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SPACELIKE AND TIMELIKE KERNEL FUNCTIONS FOR THE HADRONIC VACUUM POLARIZATION CONTRIBUTIONS TO THE MUON ANOMALOUS MAGNETIC MOMENT

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INTRODUCTION

The theoretical description of $a_\mu = (g_\mu - 2)/2$ is a long–standing challenging issue of the elementary particle physics.

Theory: $a_\mu^{\text{theor}} = a_\mu^{\text{QED}} + a_\mu^{\text{EW}} + a_\mu^{\text{HVP}} + a_\mu^{\text{HLbL}} = (11659181.0 \pm 4.3) \times 10^{-10}$ (0.37 ppm)

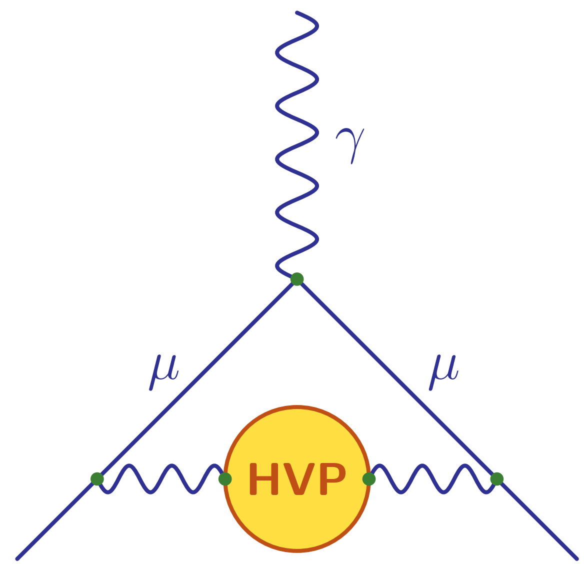
- Aoyama et al., Phys. Rept. 887, 1 (2020) [and references therein].

Experiment: $a_\mu^{\text{exp}} = (11659206.1 \pm 4.1) \times 10^{-10}$ (0.35 ppm)

- BNL E821 (2002–2006); FNAL E989 Run–1 (2021).

The discrepancy $a_\mu^{\text{exp}} - a_\mu^{\text{theor}} = (25.1 \pm 5.9) \times 10^{-10}$ $(4.2\,\sigma)$ may be an evidence for the existence of a new physics beyond the Standard Model.

The uncertainty of evaluation of a_μ^{theor} is largely dominated by the a_μ^{HVP} term.



SPACELIKE APPROACH

$$a_\mu^{\mathrm{HVP}} = A_0 \int_0^\infty K_\Pi(Q^2)\bar{\Pi}(Q^2)\frac{dQ^2}{4m_\mu^2} = A_0 \int_0^\infty \tilde{K}_\Pi(\zeta)\bar{\Pi}(4\zeta m_\mu^2)d\zeta =$$

$$= A_0 \int_0^\infty K_D(Q^2)D(Q^2)\frac{dQ^2}{4m_\mu^2} = A_0 \int_0^\infty \tilde{K}_D(\zeta)D(4\zeta m_\mu^2)d\zeta, \qquad \zeta = \frac{Q^2}{4m_\mu^2}.$$

In this equation A_0 is a constant prefactor, $Q^2 = -q^2 \geq 0$ denotes a spacelike kinematic variable, $\bar{\Pi}(Q^2) = -\Pi(-Q^2)$ stands for the subtracted at zero hadronic vacuum polarization function, $D(Q^2)$ is the Adler function, $K_\Pi(Q^2)$ and $K_D(Q^2)$ denote the corresponding spacelike kernel functions.

Here the perturbative results for $\bar{\Pi}(Q^2)$ and $D(Q^2)$ have to be supplemented with the relevant nonperturbative inputs, that can be provided by

- MUonE @ CERN measurements
- lattice simulations
- reliable phenomenological models

TIMELIKE APPROACH

$$a_\mu^{\text{HVP}} = A_0 \int_{s_0}^{\infty} K_R(s) R(s) \frac{ds}{4m_\mu^2} = A_0 \int_{\chi}^{\infty} \tilde{K}_R(\eta) R(4\eta m_\mu^2)\, d\eta, \quad \eta = \frac{s}{4m_\mu^2}, \quad \chi = \frac{s_0}{4m_\mu^2}.$$

In this equation $s = q^2 \geq 0$ stands for a timelike kinematic variable, $s_0 = 4m_\pi^2$ denotes the hadronic threshold, $R(s)$ is the R–ratio of electron–positron annihilation into hadrons, and $K_R(s)$ stands for the respective timelike kernel function.

Here the perturbative results for $R(s)$ are usually complemented by the low–energy experimental data on the R–ratio, that constitutes the data–driven method of evaluation of a_μ^{HVP}.

The timelike kernel functions $K_R(s)$ have been extensively studied over the past decades, whereas the corresponding spacelike kernel functions $K_\Pi(Q^2)$ and $K_D(Q^2)$ remain largely unavailable.

GENERAL DISPERSION RELATIONS

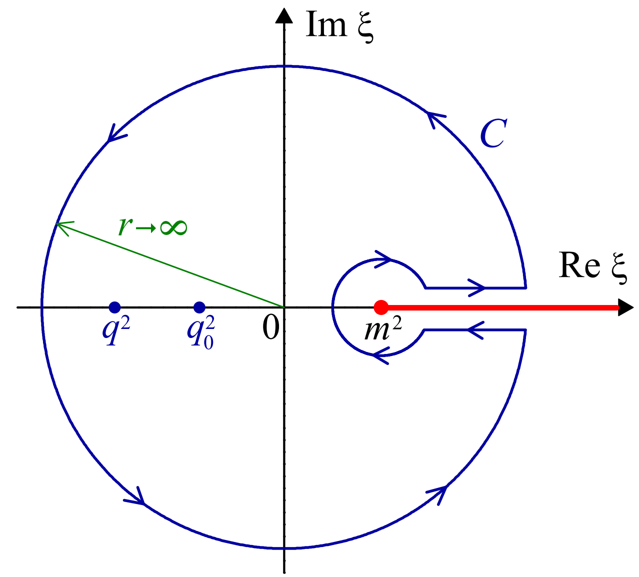
The hadronic vacuum polarization function $\Pi(q^2)$ is defined as the scalar part of the hadronic vacuum polarization tensor

$$\Pi_{\mu\nu}(q^2) = i\int d^4x\, e^{iqx}\langle 0|\, T\left\{J_\mu(x)\, J_\nu(0)\right\}|0\rangle = \frac{i}{12\pi^2}(q_\mu q_\nu - g_{\mu\nu}q^2)\Pi(q^2), \quad q^2 < 0.$$

The physical kinematic restrictions imply that $\Pi(q^2)$ has the only cut starting at the hadronic threshold $q^2 \geq s_0$ ■ Feynman (1972); Adler (1974).

The once–subtracted Cauchy's integral formula yields

$$\Pi(q^2) - \Pi(q_0^2) = (q^2 - q_0^2)\int_{s_0}^{\infty} \frac{R(\sigma)}{(\sigma - q^2)(\sigma - q_0^2)}\, d\sigma,$$



where

$$R(s) = \frac{1}{2\pi i}\lim_{\varepsilon\to 0_+}\Big(\Pi(s+i\varepsilon) - \Pi(s-i\varepsilon)\Big) = \frac{\sigma(e^+e^- \to \text{hadrons}; s)}{\sigma(e^+e^- \to \mu^+\mu^-; s)}$$

denotes the R–ratio of electron–positron annihilation into hadrons.

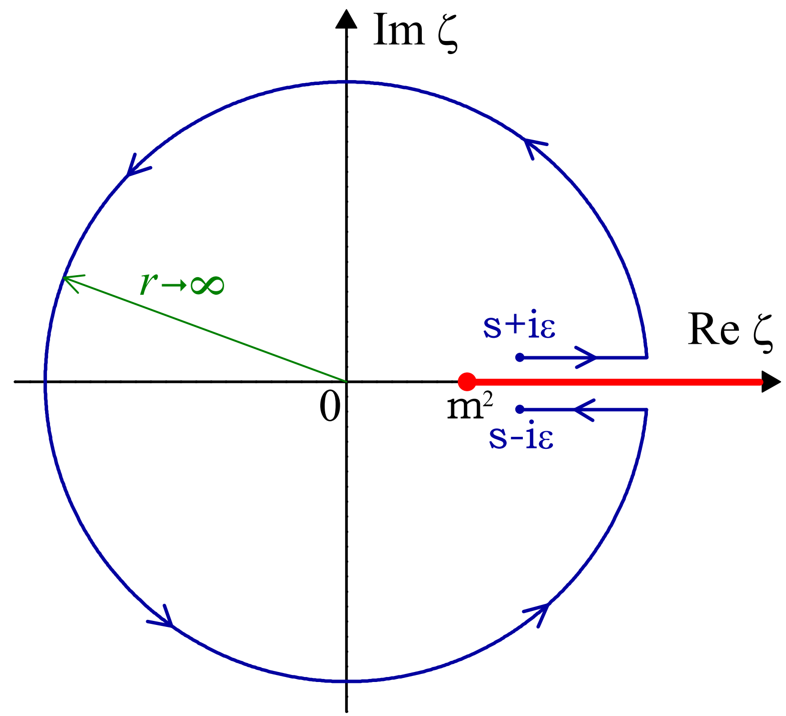
For practical purposes it proves to be particularly convenient to deal with the Adler function

$$D(Q^2) = -\frac{d\,\Pi(-Q^2)}{d\ln Q^2}, \qquad D(Q^2) = Q^2 \int_{s_0}^{\infty} \frac{R(\sigma)}{(\sigma + Q^2)^2}\, d\sigma, \qquad Q^2 = -q^2 > 0$$

- Adler (1974); De Rujula, Georgi (1976); Bjorken (1989).

This dispersion relation enables one to extract the experimental prediction for the Adler function from the respective data on the R-ratio.

The inverse relations between the functions on hand read



$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta)\, \frac{d\zeta}{\zeta}$$

- Radyushkin (1982); Krasnikov, Pivovarov (1982)

$$\Pi(-Q^2) - \Pi(-Q_0^2) = -\int_{Q_0^2}^{Q^2} D(\xi)\, \frac{d\xi}{\xi}$$

- Pennington, Ross (1977), (1981), (1984); Pivovarov (1992).

