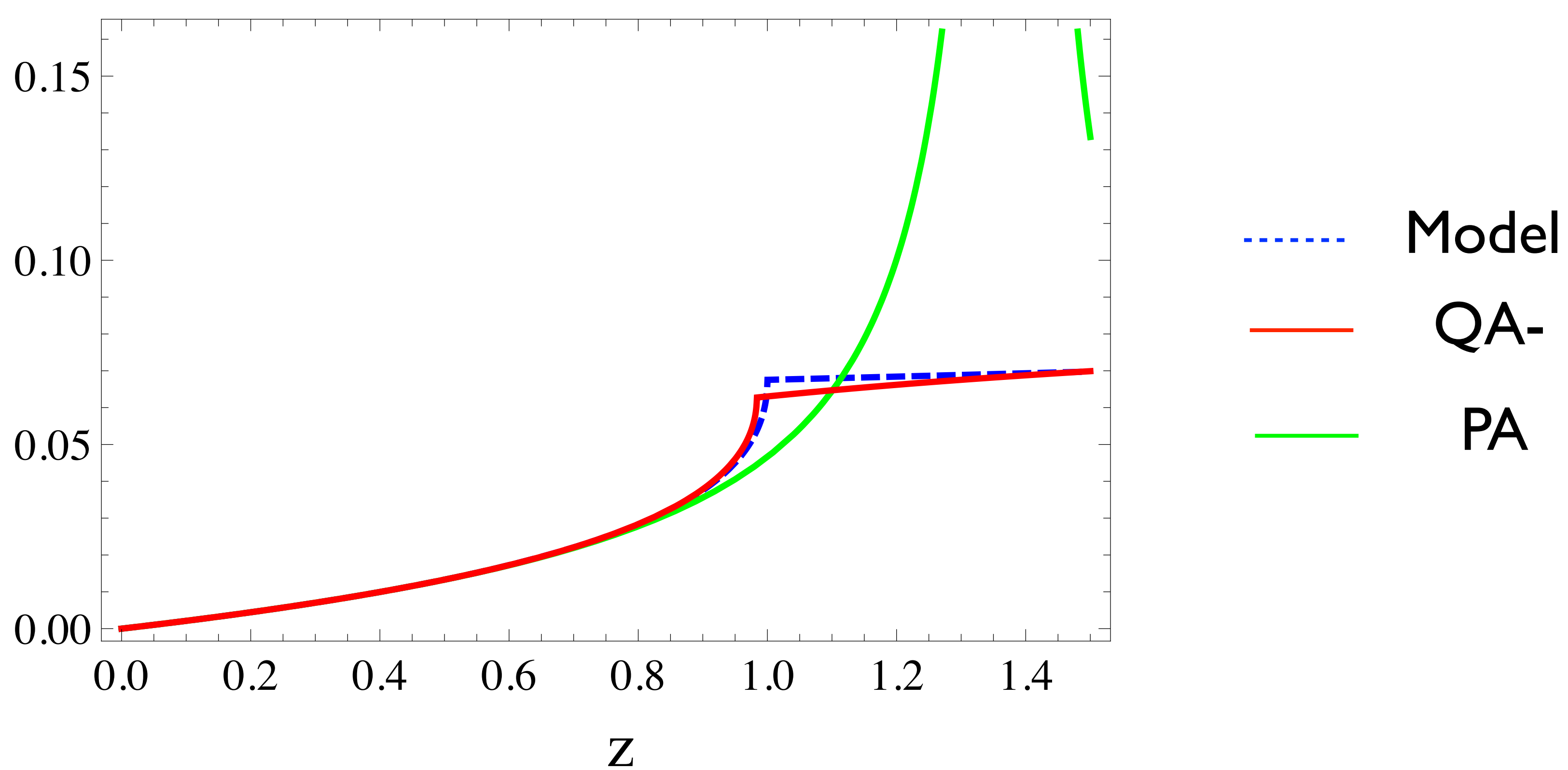
Example: vacuum polarization function

$$\Pi^{(0)}(z)=\frac{3}{16\pi^2}\left(\frac{4}{3z}+\frac{20}{9}-\frac{4(1-z)(2z+1)G(z)}{3z}\right)$$



Quadratic Approximants

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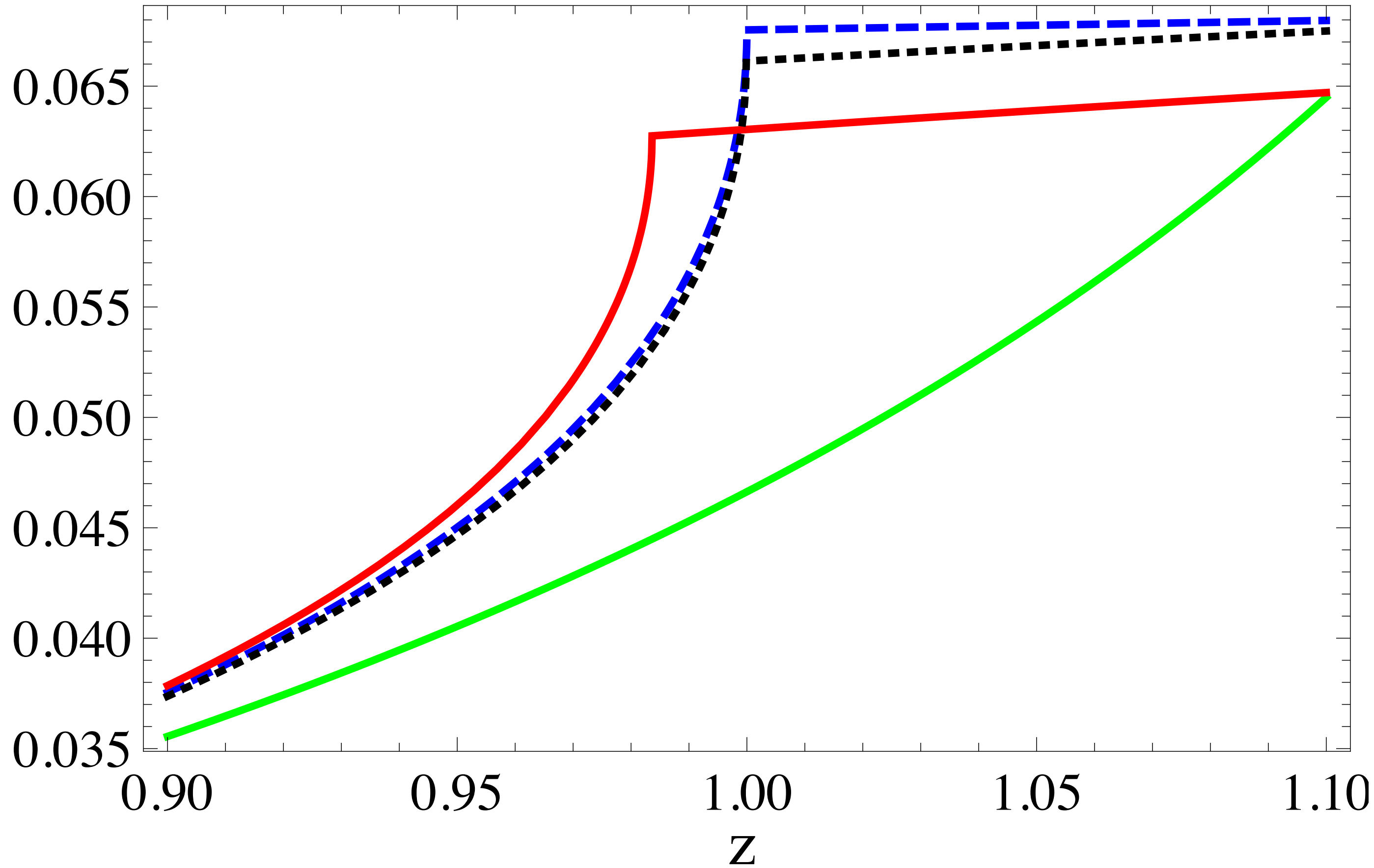


Including threshold info:

$$\Pi^{(0)}(z) \sim \mathrm{Const}$$

Model

QA-

QA threshold

PA

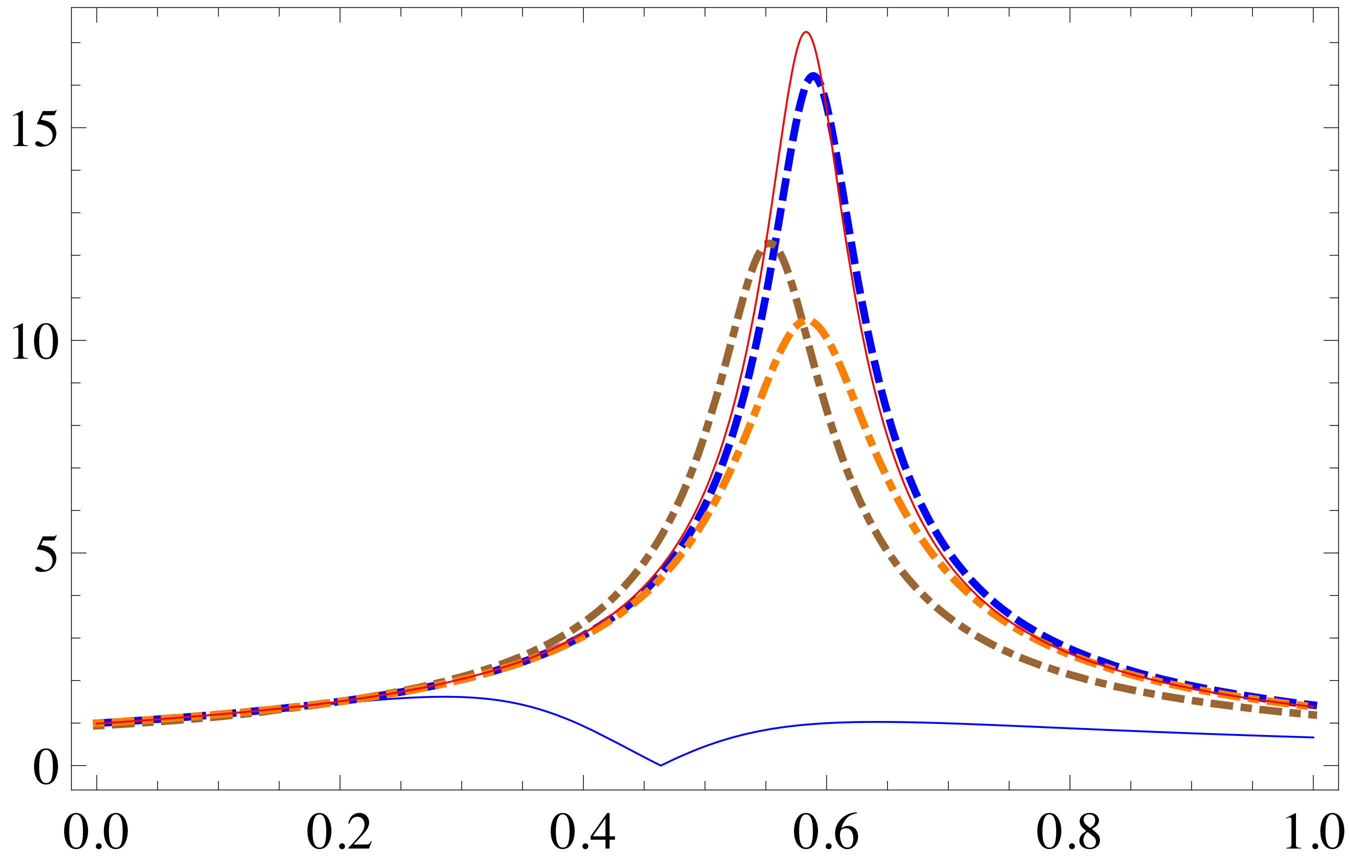
Quadratic Approximants

Quadratic approx: $Q(z)f^2(z)+2R(z)f(z)+S(z)=\mathcal{O}\left(z^{q+r+s+2}\right)$