RELATIONS BETWEEN THE KERNEL FUNCTIONS

In the ℓ–th order in the electromagnetic coupling the hadronic vacuum polarization contribution to the muon anomalous magnetic moment reads

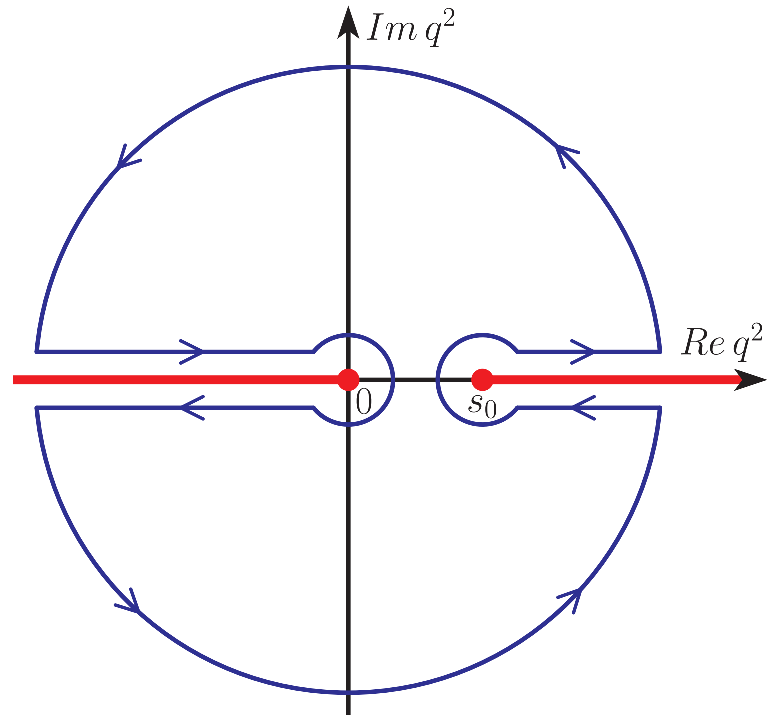
$$
\left.\begin{aligned}
a_{\mu}^{\mathrm{HVP}(\ell)} &= A_0^{(\ell)} \int_0^{\infty} K_{\Pi}^{(\ell)}(Q^2)\bar{\Pi}(Q^2)\frac{dQ^2}{4m_{\mu}^2} = \\
&= A_0^{(\ell)} \int_0^{\infty} K_{D}^{(\ell)}(Q^2)D(Q^2)\frac{dQ^2}{4m_{\mu}^2} =
\end{aligned}\right\} \quad [\,\text{spacelike}\,]
$$

$$
\left.= A_0^{(\ell)} \int_{s_0}^{\infty} K_{R}^{(\ell)}(s)R(s)\frac{ds}{4m_{\mu}^2}.\right\} \quad [\,\text{timelike}\,]
$$

The kernel functions $K_{\Pi}(Q^2)$, $K_D(Q^2)$, and $K_R(s)$ appearing in these equations can all be expressed in terms of each other

- Nesterenko, J. Phys. G **49**, 055001 (2022); arXiv:2112.05009 [hep-ph].

Kernel function $K_\Pi(Q^2)$ in terms of $K_R(s)$



$\bar{\Pi}(-q^2) = -\Pi(q^2)$: cut $q^2 \geq s_0$ ■ Feynman (1972); Adler (1974).

$K_R(q^2)$: cut $q^2 \leq 0$ ■ Barbieri, Remiddi (1975).

The contour integral of their product vanishes

$$\oint_C K_R(q^2)\bar{\Pi}(-q^2)\,dq^2 = 0,$$

that implies

$$-\frac{1}{2\pi i}\lim_{\varepsilon\to 0_+}\int_0^{-\infty}\bar{\Pi}(-p^2)\Big(K_R(p^2+i\varepsilon) - K_R(p^2-i\varepsilon)\Big)dp^2 = \int_{s_0}^{\infty} K_R(p^2)R(p^2)\,dp^2.$$

Thus, the relation, which expresses $K_\Pi(Q^2)$ in terms of $K_R(s)$, reads

$$K_\Pi(Q^2) = \frac{1}{2\pi i}\lim_{\varepsilon\to 0_+}\Big(K_R(-Q^2+i\varepsilon) - K_R(-Q^2-i\varepsilon)\Big), \qquad Q^2 \geq 0$$

■ Nesterenko, J. Phys. G **49**, 055001 (2022); arXiv:2112.05009 [hep-ph].

This relation has also been independently derived in a different way in

■ Balzani, Laporta, Passera, Phys. Lett. B **834**, 137462 (2022); arXiv:2112.05704 [hep-ph].

Kernel function $K_R(s)$ in terms of $K_\Pi(Q^2)$

Dispersion relation for the hadronic vacuum polarization function leads to

$$\int_0^\infty K_\Pi(Q^2)\bar{\Pi}(Q^2)\frac{dQ^2}{4m_\mu^2} = \int_0^\infty \frac{dQ^2}{4m_\mu^2} K_\Pi(Q^2)\, Q^2 \int_{s_0}^\infty \frac{ds}{s}\frac{R(s)}{s+Q^2} = \int_{s_0}^\infty K_R(s) R(s)\frac{ds}{4m_\mu^2}.$$

Hence, the relation, which expresses $K_R(s)$ in terms of $K_\Pi(Q^2)$, reads

$$K_R(s) = \frac{1}{s}\int_0^\infty K_\Pi(Q^2)\frac{Q^2}{s+Q^2}\,dQ^2, \qquad s \geq 0.$$

Kernel function $K_R(s)$ in terms of $K_D(Q^2)$

Dispersion relation for the Adler function yields

$$\int_0^\infty K_D(Q^2) D(Q^2)\frac{dQ^2}{4m_\mu^2} = \int_0^\infty \frac{dQ^2}{4m_\mu^2} K_D(Q^2)\, Q^2 \int_{s_0}^\infty \frac{R(s)}{(s+Q^2)^2}\,ds = \int_{s_0}^\infty K_R(s) R(s)\frac{ds}{4m_\mu^2}.$$

Therefore, the relation, which expresses $K_R(s)$ in terms of $K_D(Q^2)$, reads

$$K_R(s) = \int_0^\infty K_D(Q^2)\frac{Q^2}{(s+Q^2)^2}\,dQ^2, \qquad s \geq 0.$$

Kernel function $K_\Pi(Q^2)$ in terms of $K_D(Q^2)$

Definition of the Adler function results in

$$\int_0^\infty K_D(Q^2) D(Q^2) dQ^2 = -\int_0^\infty dQ^2\, K_D(Q^2) Q^2 \frac{d\,\Pi(-Q^2)}{d\,Q^2} =$$

$$= K_D(Q^2) Q^2\, \bar{\Pi}(Q^2)\Big|_0^\infty - \int_0^\infty dQ^2\, \bar{\Pi}(Q^2)\left(K_D(Q^2) + \frac{d\,K_D(Q^2)}{d\,\ln Q^2}\right),$$

with the integration by parts being employed. Since the first term in the second line of this equation vanishes (see also remarks given below), the relation, which expresses $K_\Pi(Q^2)$ in terms of $K_D(Q^2)$, reads

$$K_\Pi(Q^2) = -\left(K_D(Q^2) + \frac{d\,K_D(Q^2)}{d\,\ln Q^2}\right), \qquad Q^2 \geq 0$$

- Nesterenko, J. Phys. G **49**, 055001 (2022); arXiv:2112.05009 [hep-ph].

Kernel function $K_D(Q^2)$ in terms of $K_\Pi(Q^2)$

The solution to the differential equation derived on the previous page reads

$$K_D(Q^2) + \frac{d\,K_D(Q^2)}{d\,\ln Q^2} = -K_\Pi(Q^2) \quad \longrightarrow \quad K_D(Q^2) = \frac{1}{Q^2}\left(-\int K_\Pi(Q^2)\,dQ^2 + c_0\right).$$

The constant c_0 has to be chosen in the way that makes $K_D(Q^2)$ vanishing at $Q^2 \to \infty$. The relation, which expresses $K_D(Q^2)$ in terms of $K_\Pi(Q^2)$, reads

$$K_D(Q^2) = \frac{1}{Q^2}\int_{Q^2}^{\infty} K_\Pi(\xi)\,d\xi = \frac{4m_\mu^2}{Q^2}K_0 - \frac{1}{Q^2}\int_0^{Q^2} K_\Pi(\xi)\,d\xi, \qquad \xi = -p^2 \geq 0.$$

In this equation K_0 denotes the infrared limiting value of the respective spacelike and timelike (see p. 8) kernel functions, namely

$$K_0 = \lim_{Q^2\to 0_+} \frac{Q^2}{4m_\mu^2} K_D(Q^2) = \lim_{s\to 0_+} \frac{s}{4m_\mu^2} K_R(s) = \int_0^{\infty} K_\Pi(\xi)\,\frac{d\xi}{4m_\mu^2},$$

which is factually identical to the corresponding QED contribution to a_μ of the preceding order in the electromagnetic coupling

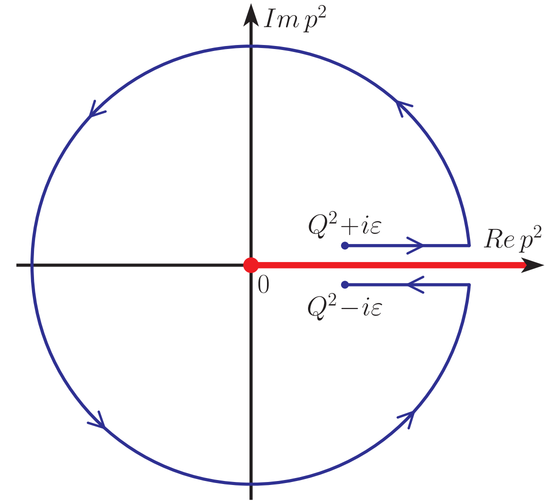
- Nesterenko, J. Phys. G **49**, 055001 (2022); arXiv:2112.05009 [hep-ph].