$R(z), S(z), Q(z)$ are polynomials

Example: vector FF (mild cut)

$$F_V(s)=\frac{M_\rho^2}{M_\rho^2-s-i\frac{\Gamma_\rho s}{M_\rho}\frac{\beta(s)^3}{\beta(M_\rho^2)^3}}$$



Model

QA threshold

QA[2,1,0]

QA[2,2,1]

PA

+ good prediction of pole

Quadratic Approximants

Examples

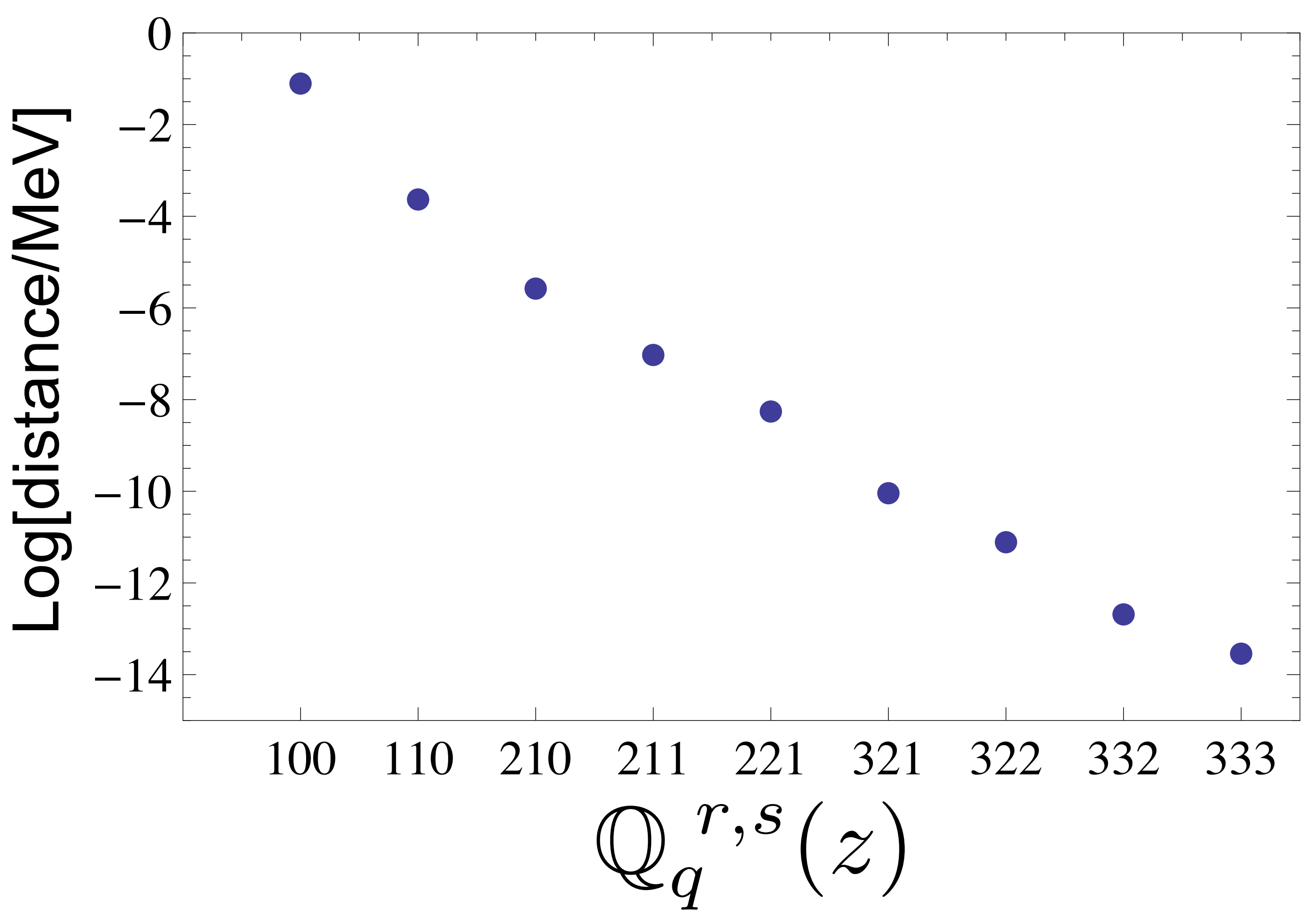
Vector FF model

$$F_V(s) = \frac{M_V^2}{M_V^2 - s + \frac{M_V \Gamma_V s}{\pi M_V^2}\left(-2\sigma(s)^2 - \sigma(s)^3 \ln\left(\frac{\sigma(s)-1}{\sigma(s)+1}\right)\right)}$$

$$m_\pi = 135 \text{ MeV}$$
$$M_V = 770 \text{ MeV}$$
$$\Gamma_V = 150 \text{ MeV}$$

Generate derivatives at 0.6 GeV^2 → fit $\mathbb{Q}_1^{1,0}(s)\,, \mathbb{Q}_2^{1,1}(s)\ \ldots$

1 pole ↗ 2 poles ↗

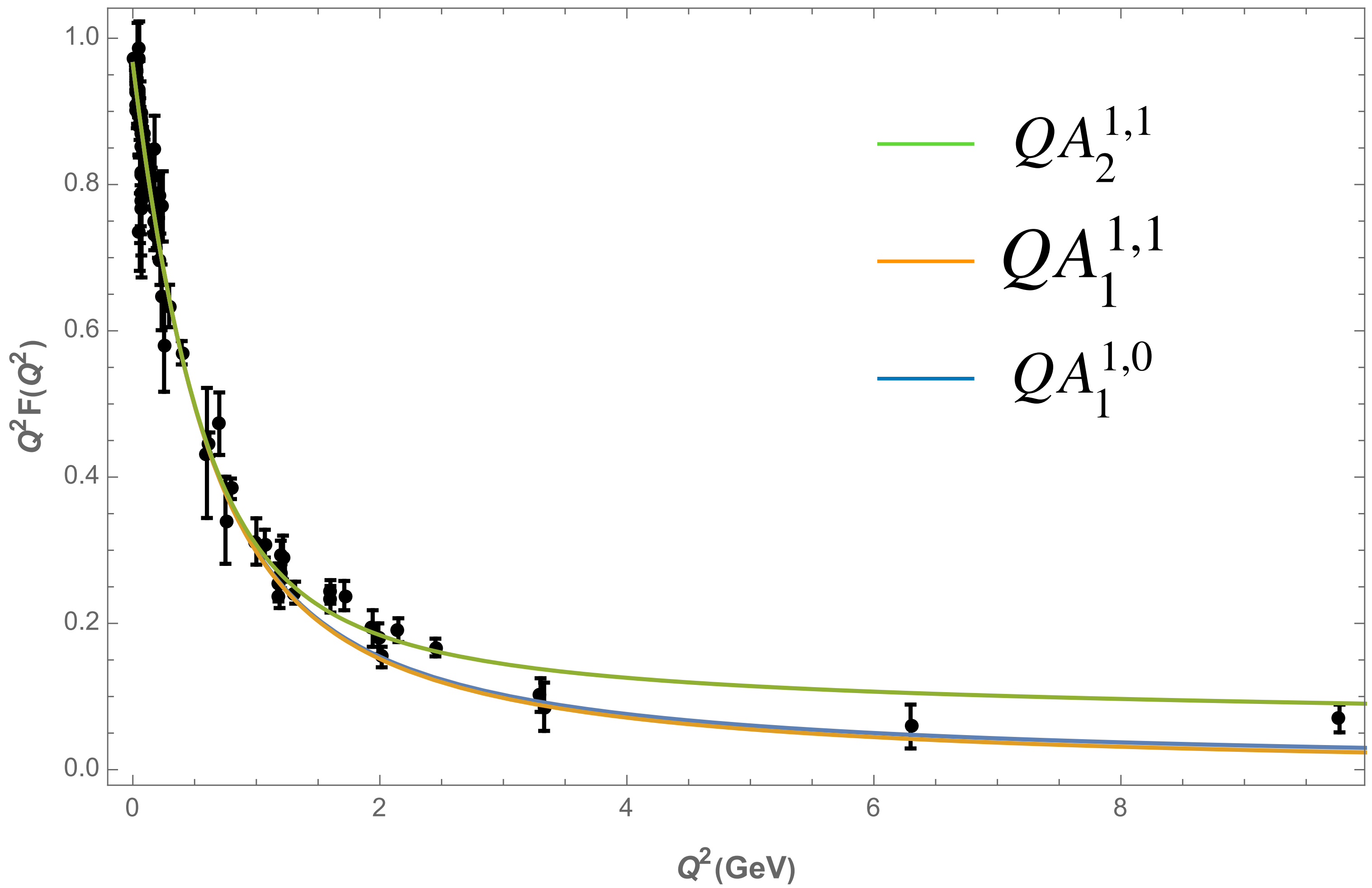


Quadratic Approximants

Examples

Pion Form Factor:

Space-like region:



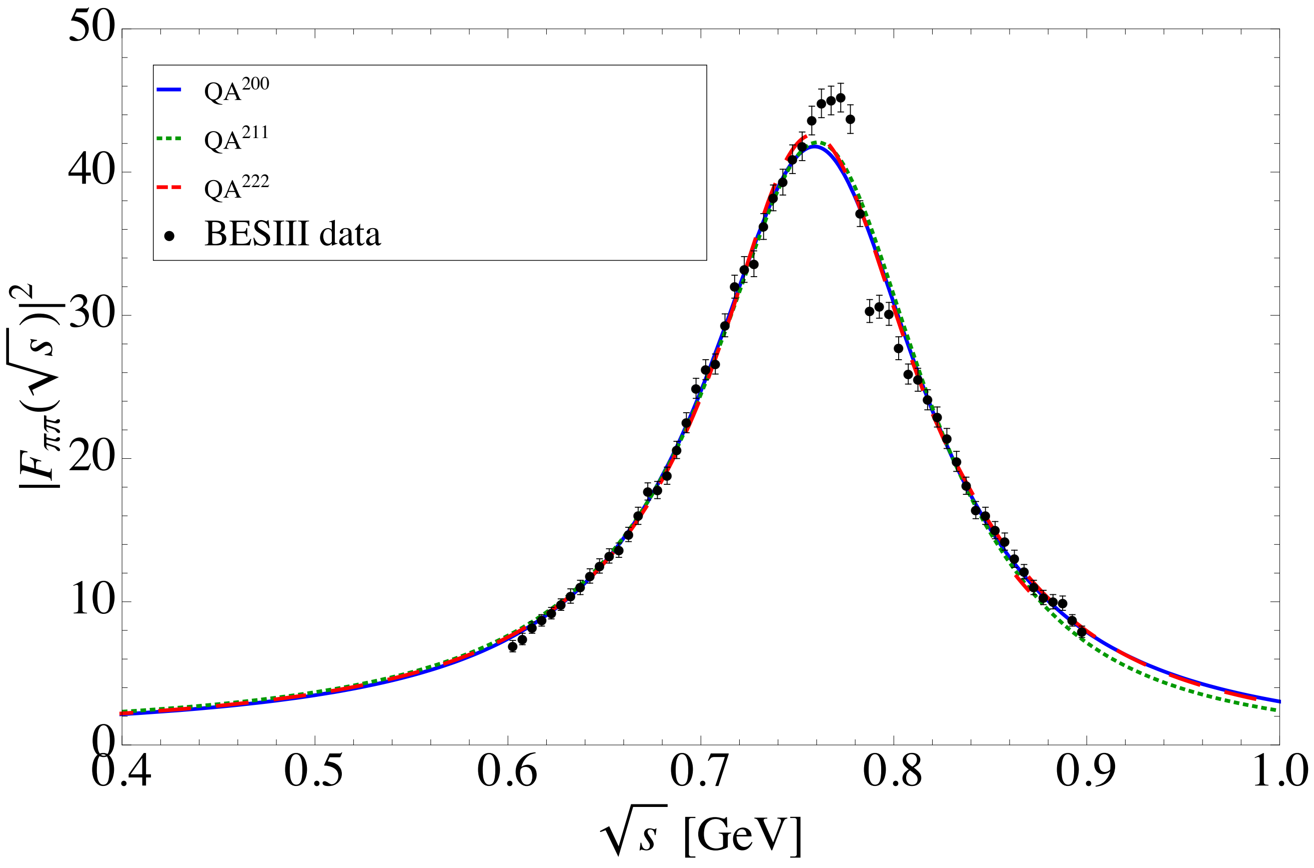
Data from NA7, and JLab Fπ coll.

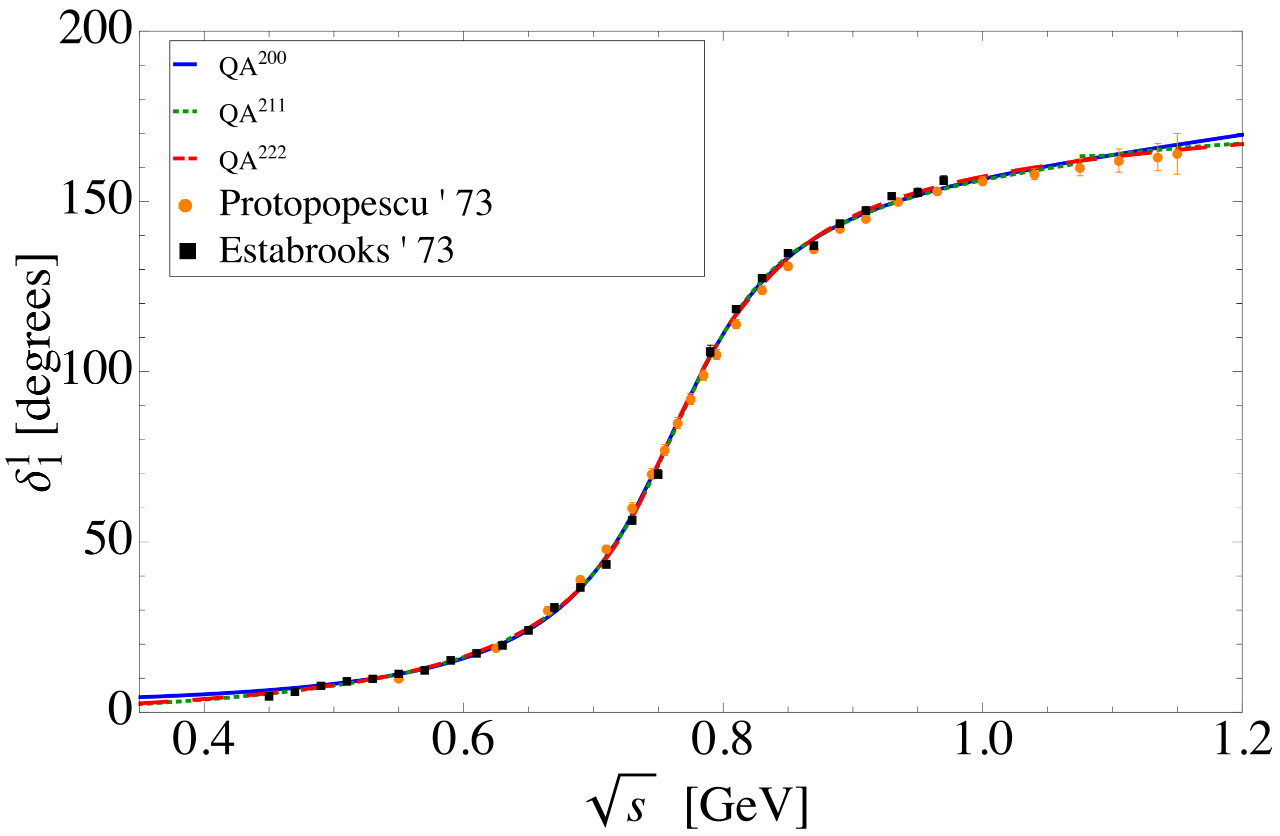
Quadratic Approximants

Examples

<u>Pion Form Factor:</u>

Time-like region:





Conclusions

- We presented a method, a TOOLKIT
- based on analyticity and unitarity, with convergence!
- Simple although further studies ongoing
- Approaches yes (improvable), assumptions no
- Systematic:
  - easy to update with new data (or derivatives)
  - error from approach
- Predictive (checkable)

Outlook

- How to use the method?
  - First: is the function of Stieltjes type?
    - Do you have experimental data to fit?
    - Do you have an expansion parameter?
    - Have you truncated something (number of resonances, small velocity, small q, …)?
  - Then: method can be used!

Outlook

- How to use the method?
  - First: is the function of Stieltjes type? (contact masjuan@ifae.es)
    - Do you have experimental data to fit?
    - Do you have an expansion parameter?
    - Have you truncated something (number of resonances, small velocity, small q, …)?

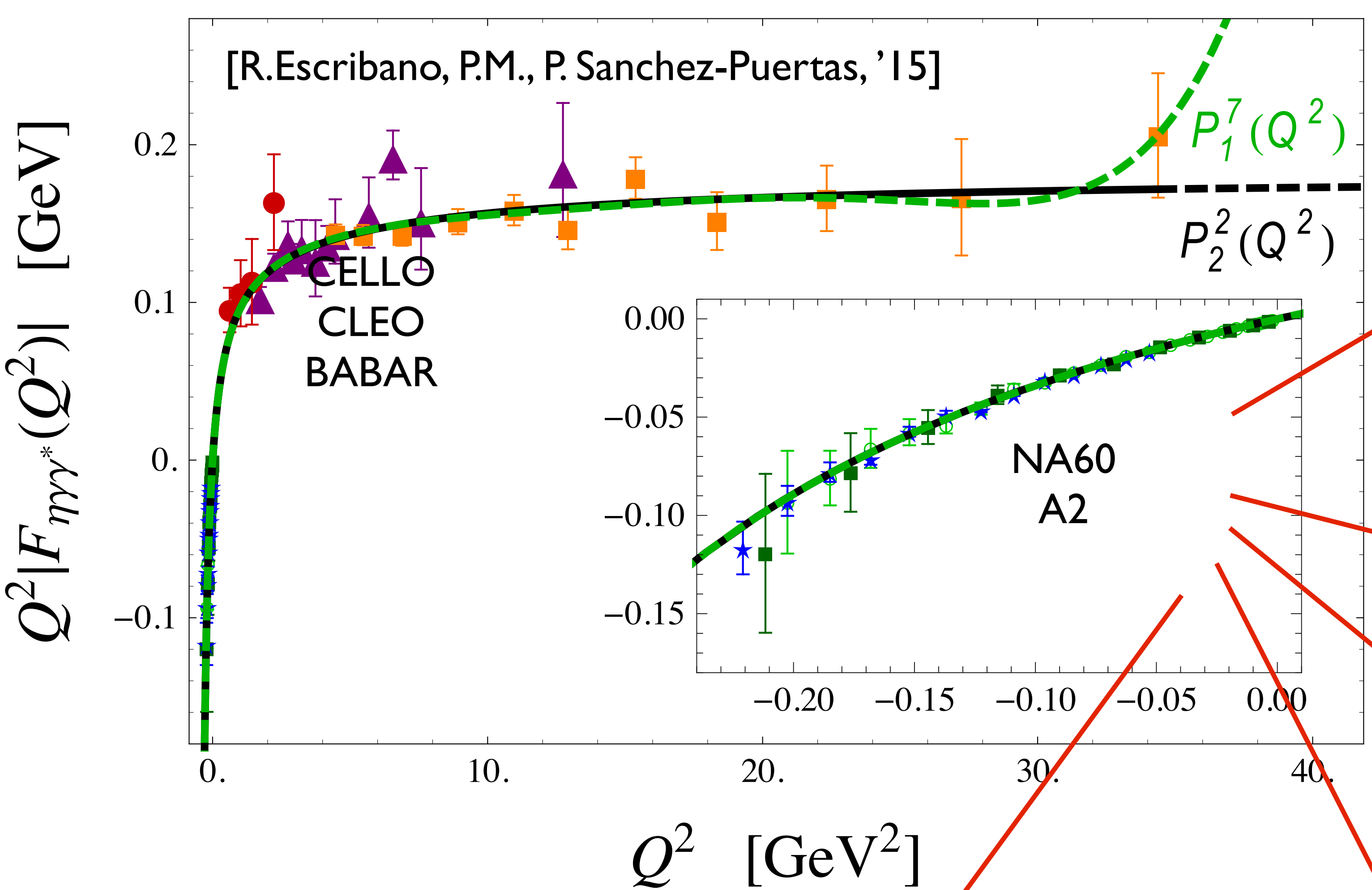
Contact us, our team of experts will be happy to help!

masjuan@ifae.es

Thanks!