Kernel function $K_D(Q^2)$ in terms of $K_R(s)$

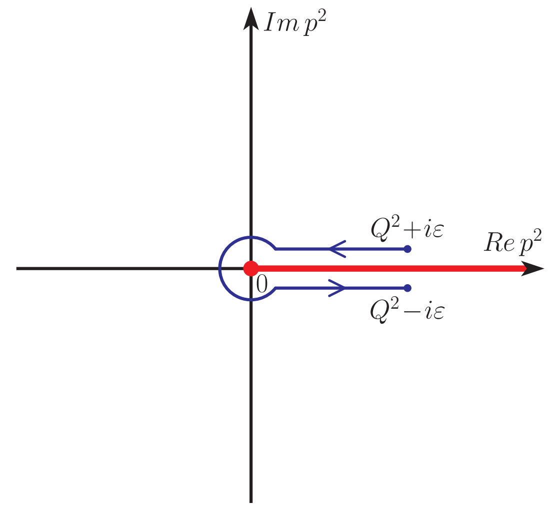
The first and the fifth derived relations between the kernel functions imply that the relation, which expresses $K_D(Q^2)$ in terms of $K_R(s)$, reads

$$K_D(Q^2) = -\frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \frac{1}{Q^2} \int_{Q^2}^{\infty} \Big(K_R(-\xi - i\varepsilon) - K_R(-\xi + i\varepsilon) \Big) d\xi =$$

$$= -\frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \frac{1}{Q^2} \int_{Q^2+i\varepsilon}^{Q^2-i\varepsilon} K_R(-p^2)\, dp^2,$$



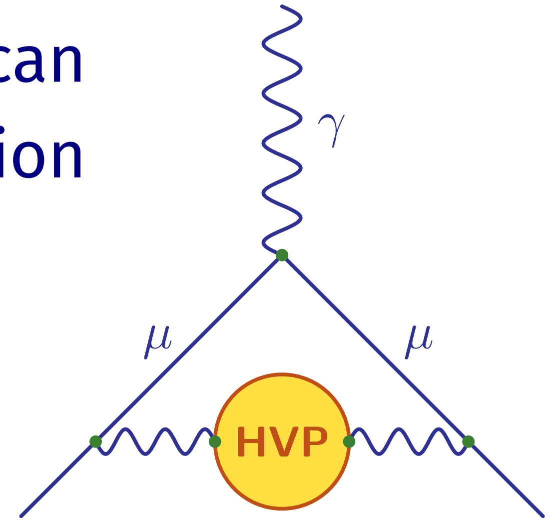
where the integration contour in the complex p^2–plane lies in the region of analyticity of the function $K_R(-p^2)$.



The obtained six equations constitute the complete set of relations, which mutually express the spacelike and timelike kernel functions $K_\Pi(Q^2)$, $K_D(Q^2)$, and $K_R(s)$ in terms of each other. The obtained relations enable one to calculate the unknown kernel functions by making use of the known one

- Nesterenko, J. Phys. G **49**, 055001 (2022); arXiv:2112.05009 [hep-ph].

KERNEL FUNCTIONS IN THE LEADING ORDER



All three leading–order kernel functions are available, that can be used to exemplify the obtained relations. The contribution $a_\mu^{\mathrm{HVP}(2)}$ in terms of the R–ratio (timelike approach) reads

$$a_\mu^{\mathrm{HVP}(2)} = A_0^{(2)} \int_{s_0}^{\infty} K_R^{(2)}(s) R(s) \frac{ds}{4m_\mu^2}, \qquad A_0^{(2)} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2,$$

$$K_R^{(2)}(s) = \frac{4m_\mu^2}{s} \int_0^1 \frac{x^2(1-x)}{x^2 + (1-x)s/m_\mu^2}\, dx, \quad s = q^2 \geq 0, \quad K_R^{(2)}(s) = \tilde{K}_R^{(2)}\left(\frac{s}{4m_\mu^2}\right), \quad \eta = \frac{s}{4m_\mu^2}$$

- Berestetskii, Krokhin, Khlebnikov (1956); Bouchiat, Michel (1961); Kinoshita, Oakes (1967).

Explicit expression for the leading–order timelike kernel function:

$$\eta \tilde{K}_R^{(2)}(\eta) = \frac{1}{2} + 4\eta\Big((2\eta-1)\ln(4\eta) - 1\Big) - 2\Big(2(2\eta-1)^2 - 1\Big)\mathrm{arctanh}\big(\psi(\eta)\big)\frac{\sqrt{\eta}}{\sqrt{\eta-1}}$$

- Berestetskii, Krokhin, Khlebnikov (1956); Durand (1962); Brodsky, de Rafael (1968); Lautrup, de Rafael (1968).

Factually, the specific form of the leading–order timelike kernel function $K_R^{(2)}(s)$ makes it possible to express $a_\mu^{\mathrm{HVP}(2)}$ in terms of the spacelike functions $\bar{\Pi}(Q^2)$ and $D(Q^2)$, namely

$$a_\mu^{\mathrm{HVP}(2)} = A_0^{(2)} \int_0^1 dx\,(1-x) \int_{s_0}^{\infty} \frac{ds}{s} \frac{R(s)\, m_\mu^2 x^2 (1-x)^{-1}}{s + m_\mu^2 x^2 (1-x)^{-1}} = A_0^{(2)} \int_0^1 (1-x)\, \bar{\Pi}\!\left(m_\mu^2 \frac{x^2}{1-x}\right) dx$$

■ Lautrup, Peterman, de Rafael, Phys. Rept. **3**, 193 (1972); de Rafael, Phys. Rev. D **96**, 014510 (2017).

In turn, its integration by parts eventually yields

$$a_\mu^{\mathrm{HVP}(2)} = A_0^{(2)} \int_0^1 (1-x)\left(1-\frac{x}{2}\right) D\!\left(m_\mu^2 \frac{x^2}{1-x}\right) \frac{dx}{x}$$

■ Knecht, Lect. Notes Phys. **629**, 37 (2004); de Rafael, Phys. Rev. D **96**, 014510 (2017).

It is necessary to emphasize here that this way of derivation of the spacelike expressions for $a_\mu^{\mathrm{HVP}(2)}$ from the timelike one entirely relies on the specific form of the leading–order kernel function $K_R^{(2)}(s)$.

The explicit form of the leading–order spacelike kernel functions in terms of the Q^2 kinematic variable can be obtained by mapping the integration range $0 \le x < 1$ in the equations given on the previous page onto the kinematic interval $0 \le Q^2 < \infty$. Specifically, the kernel function $K_\Pi^{(2)}(Q^2)$ takes the following form

$$K_\Pi^{(2)}(Q^2) = \tilde{K}_\Pi^{(2)}\left(\frac{Q^2}{4m_\mu^2}\right), \quad \zeta \tilde{K}_\Pi^{(2)}(\zeta) = \frac{1}{\zeta^2}\frac{y^5(\zeta)}{1-y(\zeta)}, \quad y(\zeta) = \zeta\left(\sqrt{1+\zeta^{-1}}-1\right), \quad \zeta = \frac{Q^2}{4m_\mu^2}$$

- Groote, Korner, Pivovarov, Eur. Phys. J. C **24**, 393 (2002); Blum, Phys. Rev. Lett. **91**, 052001 (2003); Nesterenko, J. Phys. G **42**, 085004 (2015); de Rafael, Phys. Rev. D **96**, 014510 (2017).

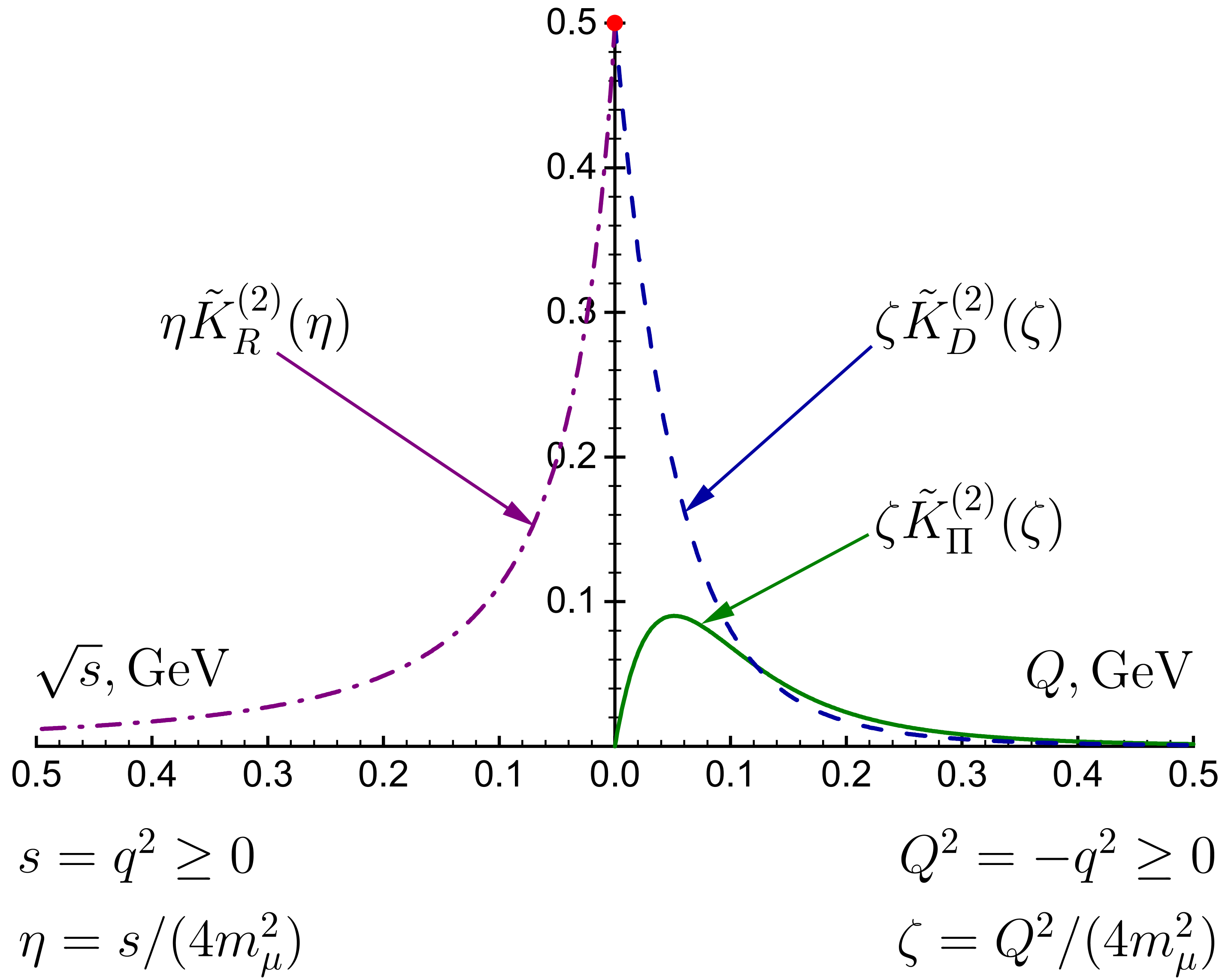
In turn, for the kernel function $K_D^{(2)}(Q^2)$ the foregoing mapping the integration range $0 \le x < 1$ onto the kinematic interval $0 \le Q^2 < \infty$ yields

$$K_D^{(2)}(Q^2) = \tilde{K}_D^{(2)}\left(\frac{Q^2}{4m_\mu^2}\right), \quad \zeta \tilde{K}_D^{(2)}(\zeta) = (2\zeta+1)^2 - 2(2\zeta+1)\sqrt{\zeta(\zeta+1)} - \frac{1}{2}, \quad \zeta = \frac{Q^2}{4m_\mu^2}$$

- Groote, Korner, Pivovarov, Eur. Phys. J. C **24**, 393 (2002); de Rafael, Phys. Rev. D **96**, 014510 (2017).