Quadratic Approximants

Before "attacking" the HVP, we need to convince about the method:



Low-energy parameters up to the third derivative!

HLBL in (g-2)$_\mu$

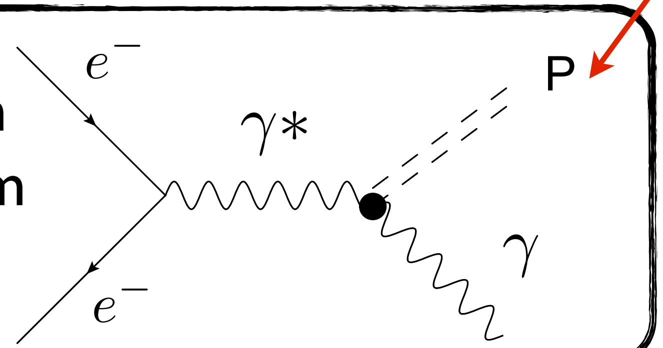
mixing parameters of the η-η' system

f_q, f_s, ϕ

Rare decays

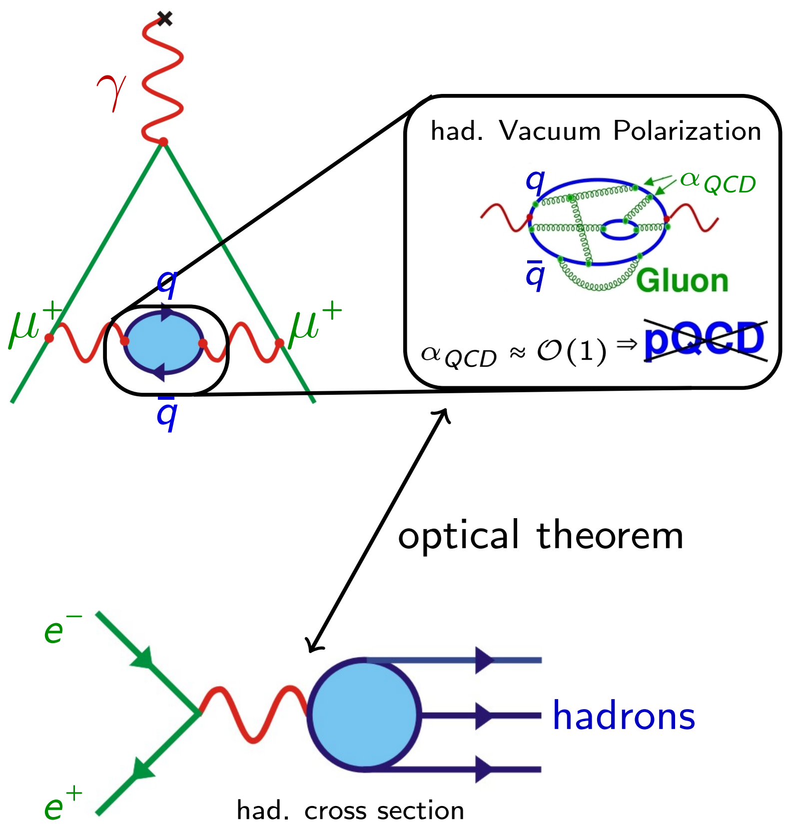
$\Gamma_{P\to l^+l^-}$

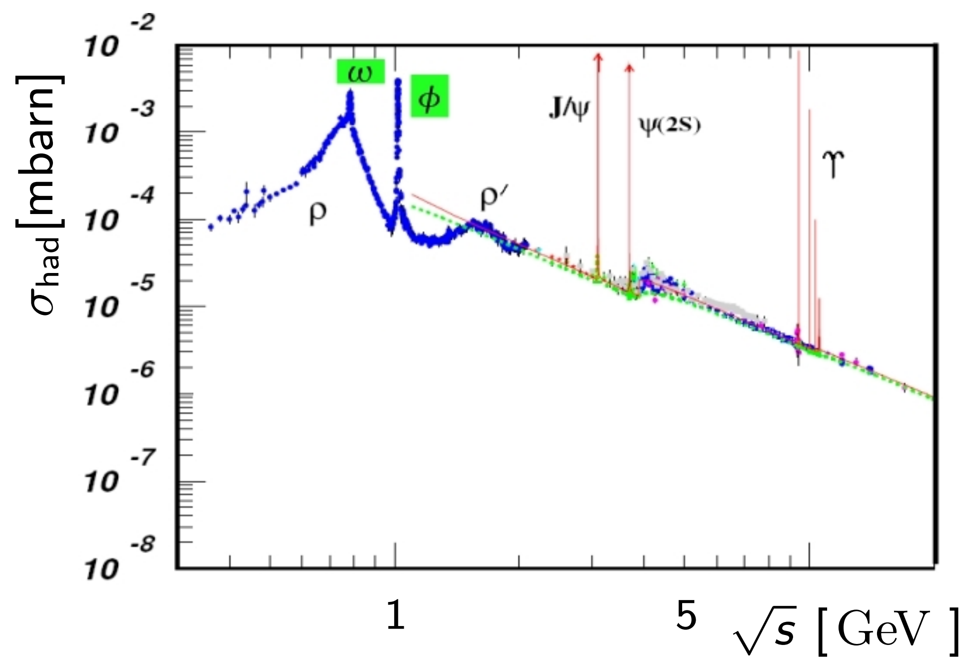
PS continuum on quarkonium region



beautiful synergy experiment - theory

Hadronic Vacuum Polarization





$$a_{\mu,LO}^{\rm had} = \frac{1}{4\pi^3}\int_{m_{\pi^0}^2}^{\infty} ds\, K(s)\, \sigma_{\rm had}(s)$$

$\sigma_{\rm had}(s) \sim 1/s$ & $K(s) \sim 1/s$

$\Downarrow$

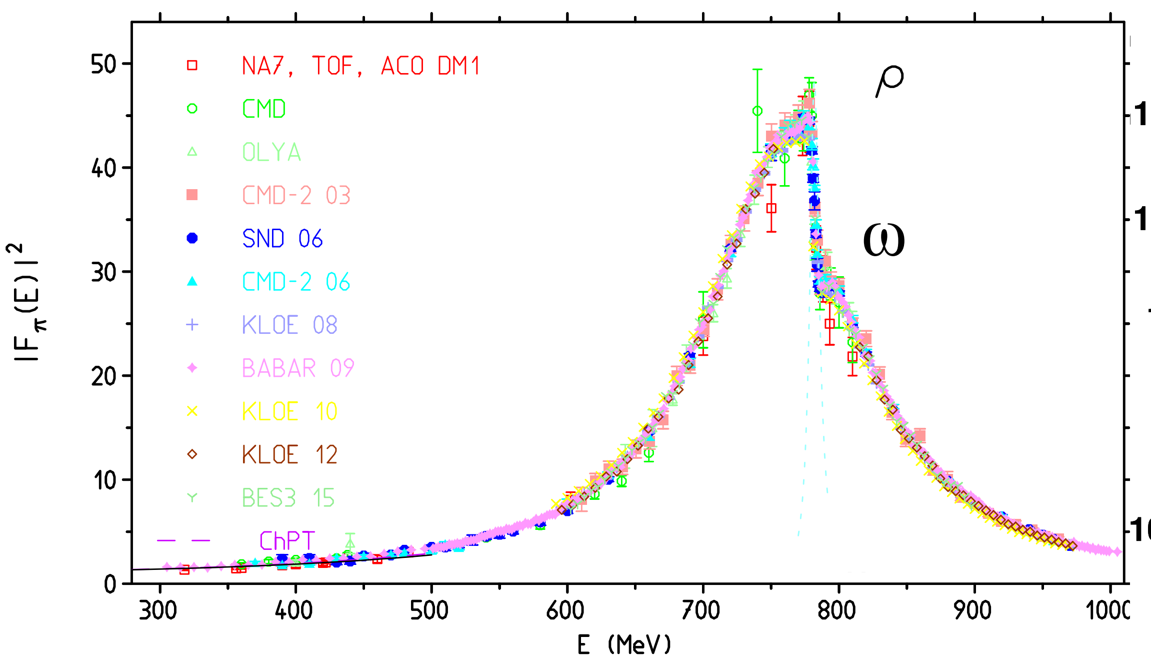
Low energy region important! $\sim 1/s^2$

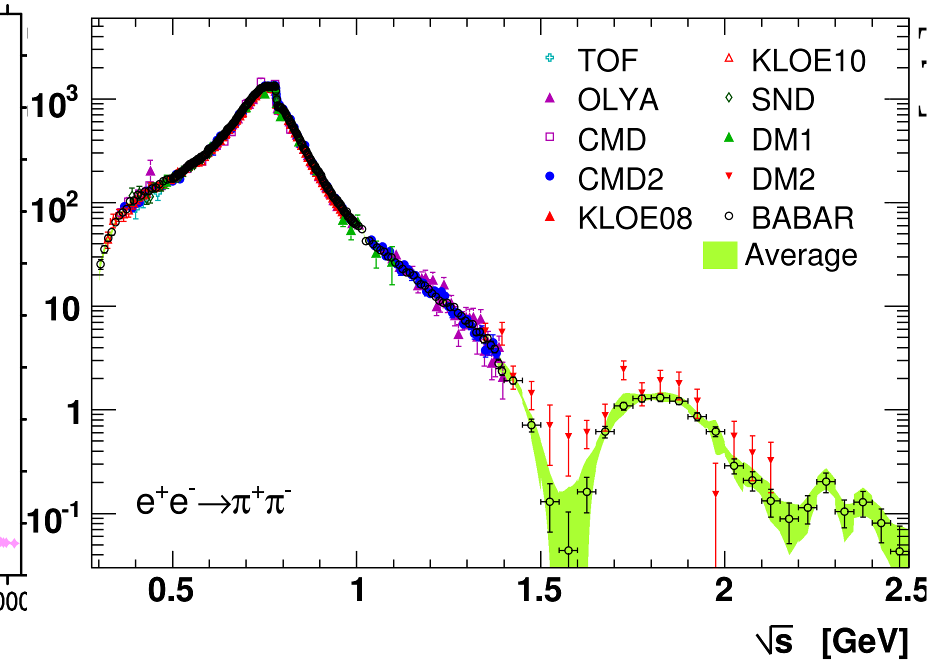
Sum of exclusive $\sigma_{\rm had}$

$\Downarrow$

Hadronic contribution of a_μ

Hadronic Vacuum Polarization





- ρ peak
- ρ-ω interference
- Contribution to a_μ(VP): 75%
- Largest error from 1-2GeV