It is straghtforward to verify that all six obtained relations for the spacelike and timelike kernel functions hold for $K_{\Pi}^{(2)}(Q^2)$, $K_D^{(2)}(Q^2)$, and $K_R^{(2)}(s)$

- Nesterenko, J. Phys. G **49**, 055001 (2022); arXiv:2112.05009 [hep-ph].



The aforementioned infrared limiting value of the spacelike and timelike kernel functions

$$K_0^{(2)} = \lim_{Q^2\to 0_+} \frac{Q^2}{4m_\mu^2} K_D^{(2)}(Q^2) = \lim_{s\to 0_+} \frac{s}{4m_\mu^2} K_R^{(2)}(s) = \int_0^\infty K_\Pi^{(2)}(\xi)\,\frac{d\xi}{4m_\mu^2} = \frac{1}{2}$$

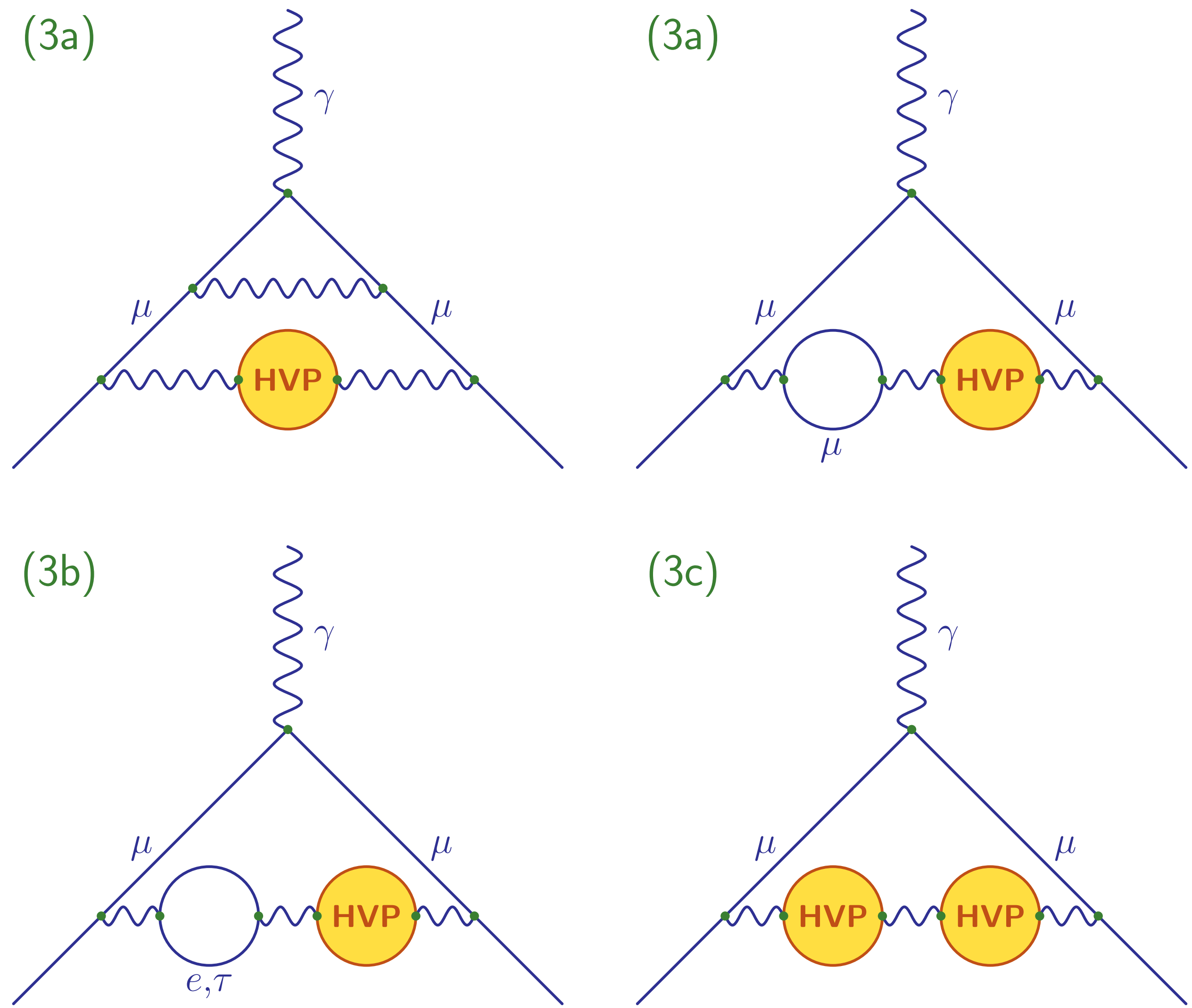
corresponds to the leading Schwinger contribution

- Schwinger, Phys. Rev. **73**, 416 (1948).

KERNEL FUNCTIONS IN THE NEXT–TO–LEADING ORDER

In the next–to–leading order of perturbation theory (i.e., in the third order in the electromagnetic coupling) the hadronic vacuum polarization contribution to the muon anomalous magnetic moment consists of three parts, namely

$$a_\mu^{\mathrm{HVP}(3)} = a_\mu^{\mathrm{HVP}(3a)} + a_\mu^{\mathrm{HVP}(3b)} + a_\mu^{\mathrm{HVP}(3c)}.$$



Kernel functions (3a)

Here the explicit expression for the timelike kernel function $K_R^{(3a)}(s)$ is available, whereas the spacelike kernel functions $K_\Pi^{(3a)}(Q^2)$ and $K_D^{(3a)}(Q^2)$ can be calculated by making use of the relations obtained above. Namely,

$$\eta \tilde{K}_R^{(3a)}(\eta) = -\frac{139}{144} + \frac{115}{18}\eta + \left(\frac{19}{12} - \frac{7}{9}\eta + \frac{23}{9}\eta^2 + \frac{1}{4(\eta-1)}\right)\ln(4\eta) +$$
$$+ \left(\frac{2}{3\,\eta} - \frac{127}{18} + \frac{115}{9}\eta - \frac{46}{9}\eta^2\right)\frac{A(\eta)}{\psi(\eta)} + \left(\frac{9}{4} + \frac{5}{6}\eta - 8\eta^2 - \frac{1}{2\,\eta}\right)\zeta_2 + \frac{5}{6}\eta^2 \ln^2(4\eta) +$$
$$+ \left(\frac{14}{3}\eta - 1\right)(\eta - 1)\frac{1}{\psi(\eta)}T_1(\eta) + \left(\frac{19}{6} + \frac{53}{3}\eta - \frac{58}{3}\eta^2 - \frac{1}{3\,\eta} + \frac{2}{\eta - 1}\right)A^2(\eta) +$$
$$+ \left(\frac{13}{12\,\eta} - \frac{7}{6} + \eta - \frac{8}{3}\eta^2 - \frac{1}{4\eta(\eta-1)}\right)\frac{T_2(\eta)}{\psi(\eta)} + \left(\frac{1}{2} - \frac{14}{3}\eta + 8\eta^2\right)T_3(\eta), \quad \eta = \frac{s}{4m_\mu^2},$$

with $s = q^2 \geq 0$ being the timelike kinematic variable, $A_0^{(3a)} = (2/3)(\alpha/\pi)^3$,

$$T_1(\eta) = A(\eta)\ln(4\eta) + 2\Big\{\mathrm{Li}_2\Big(1 - B(\eta)\Big) + A^2(\eta)\Big\}, \qquad T_2(\eta) = \mathrm{Li}_2\Big(-B(\eta)\Big) + A^2(\eta) + \frac{1}{2}\zeta_2,$$

$$T_3(\eta) = -6\mathrm{Li}_3\Big(B(\eta)\Big) - 3\mathrm{Li}_3\Big(-B(\eta)\Big) + 4\ln\Big(1 - B(\eta)\Big)A^2(\eta) +$$
$$+ \Big(2A^2(\eta) + 3\zeta_2\Big)\ln\Big(1 + B(\eta)\Big) - 4\Big\{\mathrm{Li}_2\Big(-B(\eta)\Big) + 2\mathrm{Li}_2\Big(-B(\eta)\Big)\Big\}A(\eta),$$

$$A(\eta) = \mathrm{arctanh}\Big(\psi(\eta)\Big), \qquad B(\eta) = \frac{1 - \psi(\eta)}{1 + \psi(\eta)}, \qquad \psi(\eta) = \frac{\sqrt{\eta - 1}}{\sqrt{\eta}},$$

$$\mathrm{Li}_2(y) = -\int_0^y \ln(1 - t)\frac{dt}{t}, \qquad \mathrm{Li}_3(y) = \int_0^y \mathrm{Li}_2(t)\frac{dt}{t}, \qquad \zeta_t = \sum_{n=1}^{\infty}\frac{1}{n^t}$$

- Barbieri, Remiddi, Nucl. Phys. B **90**, 233 (1975).