Padé approximants

Padé approx: $Q(z)f(z) + R(z) = \mathcal{O}\left(z^{q+r+1}\right)$

$R(z), Q(z)$ are polynomials

Example: vacuum polarization function

$$\Pi(q^2) = \Pi^{(0)}(q^2) + \left(\frac{\alpha_s}{\pi}\right)\Pi^{(1)}(q^2) + \mathcal{O}(\alpha_s^2)$$

let me define $z = \dfrac{q^2}{4m^2}$

$$\Pi^{(0)}(z) = \frac{3}{16\pi^2}\left(\frac{4}{3z} + \frac{20}{9} - \frac{4(1-z)(2z+1)G(z)}{3z}\right)$$

$$G(z) = 2\frac{u\log(u)}{u^2-1} \qquad \text{where} \qquad u \to \frac{\sqrt{1-z^{-1}}-1}{\sqrt{1-z^{-1}}+1}$$

Quadratic Approximants

Quadratic approx: $Q(z)f^2(z) + 2R(z)f(z) + S(z) = \mathcal{O}\left(z^{q+r+s+2}\right)$

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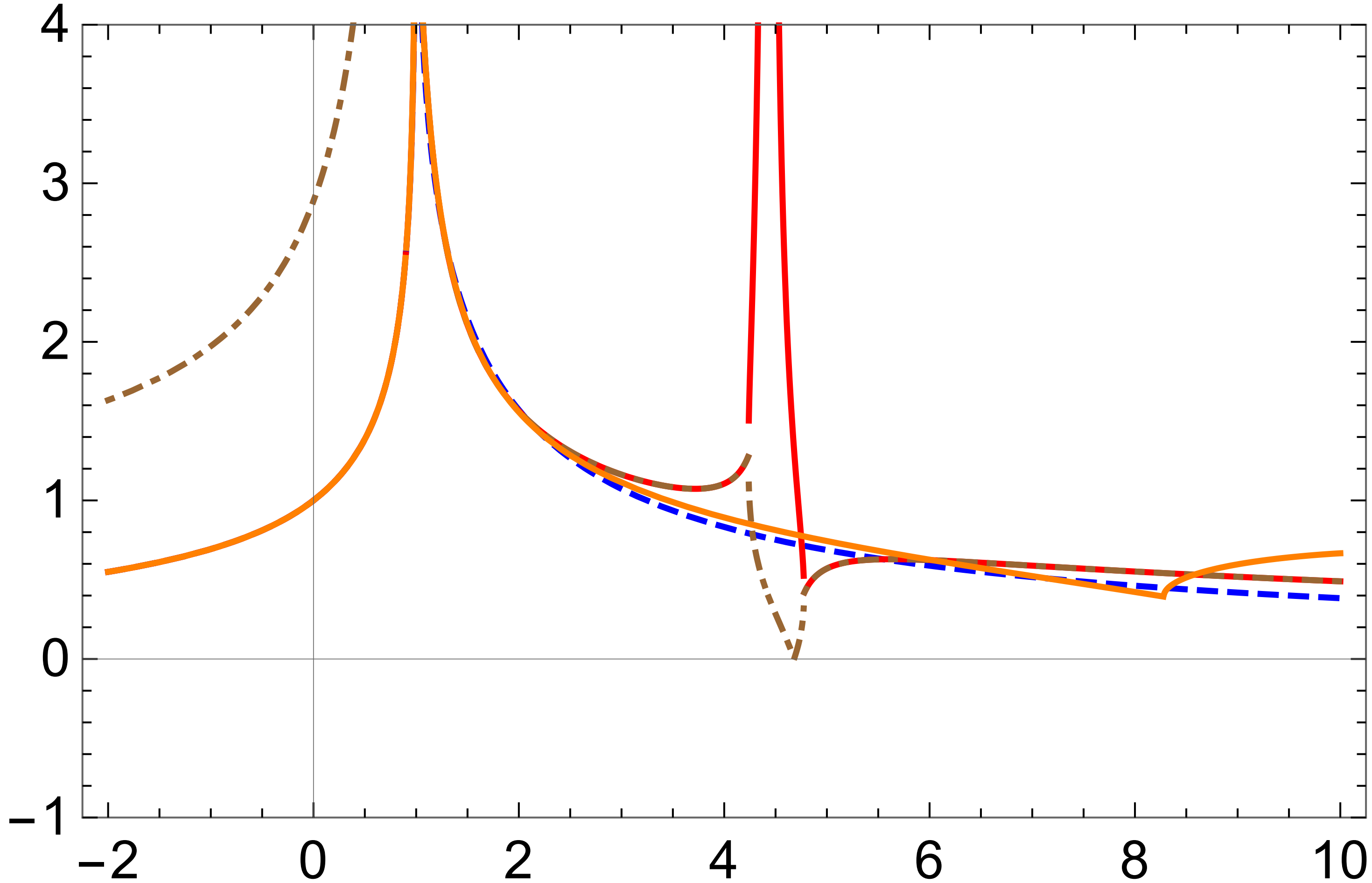
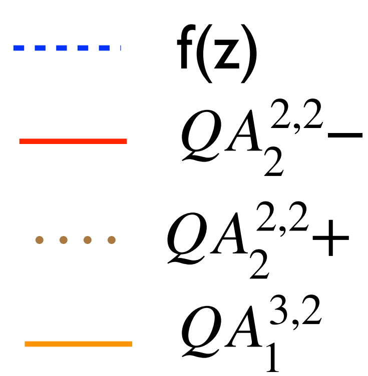


Higher orders:



Good approach even along the cut

Padé approximants

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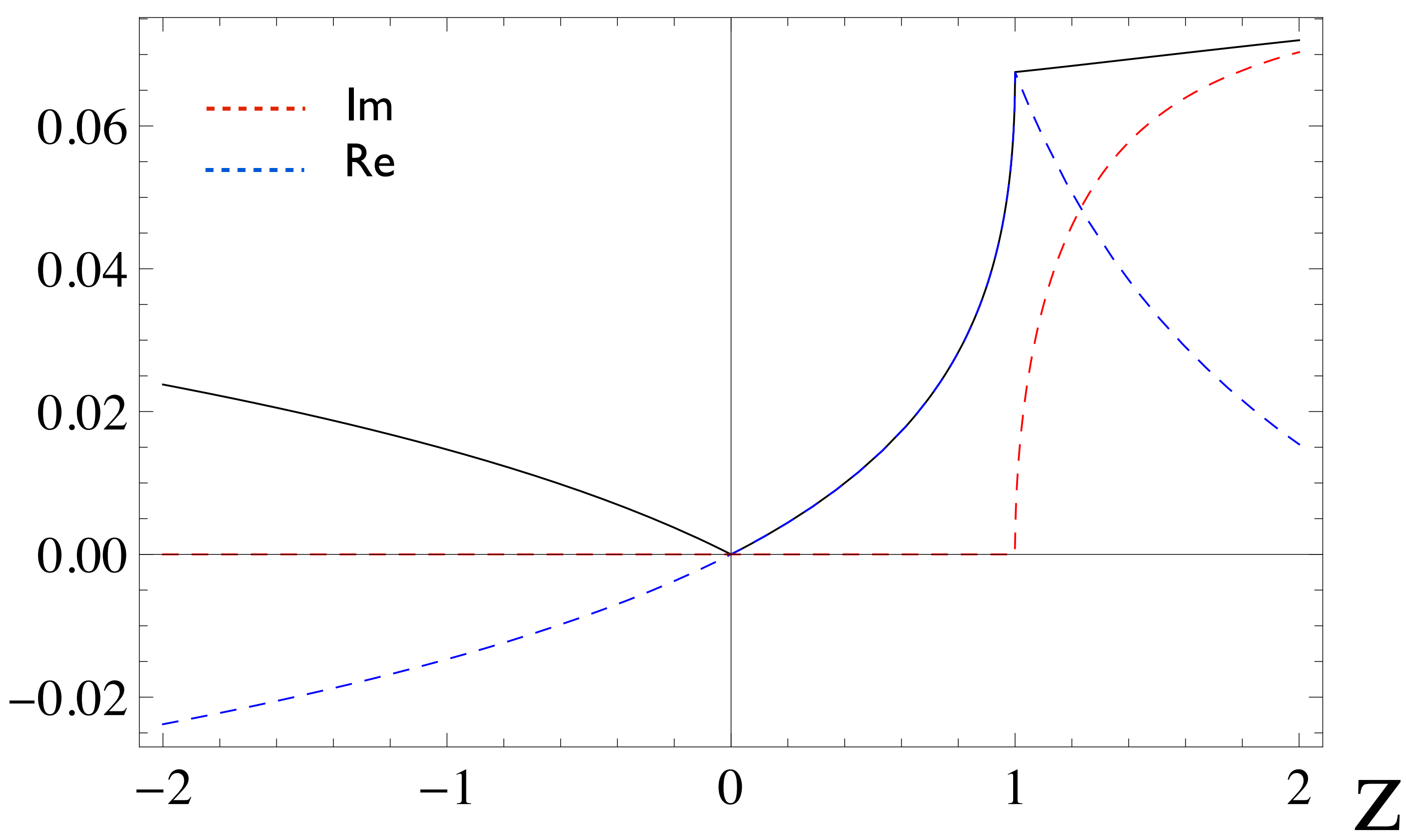

$$z = \frac{q^2}{4m^2}$$

Padé approximants

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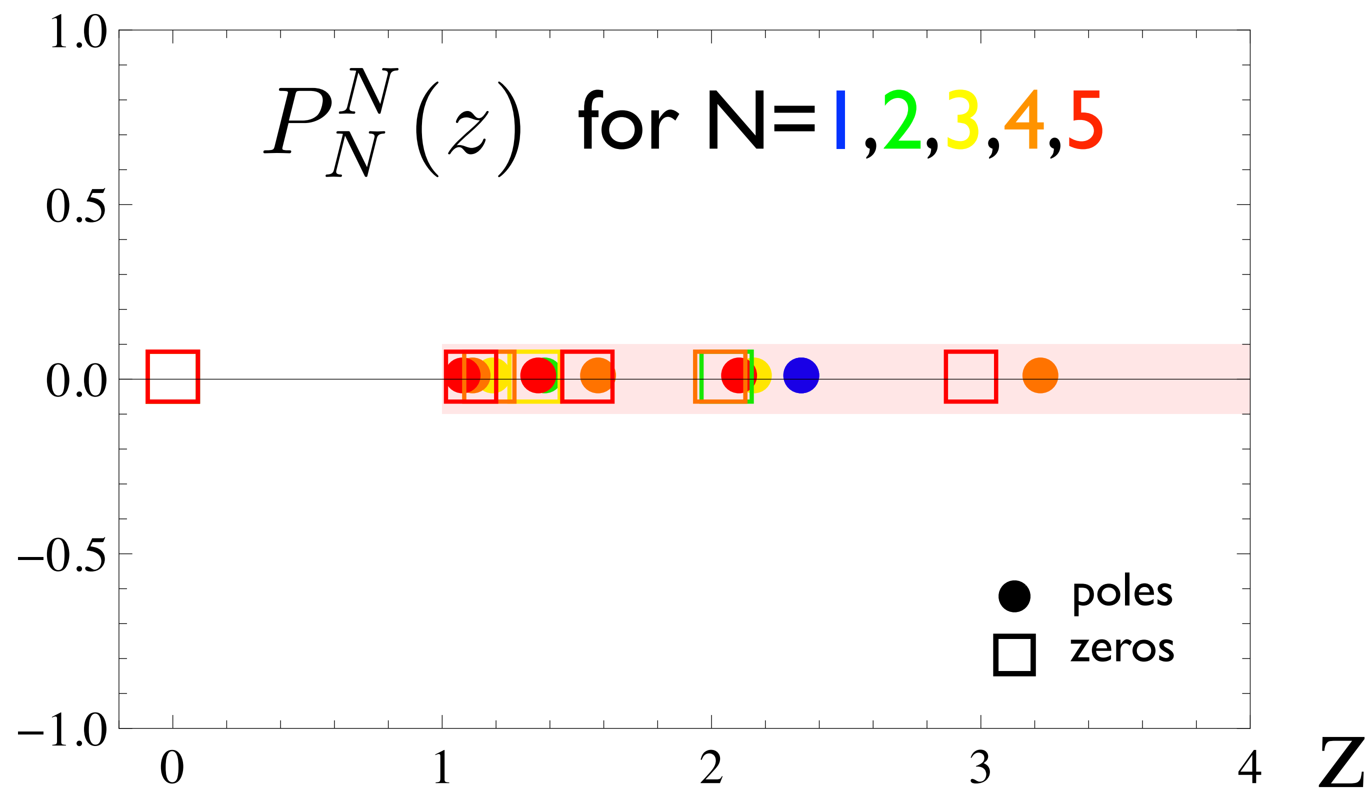

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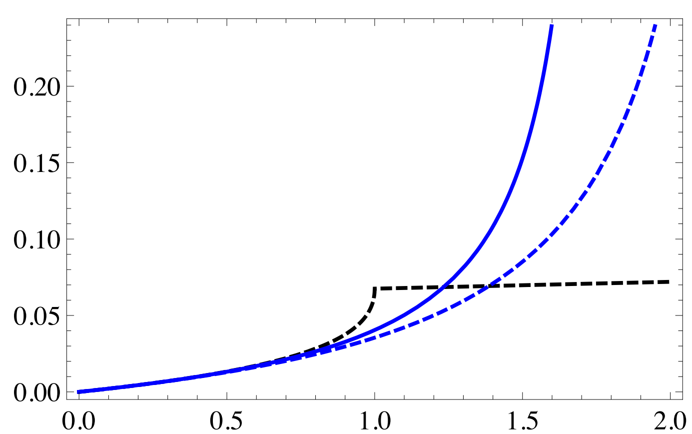
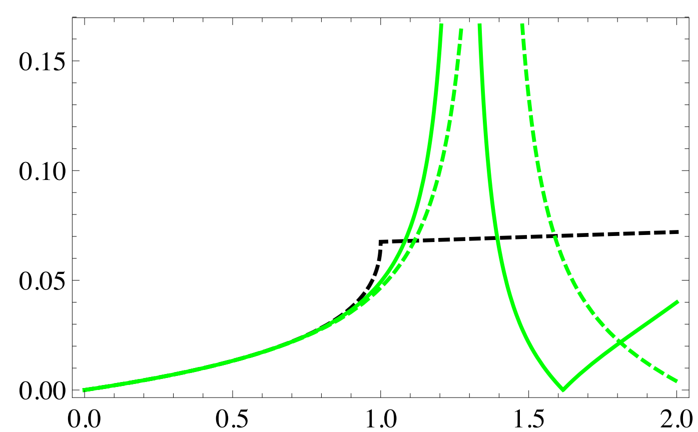
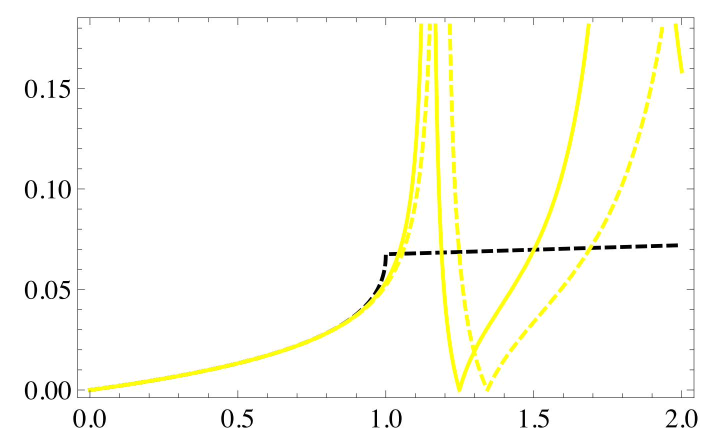
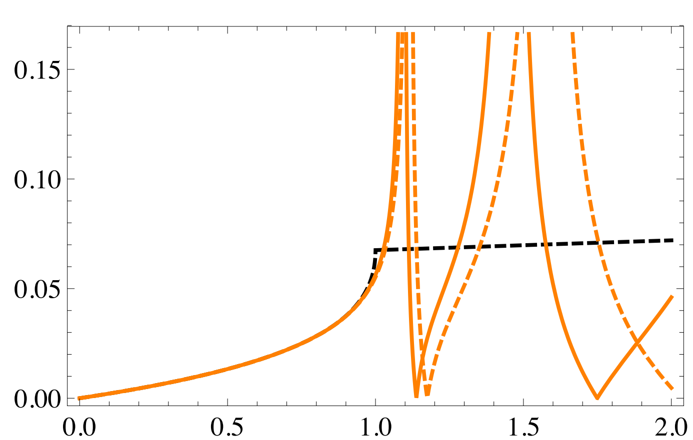
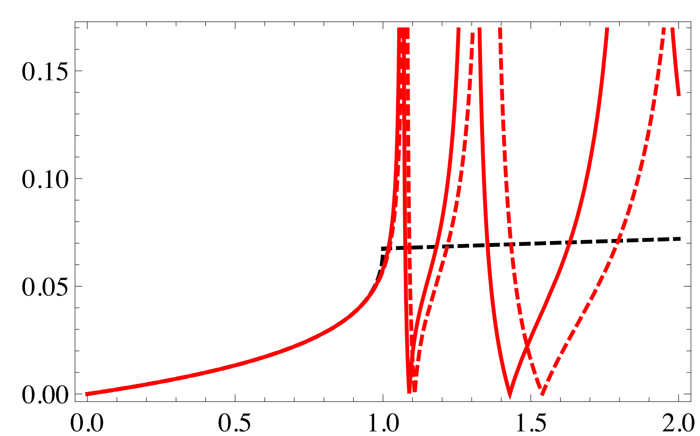






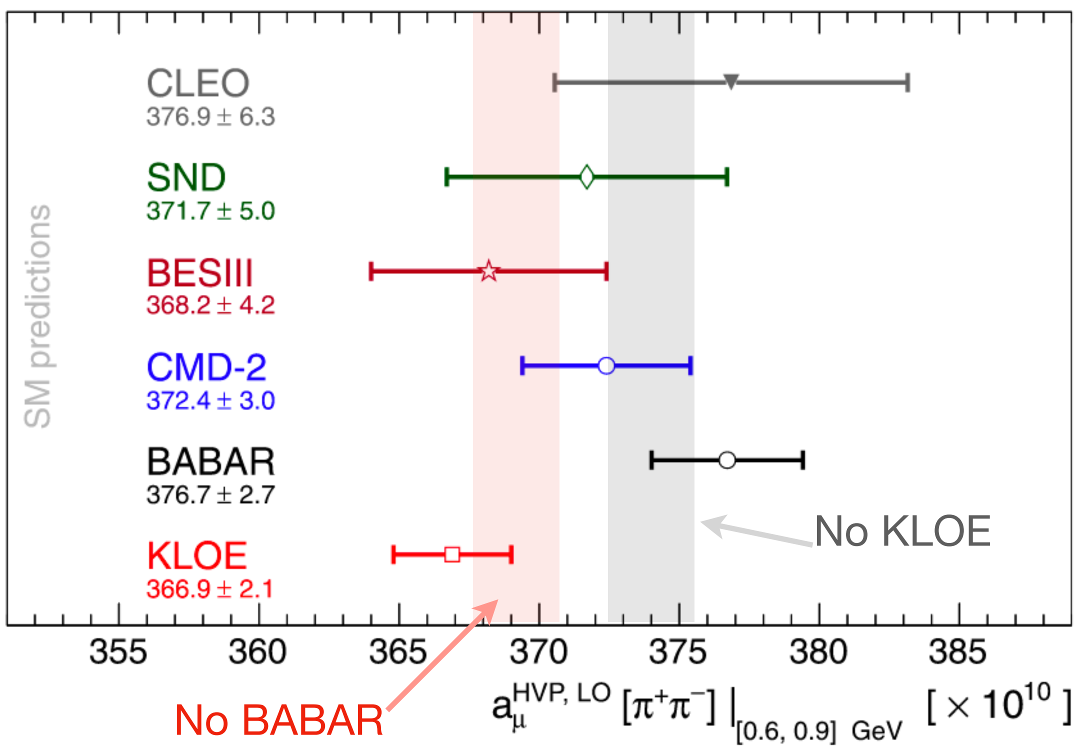


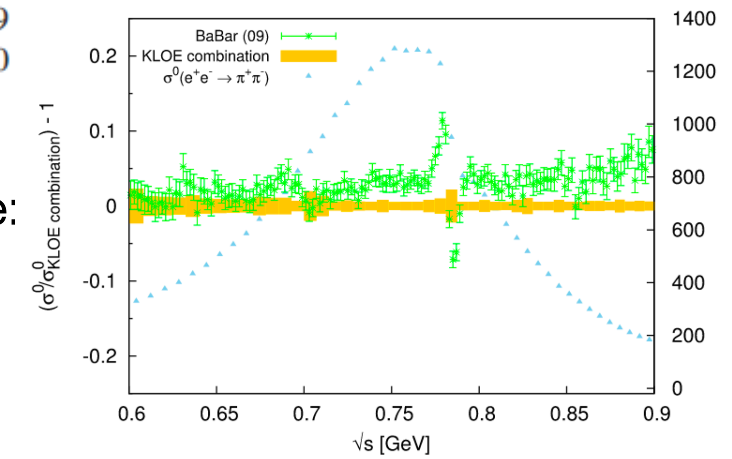


Hadronic Vacuum Polarization

[Keshavarzi et al 18]

Channel	This work (KNT18)	DHMZ17 [78]	Difference
	Data based channels ($\sqrt{s} \le 1.8$ GeV)		
$\pi^0\gamma$ (data + ChPT)	4.58 ± 0.10	4.29 ± 0.10	0.29
$\pi^+\pi^-$ (data + ChPT)	503.74 ± 1.96	507.14 ± 2.58	−3.40





Combined in quadrature:
4.0×10^{-10}

Combined linearly:
6.5×10^{-10}

What about
$\Delta a_\mu^{HVP,LO}[\pi^+\pi^-]|_{\mathrm{KLOEvsBABAR}} = 10.1(3.4) \times 10^{-10}$