Spacelike kernel functions in terms of the timelike one (see p. 7 and p. 10):

$$K_\Pi^{(3a)}(Q^2) = \frac{1}{2\pi i}\lim_{\varepsilon\to 0_+}\Big(K_R^{(3a)}(-Q^2 + i\varepsilon) - K_R^{(3a)}(-Q^2 - i\varepsilon)\Big), \qquad Q^2 \geq 0,$$

$$K_D^{(3a)}(Q^2) = \frac{1}{Q^2}\int_{Q^2}^{\infty} K_\Pi^{(3a)}(\xi)\,d\xi = \frac{4m_\mu^2}{Q^2}K_0^{(3a)} - \frac{1}{Q^2}\int_0^{Q^2} K_\Pi^{(3a)}(\xi)\,d\xi, \quad \xi = -p^2 \geq 0.$$

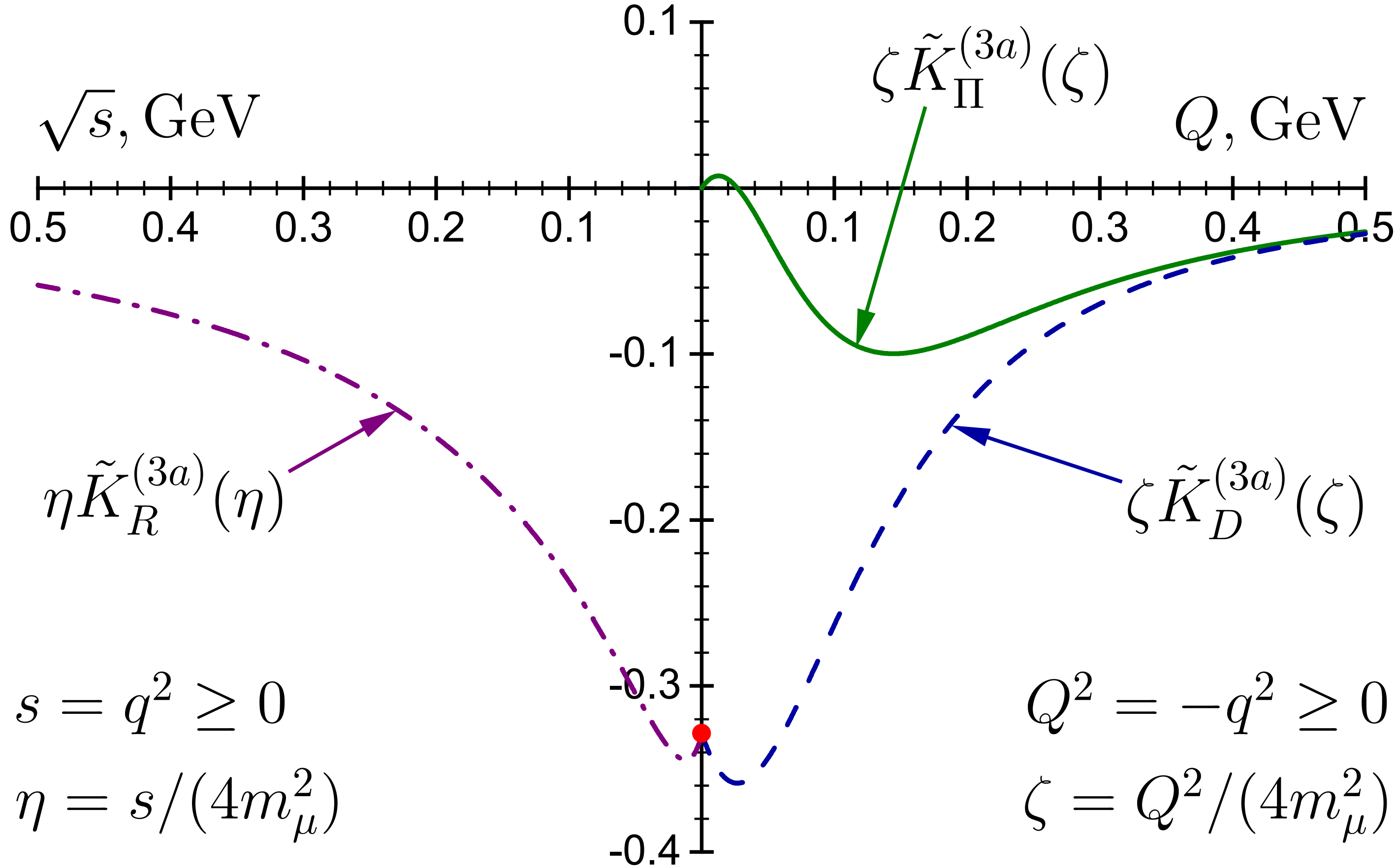
The explicit expression for the spacelike kernel function $K_{\Pi}^{(3a)}(Q^2)$, which can be employed in such methods as MUonE and lattice, reads

$$\zeta \tilde{K}_{\Pi}^{(3a)}(\zeta) = -\left(\frac{19}{12} + \frac{7}{9}\zeta + \frac{23}{9}\zeta^2 - \frac{1}{4(\zeta+1)}\right) + \left(\frac{1}{3\zeta} + \frac{127}{36} + \frac{115}{18}\zeta + \frac{23}{9}\zeta^2\right)\psi(\zeta+1) -$$
$$- \left(\frac{14}{3}\zeta + 1\right)(\zeta+1)\psi(\zeta+1)\left\{\frac{1}{2}\ln(4\zeta) + 3A(\zeta+1) + 2\ln\Bigl(1 + B(\zeta+1)\Bigr)\right\} +$$
$$+ \left(-\frac{19}{6} + \frac{53}{3}\zeta + \frac{58}{3}\zeta^2 - \frac{1}{3\zeta} + \frac{2}{\zeta+1}\right)A(\zeta+1) - \frac{5}{3}\zeta^2\ln(4\zeta) +$$
$$+ \left(\frac{13}{12\,\zeta} + \frac{7}{6} + \zeta + \frac{8}{3}\zeta^2 + \frac{1}{4\zeta(\zeta+1)}\right)\psi(\zeta+1)A(\zeta+1) -$$
$$- \left(\frac{1}{2} + \frac{14}{3}\zeta + 8\zeta^2\right)\left\{2A(\zeta+1)\left\{2\ln\Bigl(1 + B(\zeta+1)\Bigr) + \ln\Bigl(1 - B(\zeta+1)\Bigr)\right\} -\right.$$
$$\left. - 2\left\{\mathrm{Li}_2\Bigl(B(\zeta+1)\Bigr) + 2\mathrm{Li}_2\Bigl(-B(\zeta+1)\Bigr)\right\}\right\}, \qquad \zeta = \frac{Q^2}{4m_\mu^2}$$

- Nesterenko, J. Phys. G **49**, 055001 (2022); arXiv:2112.05009 [hep-ph].

An equivalent form of this equation has also been independently derived in

- Balzani, Laporta, Passera, Phys. Lett. B **834**, 137462 (2022); arXiv:2112.05704 [hep-ph].



The infrared limiting value of the spacelike and timelike kernel functions

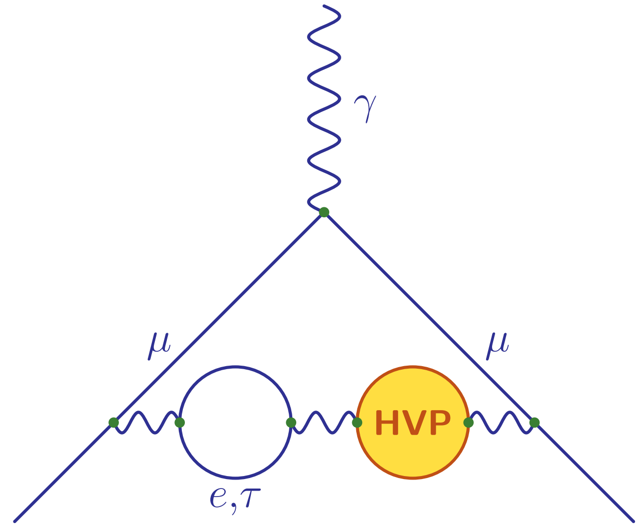
$$K_0^{(3a)} = \lim_{Q^2 \to 0_+} \frac{Q^2}{4m_\mu^2} K_D^{(3a)}(Q^2) = \lim_{s \to 0_+} \frac{s}{4m_\mu^2} K_R^{(3a)}(s) = \int_0^\infty K_\Pi^{(3a)}(\xi) \frac{d\xi}{4m_\mu^2} =$$

$$= \frac{197}{144} + \frac{1}{2}\zeta_2 - 3\zeta_2 \ln(2) + \frac{3}{4}\zeta_3 \simeq -0.328479$$

corresponds to the QED contribution ■ Sommerfield (1957), (1958); Petermann (1957), (1958).

Kernel functions (3b)

For the timelike kernel function $K_R^{(3b)}(s)$ the integral representation of the following form is available:



$$a_\mu^{\mathrm{HVP}(3b)} = A_0^{(3b)} \int_{s_0}^{\infty} K_R^{(3b)}(s) R(s) \frac{ds}{4m_\mu^2}, \qquad A_0^{(3b)} = \frac{2}{3}\left(\frac{\alpha}{\pi}\right)^3,$$

$$K_R^{(3b)}(s) = \frac{4m_\mu^2}{s} \int_0^1 \frac{x^2(1-x)}{x^2 + (1-x)s/m_\mu^2} \bar{\Pi}_\ell\left(m_\mu^2 \frac{x^2}{1-x}\right) dx, \qquad s = q^2 \geq 0, \qquad \eta = \frac{s}{4m_\mu^2}$$

■ Calmet, Narison, Perrottet, de Rafael, Phys. Lett. B **61**, 283 (1976)

where

$$\bar{\Pi}_\ell(Q^2) = 2\int_0^1 y(1-y) \ln\left[1 + z_\ell\, y(1-y)\right] dy =$$

$$= -\frac{5}{9} + \frac{4}{3z_\ell} + \frac{2}{3}\left(1 - \frac{2}{z_\ell}\right)\sqrt{1 + \frac{4}{z_\ell}}\,\mathrm{arctanh}\left(\frac{1}{\sqrt{1 + 4/z_\ell}}\right), \qquad z_\ell = \frac{Q^2}{m_\ell^2} \geq 0$$

and m_ℓ is the mass of the corresponding lepton ■ Akhiezer, Berestetskii, (1965).

Since the diagram on hand factually constitutes an additional lepton loop insertion into the only internal photon line of the leading–order diagram (see p. 12), the spacelike kernel function $K_{\Pi}^{(3b)}(Q^2)$ is the product of the kernel function of the preceding order $K_{\Pi}^{(2)}(Q^2)$ (see p. 14) and the leptonic vacuum polarization function $\bar{\Pi}_{\ell}(Q^2)$ (see p. 21), namely

$$K_{\Pi}^{(3b)}(Q^2) = K_{\Pi}^{(2)}(Q^2)\bar{\Pi}_{\ell}(Q^2).$$

Note that the spacelike kernel function $K_{\Pi}^{(3b)}(Q^2)$ has also been derived from the timelike one $K_{R}^{(3b)}(s)$ by making use of the relevant dispersion relation

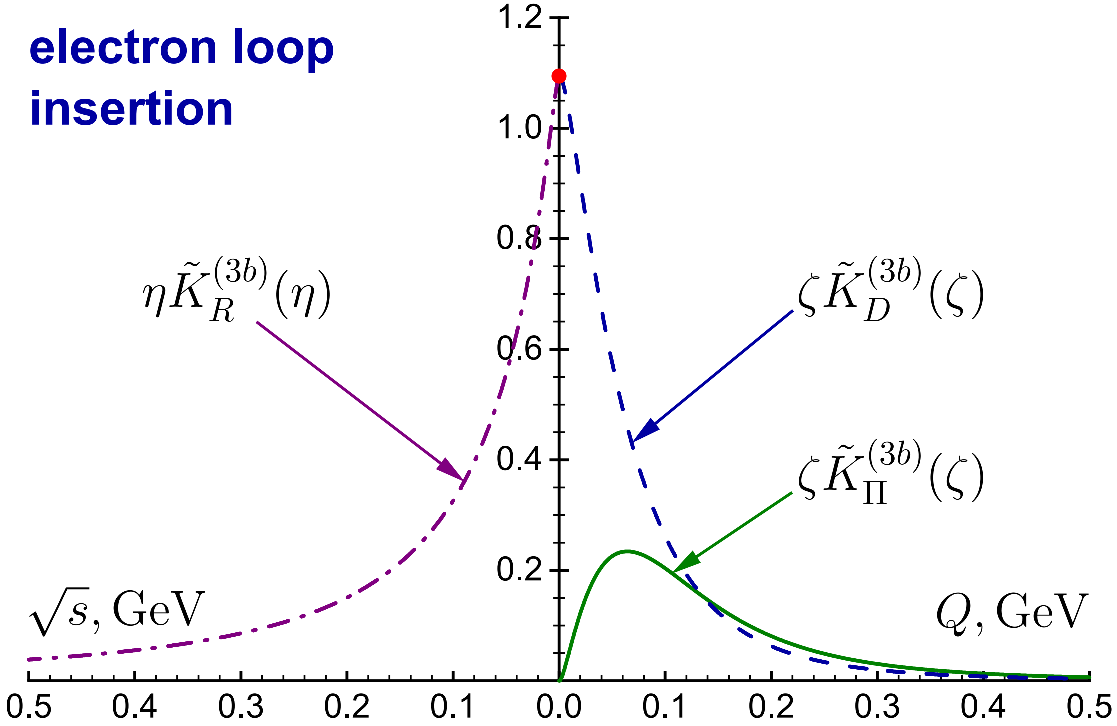
■ Chakraborty *et al*, (2018); Balzani, Laporta, Passera, Phys. Lett. B **834**, 137462 (2022); arXiv:2112.05704 [hep-ph].

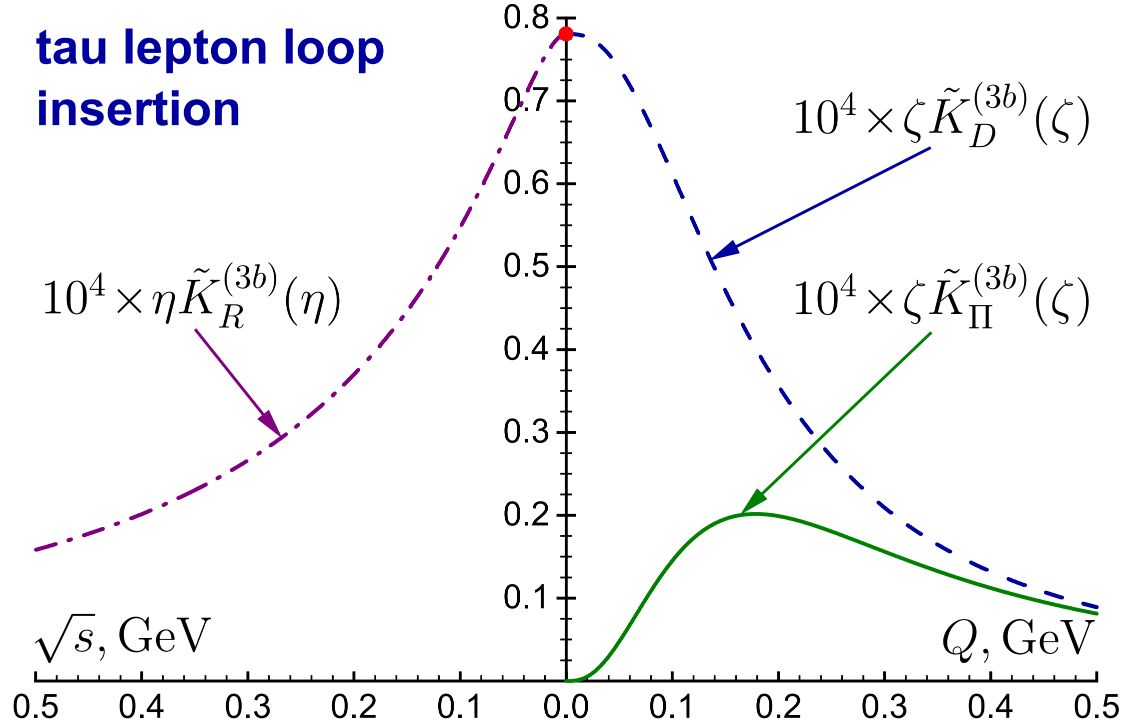
In turn, the spacelike kernel function $K_{D}^{(3b)}(Q^2)$ can be calculated by making use of the relation obtained earlier (see p. 10):

$$K_{D}^{(3b)}(Q^2) = \frac{1}{Q^2}\int_{Q^2}^{\infty} K_{\Pi}^{(3b)}(\xi)\, d\xi = \frac{4m_{\mu}^2}{Q^2} K_{0}^{(3b)} - \frac{1}{Q^2}\int_{0}^{Q^2} K_{\Pi}^{(3b)}(\xi)\, d\xi, \quad \xi = -p^2 \geq 0.$$

All the relevant details can be found in

■ Nesterenko, J. Phys. G **49**, 055001 (2022); arXiv:2112.05009 [hep-ph]; arXiv:2209.03217 [hep-ph].





$s = q^2 \geq 0$ $\qquad$ $Q^2 = -q^2 \geq 0$ $\qquad$ $s = q^2 \geq 0$ $\qquad$ $Q^2 = -q^2 \geq 0$

$\eta = s/(4m_\mu^2)$ $\qquad$ $\zeta = Q^2/(4m_\mu^2)$ $\qquad$ $\eta = s/(4m_\mu^2)$ $\qquad$ $\zeta = Q^2/(4m_\mu^2)$

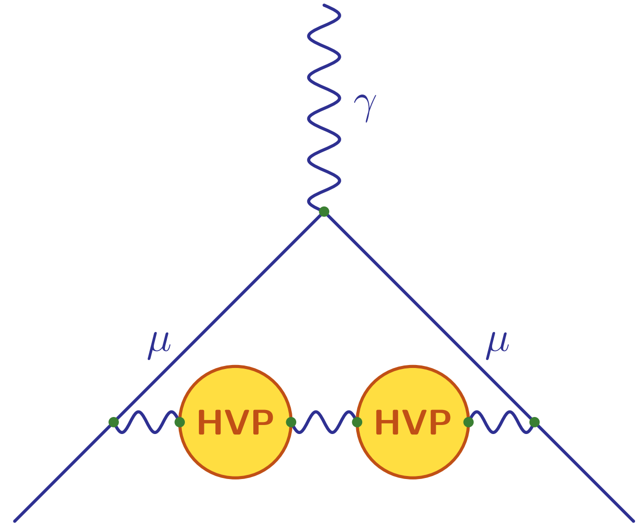
The infrared limiting value of the spacelike and timelike kernel functions

$$K_0^{(3b)} = \lim_{Q^2 \to 0_+} \frac{Q^2}{4m_\mu^2} K_D^{(3b)}(Q^2) = \lim_{s \to 0_+} \frac{s}{4m_\mu^2} K_R^{(3b)}(s) = \int_0^\infty K_\Pi^{(3b)}(\xi) \frac{d\xi}{4m_\mu^2} \simeq \begin{cases} 1.094258, & \text{[electron]} \\ 0.780758 \times 10^{-4}, & [\tau\text{–lepton}] \end{cases}$$

corresponds to the QED contribution ■ Elend, Phys. Lett. **20**, 682 (1966); **21**, 720(E) (1966).

Kernel functions (3c)

The contribution $a_\mu^{\mathrm{HVP}(3c)}$ takes a particularly simple form in terms of the hadronic vacuum polarization function $\bar{\Pi}(Q^2)$



$$a_\mu^{\mathrm{HVP}(3c)} = A_0^{(3c)} \int_0^\infty K_\Pi^{(2)}(Q^2) \Big(\bar{\Pi}(Q^2)\Big)^2 \frac{dQ^2}{4m_\mu^2}, \qquad A_0^{(3c)} = \frac{1}{9}\left(\frac{\alpha}{\pi}\right)^3,$$

where $K_\Pi^{(2)}(Q^2)$ stands for the spacelike kernel function of the preceding order (see p. 14). The contribution $a_\mu^{\mathrm{HVP}(3c)}$ can also be represented in terms of the Adler function $D(Q^2)$ by making use of the relevant dispersion relation (see p. 5):

$$a_\mu^{\mathrm{HVP}(3c)} = A_0^{(3c)} \int_0^\infty \frac{dQ^2}{4m_\mu^2} K_\Pi^{(2)}(Q^2) \left(\int_0^{Q^2} \frac{d\xi}{\xi} D(\xi) \right)^2,$$

with $\xi = -p^2 \geq 0$ being a spacelike kinematic variable.