Superconservative?:
$(6.5 + 5.0) \times 10^{-10} = 11.5 \times 10^{-10}$

$$a_\mu^{\mathrm{HVP}} = 684.5(11.5) \times 10^{-10}$$

Quadratic Approximants

Examples

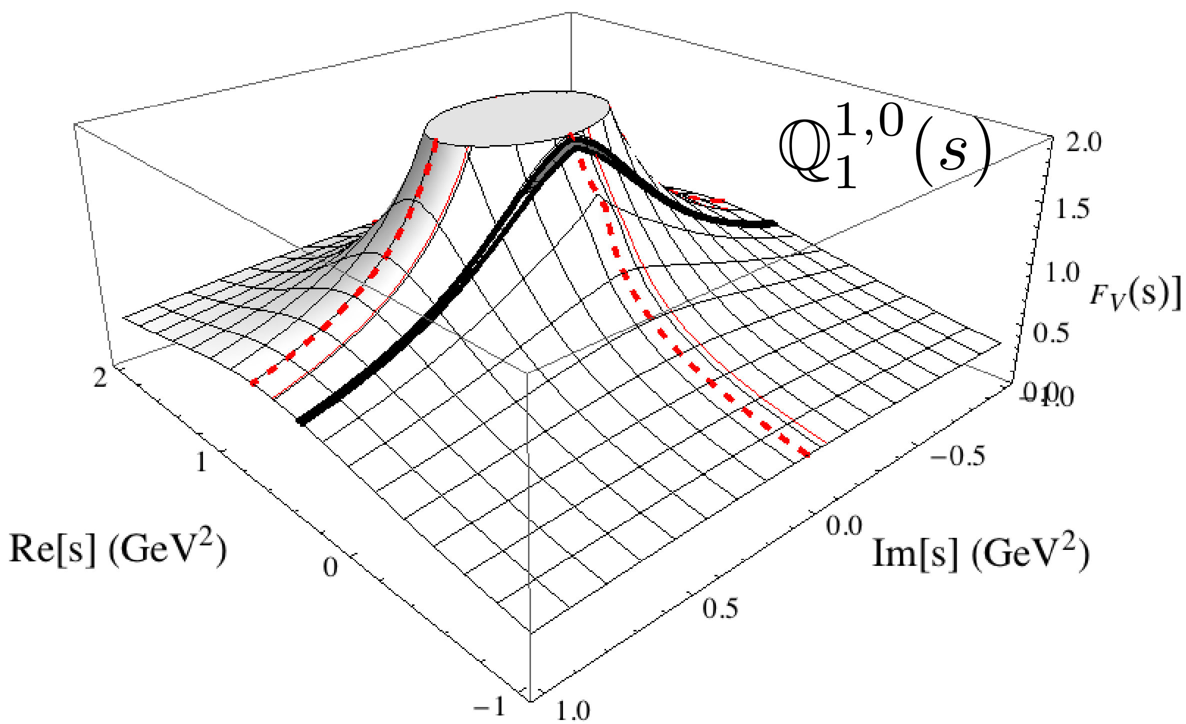
<u>Vector FF model</u>

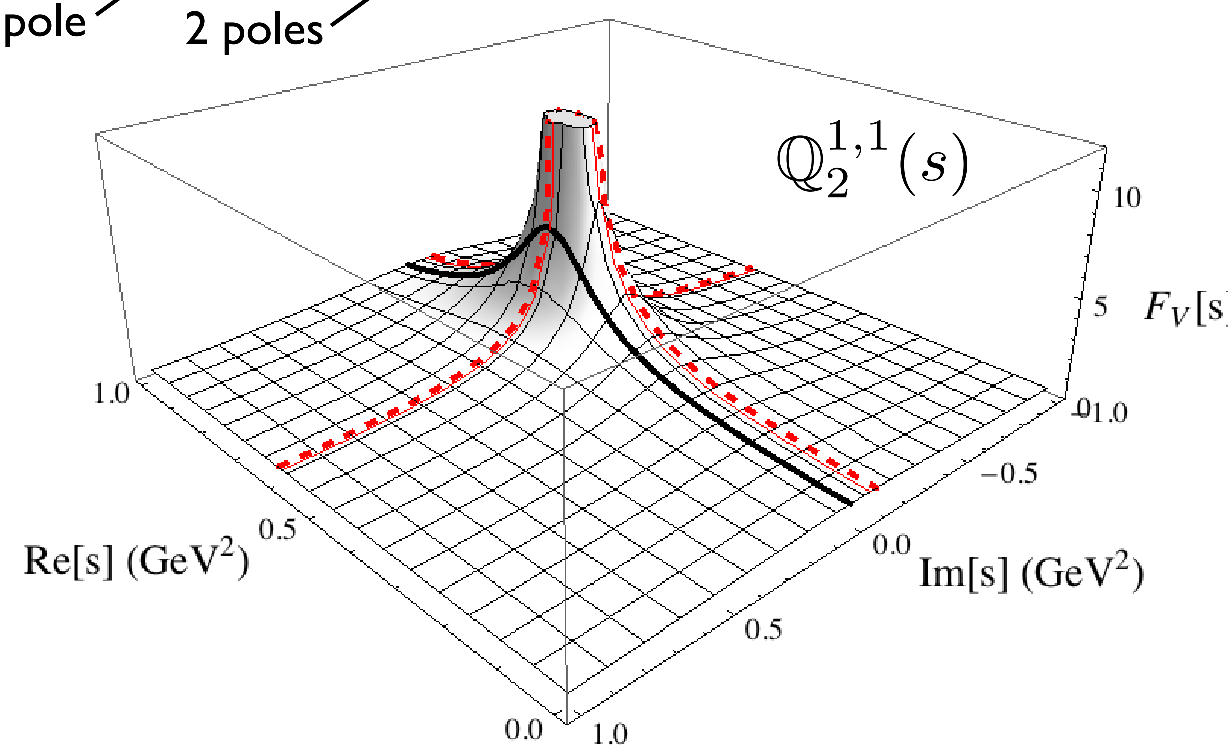
$$F_V(s) = \frac{M_\rho^2}{M_\rho^2 - s - i\frac{\Gamma_\rho s}{M_\rho}\frac{\beta(s)^3}{\beta(M_\rho^2)^3}}$$

$$m_\pi = 135 \text{ MeV}$$
$$M_V = 770 \text{ MeV}$$
$$\Gamma_V = 150 \text{ MeV}$$

Generate Pseudodata in Space like ⟶ fit $\mathbb{Q}_1^{1,0}(s)\,,\mathbb{Q}_2^{1,1}(s)\ \ldots$

1 pole ↗ 2 poles ↗





Quadratic Approximants

Examples

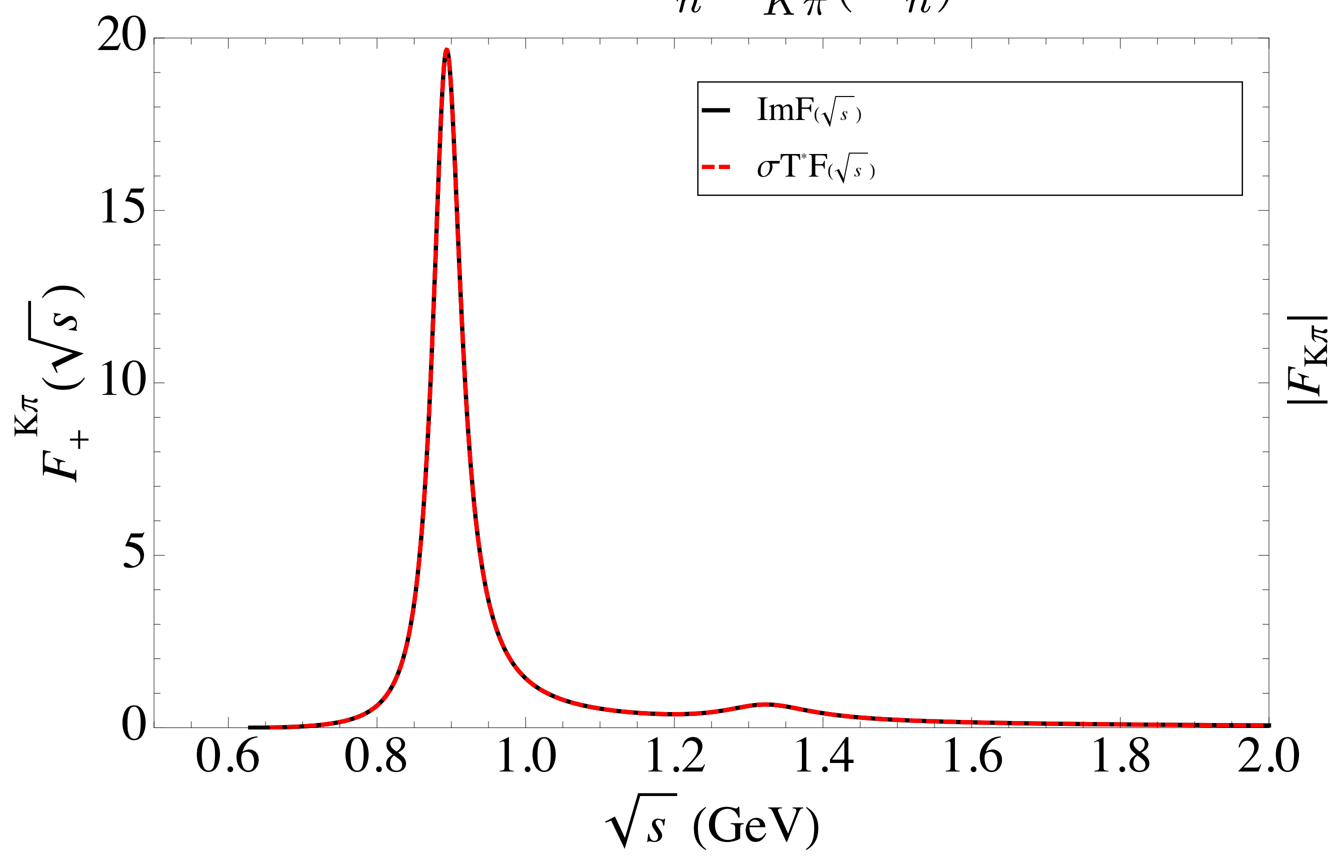
<u>Vector FF model</u> $\tau^- \to K_S \pi^- \nu_\tau$