In turn, the contribution $a_\mu^{\mathrm{HVP}(3c)}$ can be expressed in terms of the R–ratio of electron–positron annihilation into hadrons by making use of the pertinent dispersion relation (see p. 4):

$$a_\mu^{\mathrm{HVP}(3c)} = A_0^{(3c)} \int_{s_0}^{\infty} \frac{ds_1}{s_1} \int_{s_0}^{\infty} \frac{ds_2}{s_2} K_R^{(3c)}(s_1, s_2) R(s_1) R(s_2),$$

where

$$K_R^{(3c)}(s_1, s_2) = \int_0^{\infty} \frac{K_\Pi^{(2)}(Q^2)\, Q^4}{(s_1 + Q^2)(s_2 + Q^2)} \frac{dQ^2}{4m_\mu^2}.$$

- Calmet, Narison, Perrottet, de Rafael, Phys. Lett. B **61**, 283 (1976).

The explicit form of the timelike kernel function $K_R^{(3c)}(s_1, s_2)$ was given in

- Krause, Phys. Lett. B **390**, 392 (1997).

DISPERSIVELY IMPROVED PERTURBATION THEORY

Underlying concept: merge the nonperturbative constrains, which relevant dispersion relations impose on $\Pi(q^2)$, $R(s)$, and $D(Q^2)$, with corresponding perturbative input in a self–consistent way. This can be achieved by expressing the functions on hand in terms of the common spectral density:

$$\Delta\Pi(q^2,\, q_0^2) = N_c \sum_{f=1}^{n_f} Q_f^2 \left\{ \Delta\Pi^{(0)}(q^2,\, q_0^2) + \int_{s_0}^{\infty} \rho(\sigma) \ln\left(\frac{\sigma - q^2}{\sigma - q_0^2} \, \frac{s_0 - q_0^2}{s_0 - q^2} \right) \frac{d\sigma}{\sigma} \right\},$$

$$R(s) = N_c \sum_{f=1}^{n_f} Q_f^2 \left\{ R^{(0)}(s) + \theta(s - s_0) \int_{s}^{\infty} \rho(\sigma) \frac{d\sigma}{\sigma} \right\},$$

$$D(Q^2) = N_c \sum_{f=1}^{n_f} Q_f^2 \left\{ D^{(0)}(Q^2) + \frac{Q^2}{Q^2 + s_0} \int_{s_0}^{\infty} \rho(\sigma) \frac{\sigma - s_0}{\sigma + Q^2} \frac{d\sigma}{\sigma} \right\},$$

$$\rho(\sigma) = \frac{1}{\pi} \frac{d}{d \ln \sigma} \operatorname{Im} \lim_{\varepsilon \to 0_+} p(\sigma - i\varepsilon) = -\frac{d\, r(\sigma)}{d \ln \sigma} = \frac{1}{\pi} \operatorname{Im} \lim_{\varepsilon \to 0_+} d(-\sigma - i\varepsilon)$$

- Nesterenko, Phys. Rev. D **88**, 056009 (2013); J. Phys. G **42**, 085004 (2015); ISBN: 9780128034392 (2017).

Derivation of the obtained integral representations for $\Pi(q^2)$, $R(s)$, $D(Q^2)$ involves neither additional approximations nor model–dependent assumptions, with all the nonperturbative constraints being embodied.

The leading–order terms of the functions on hand read

$$\Delta\Pi^{(0)}(q^2,\, q_0^2) = 2\,\frac{\varphi - \tan\varphi}{\tan^3\varphi} - 2\,\frac{\varphi_0 - \tan\varphi_0}{\tan^3\varphi_0}, \qquad \sin^2\varphi = \frac{q^2}{s_0},$$

$$R^{(0)}(s) = \theta(s - s_0)\left(1 - \frac{s_0}{s}\right)^{3/2}, \qquad \sin^2\varphi_0 = \frac{q_0^2}{s_0},$$

$$D^{(0)}(Q^2) = 1 + \frac{3}{\xi}\left(1 - \sqrt{1 + \xi^{-1}}\,\mathrm{arcsinh}\sqrt{\xi}\,\right), \qquad \xi = \frac{Q^2}{s_0}$$

■ Feynman (1972); Akhiezer, Berestetsky (1965).

The spectral density $\rho(\sigma)$ brings in the perturbative input:

$$\rho_{\text{pert}}(\sigma) = \frac{1}{\pi}\,\frac{d}{d\,\ln\sigma}\,\mathrm{Im}\,p_{\text{pert}}(\sigma - i0_+) = -\frac{d}{d\,\ln\sigma}\,r_{\text{pert}}(\sigma) = \frac{1}{\pi}\,\mathrm{Im}\,d_{\text{pert}}(-\sigma - i0_+).$$

One–loop level: $\rho^{(1)}_{\text{pert}}(\sigma) = 4/\left[\beta_0\left(\ln^2(\sigma/\Lambda^2) + \pi^2\right)\right]$.

First few loop levels: ■ Nesterenko, Simolo (2010, 2011); Bakulev (2013); Cvetic (2015).

The perturbative spectral function at the ℓ–loop level:

$$\rho^{(\ell)}_{\text{pert}}(\sigma) = \sum_{j=1}^{\ell} d_j\, \bar{\rho}^{(\ell)}_j(\sigma), \qquad \bar{\rho}^{(\ell)}_j(\sigma) = \frac{1}{2\pi i} \lim_{\varepsilon\to 0_+} \left\{ \left[a^{(\ell)}_s(-\sigma - i\varepsilon)\right]^j - \left[a^{(\ell)}_s(-\sigma + i\varepsilon)\right]^j \right\}.$$

Explicit expression for $\rho^{(\ell)}_{\text{pert}}(\sigma)$ valid at an arbitrary loop level:

$$\rho^{(\ell)}_{\text{pert}}(\sigma) = \sum_{j=1}^{\ell} d_j \sum_{k=0}^{K(j)} \binom{j}{2k+1} (-1)^k\, \pi^{2k} \left[\sum_{n=1}^{\ell} \sum_{m=0}^{n-1} b^m_n\, u^m_n(\sigma)\right]^{j-2k-1} \left[\sum_{n=1}^{\ell} \sum_{m=0}^{n-1} b^m_n\, v^m_n(\sigma)\right]^{2k+1}$$

■ Nesterenko, Eur. Phys. J. C **77**, 844 (2017).

This makes the higher–loop calculations within Dispersively improved perturbation theory easily accessible. Here ℓ denotes the loop level,

$$u^m_n(\sigma) = \begin{cases} u^0_n(\sigma), & \text{if } m = 0, \\ u^0_n(\sigma) u^m_0(\sigma) - \pi^2 v^0_n(\sigma) v^m_0(\sigma), & \text{if } m \ge 1, \end{cases}$$

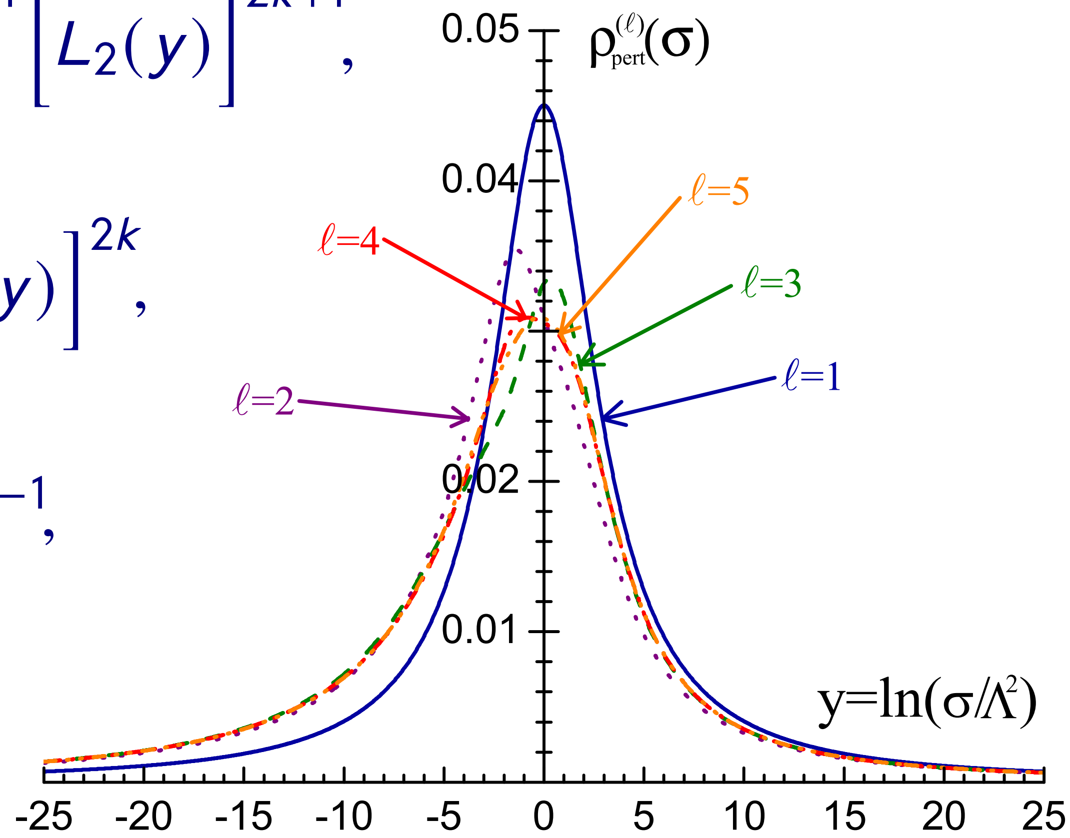
$$v^m_n(\sigma) = \begin{cases} v^0_n(\sigma), & \text{if } m = 0, \\ v^0_n(\sigma) u^m_0(\sigma) + u^0_n(\sigma) v^m_0(\sigma), & \text{if } m \ge 1, \end{cases}$$

$$v_0^m(\sigma) = \sum_{k=0}^{K(m)} \binom{m}{2k+1} (-1)^{k+1} \pi^{2k} \Big[L_1(y)\Big]^{m-2k-1} \Big[L_2(y)\Big]^{2k+1},$$

$$u_0^m(\sigma) = \sum_{k=0}^{K(m+1)} \binom{m}{2k} (-1)^{k} \pi^{2k} \Big[L_1(y)\Big]^{m-2k} \Big[L_2(y)\Big]^{2k},$$

$$v_n^0(\sigma) = \frac{1}{(y^2+\pi^2)^n} \sum_{k=0}^{K(n)} \binom{n}{2k+1} (-1)^{k} \pi^{2k} y^{n-2k-1},$$

$$u_n^0(\sigma) = \frac{1}{(y^2+\pi^2)^n} \sum_{k=0}^{K(n+1)} \binom{n}{2k} (-1)^{k} \pi^{2k} y^{n-2k},$$



$$K(n) = \frac{n-2}{2} + \frac{n \bmod 2}{2}, \qquad \binom{n}{m} = \frac{n!}{m!\,(n-m)!}, \qquad y = \ln\!\left(\frac{\sigma}{\Lambda^2}\right),$$

$L_1(y) = \ln\sqrt{y^2+\pi^2}$, $L_2(y) = (1/2) - \arctan(y/\pi)/\pi$, b_n^m is a combination of the perturbative β function coefficients ($b_1^0 = 1$, $b_2^0 = 0$, $b_2^1 = -\beta_1/\beta_0^2$, etc.), and d_j is a perturbative Adler function coefficient ■ Nesterenko, Eur. Phys. J. C **77**, 844 (2017).