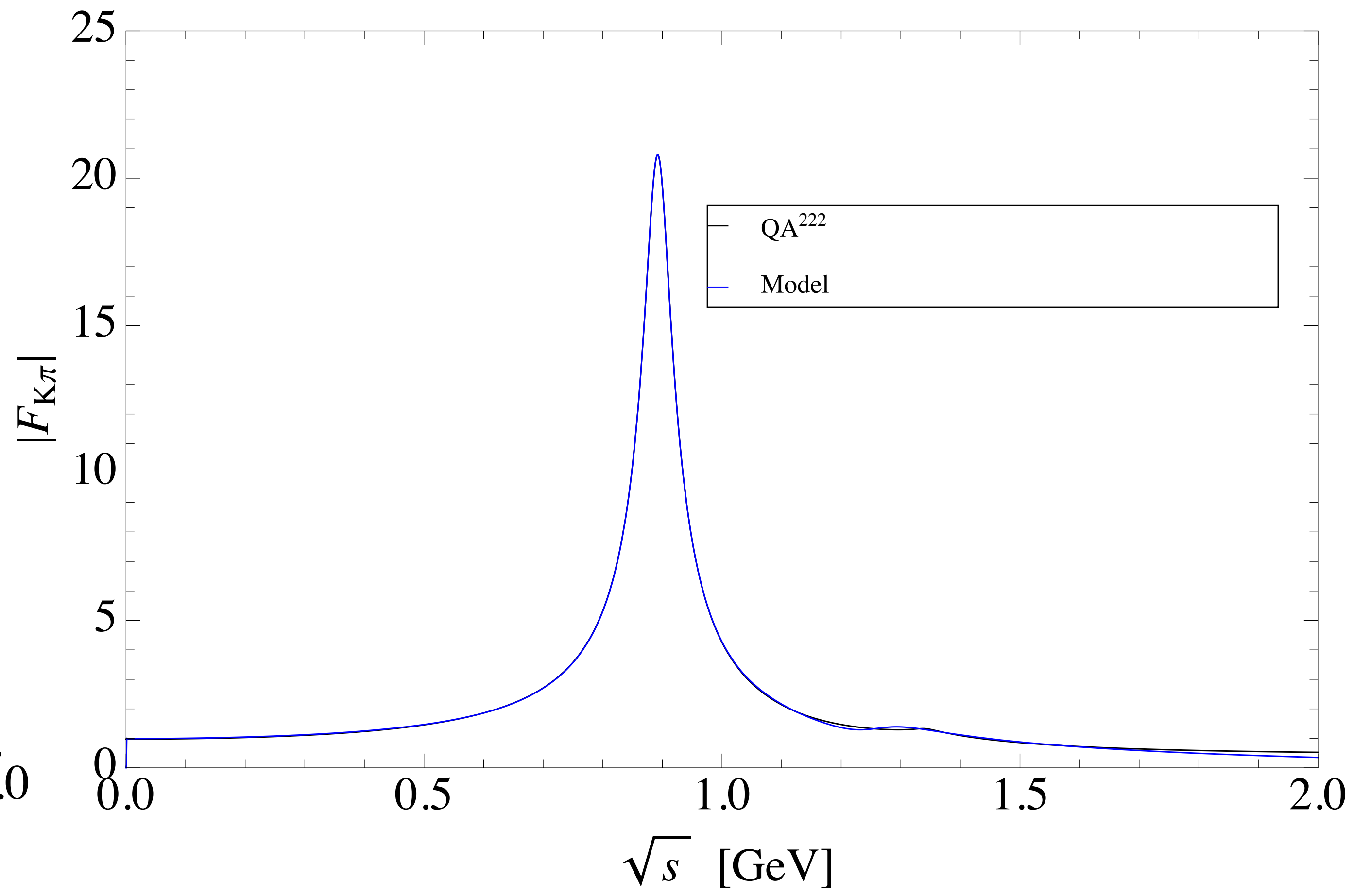
$$f_+^{K\pi}(s) = \left[\frac{m_{K^*}^2 + \gamma s}{D(m_{K^*}, \gamma_{K^*})} - \frac{\gamma s}{D(m_{K^{*\prime}}, \gamma_{K^{*\prime}})} \right] e^{\frac{3}{2}\mathrm{Re}\widetilde{H}_{K\pi}(s)}$$

$$M_{K^*} = 892 \text{ MeV}\,,\; \Gamma_{K^*} = 46 \text{ MeV}$$
$$M_{K^{*\prime}} = 1304 \text{ MeV}\,,\; \Gamma_{K^{*\prime}} = 171 \text{ MeV}$$

$$D(m_n, \gamma_n) = m_n^2 - s - i\, m_n \gamma_n(s)$$

$$\gamma_n(s) = \gamma_n \frac{s}{m_n^2} \frac{\sigma_{K\pi}^3(s)}{\sigma_{K\pi}^3(m_n^2)}$$





Quadratic Approximants

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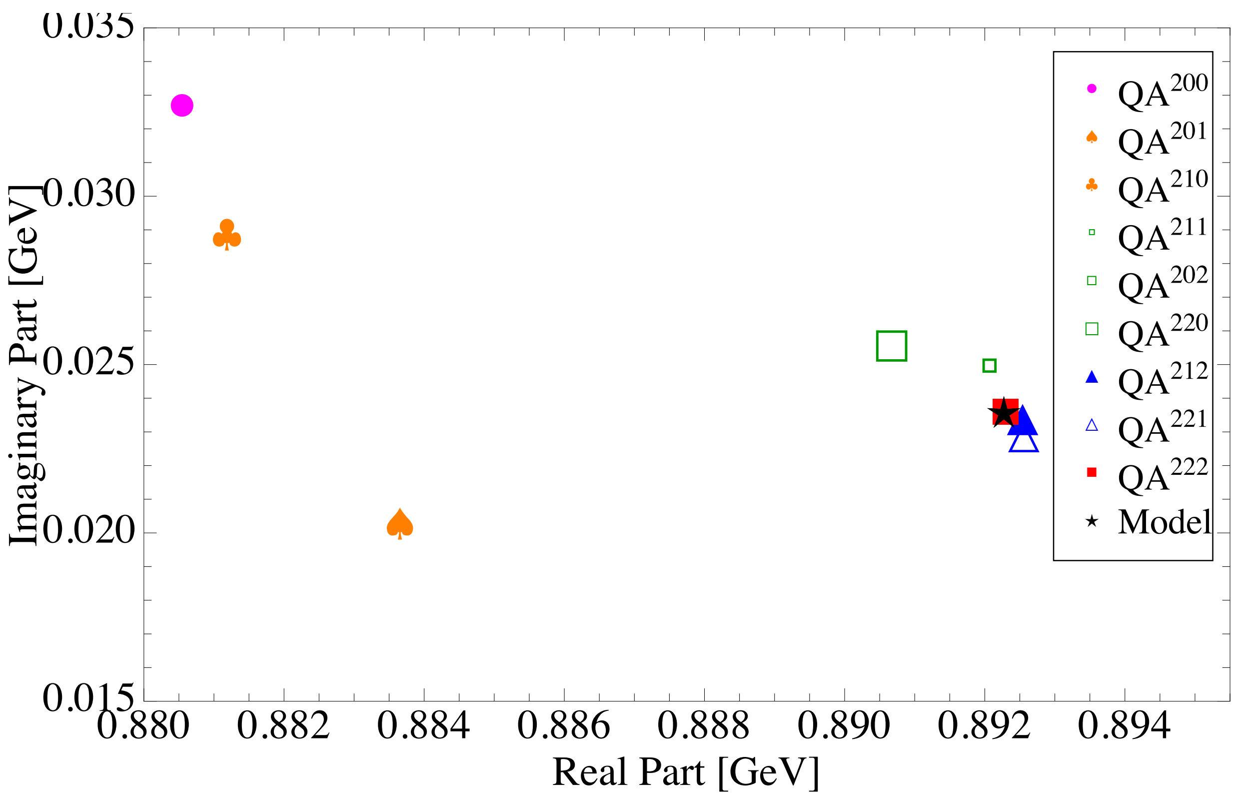
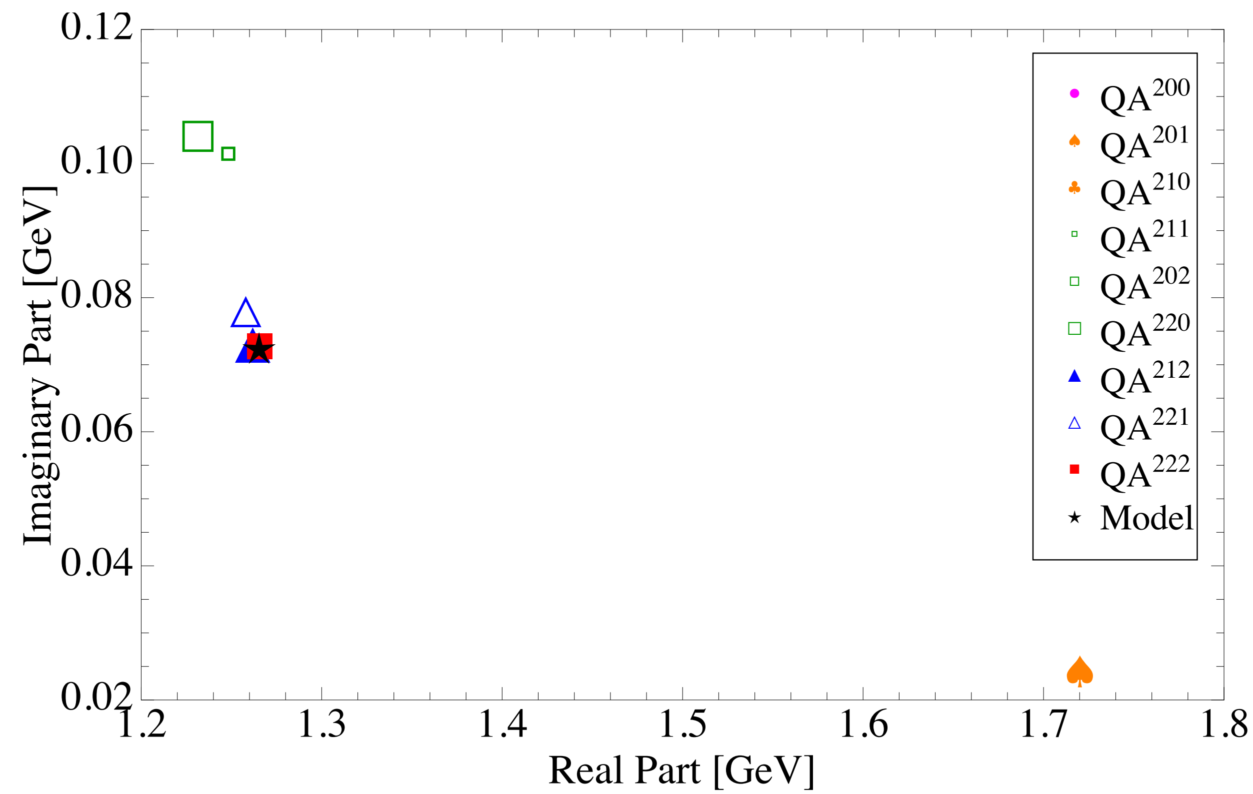
(derivatives)





Quadratic Approximants

Examples

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$$\gamma_n(s) = \gamma_n \frac{s}{m_n^2} \frac{\sigma_{K\pi}^3(s)}{\sigma_{K\pi}^3(m_n^2)}$$

(pseudodata fit)