Application of DPT to the muon $g-2$

The DPT provides $\bar{\Pi}(Q^2) = \Delta\Pi(0, -Q^2)$, which contains no unphysical singularities in the entire energy range, that makes it applicable to the muon $g-2$.

The DPT assessment of $a_\mu^{\text{HVP}(2)}$ [direct integration of $\bar{\Pi}(Q^2)$, PDG20 $\alpha_s(M_Z^2)$ as input, no data on $R(s)$ are used] yields

$$a_\mu^{\text{HVP}(2)} = (695.1 \pm 7.6) \times 10^{-10} \quad \text{[4–loop]}$$
$$= (694.9 \pm 7.7) \times 10^{-10} \quad \text{[5–loop]}.$$

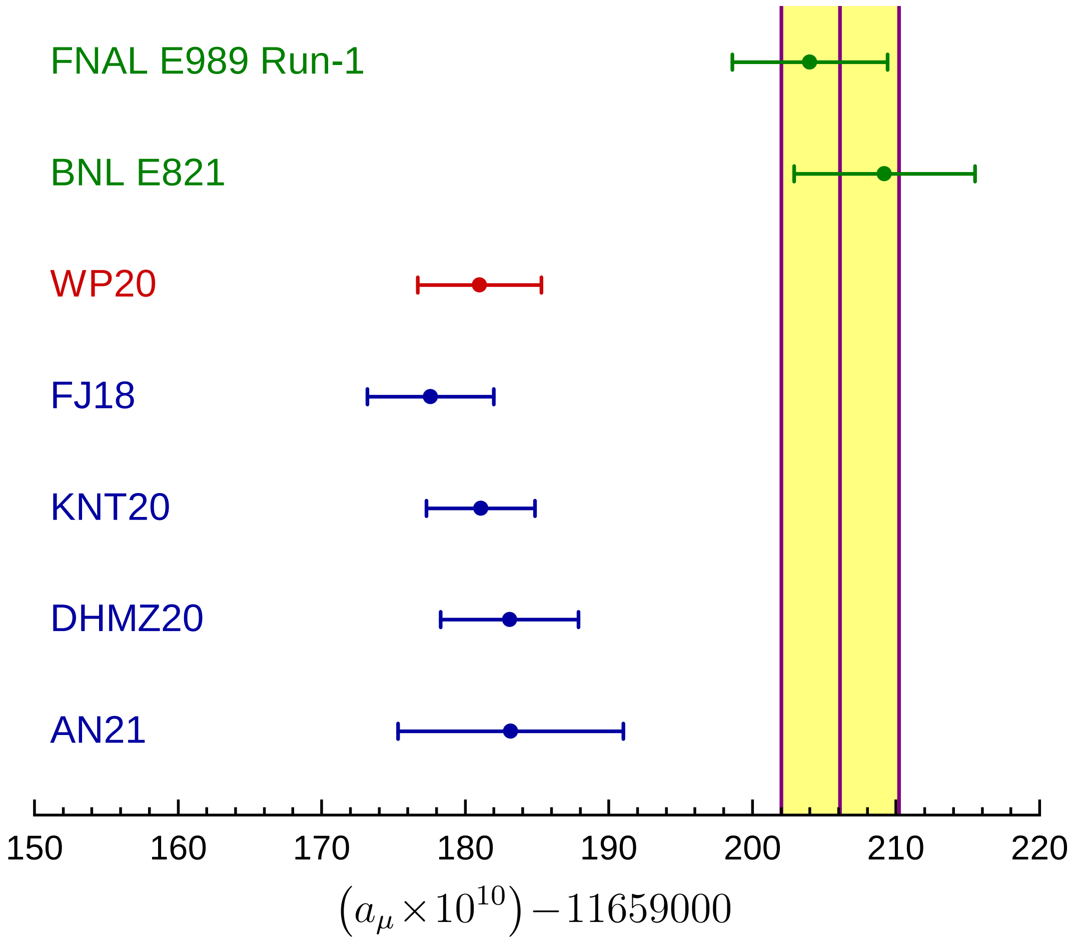
The complete SM prediction

$$a_\mu = (11659183.2 \pm 7.8) \times 10^{-10}$$

differs from a_μ^{exp} by $2.6\,\sigma$

$$a_\mu^{\text{HVP}(2)} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_0^1 (1-x)\bar{\Pi}\left(m_\mu^2 \frac{x^2}{1-x}\right) dx$$

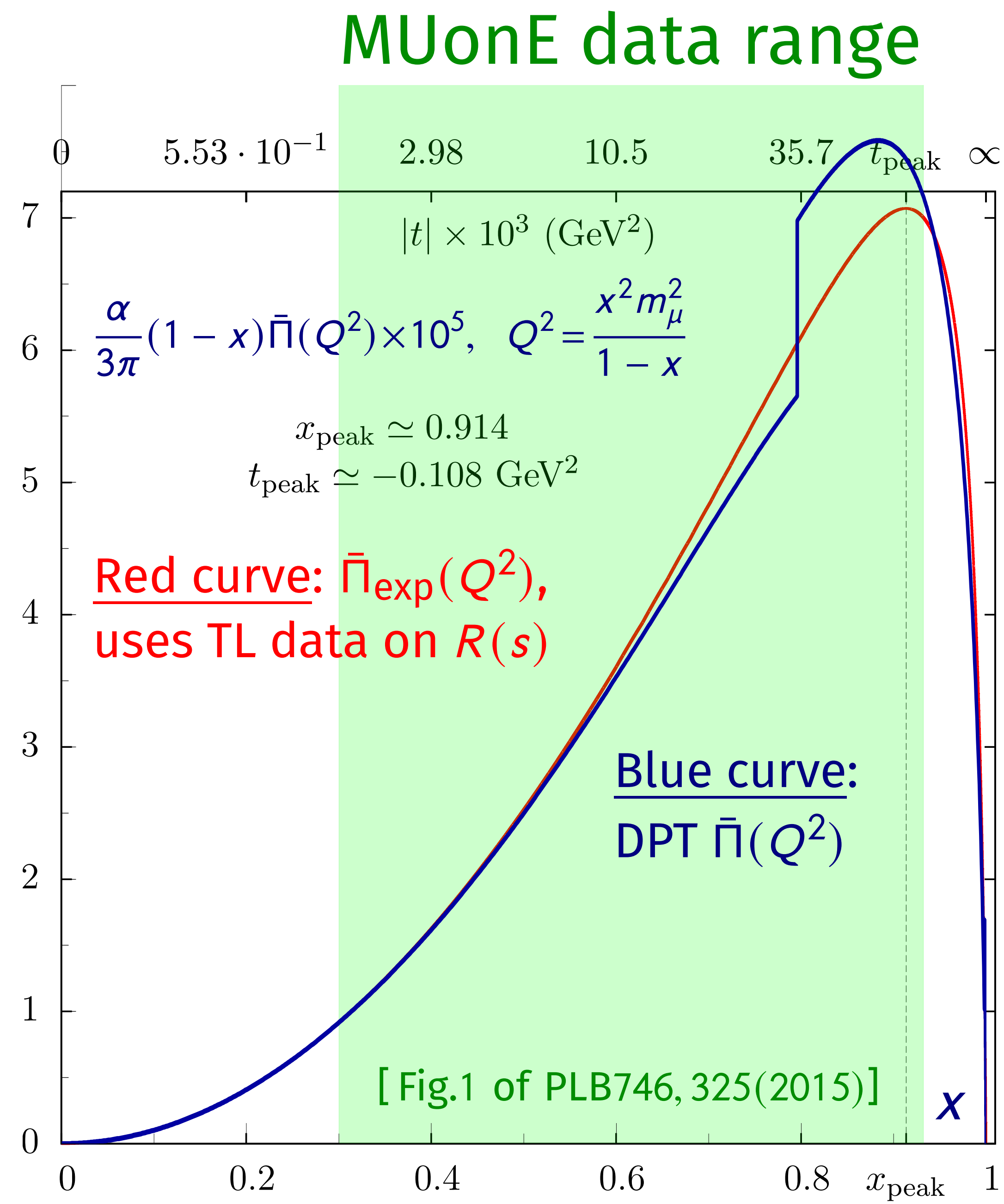
- Lautrup, Peterman, de Rafael, Phys. Rept. **3**, 193 (1972).



- Nesterenko, J. Phys. G **42**, 085004 (2015); Updated in: Kupsc, Venanzoni *et al*, arXiv:2201.12102 [hep-ph].

Application of DPT to the MUonE project

- The spacelike MUonE data on $\bar{\Pi}(Q^2)$ are exceptionally valuable and there is no need in using the fit in the data range.
- The data-driven reconstruction of $\bar{\Pi}(Q^2)$ [red curve] averages out the resonances and quark flavour thresholds of the time-like R–ratio, whereas the latter are also present in the spacelike domain.
- At low energies ($x \lesssim 0.4$) the DPT $\bar{\Pi}(Q^2)$ [blue curve, PDG20 $\alpha_s(M_Z^2)$ is used] can also be employed as a supplementing infrared input for the MUonE project, lattice studies, etc.



SUMMARY

- The complete set of six relations, which mutually express the spacelike and timelike kernel functions for a_{μ}^{HVP} in terms of each other, is obtained.
- It is shown that the infrared limiting value of the spacelike $Q^2 K_D(Q^2)$ and timelike $s K_R(s)$ kernel functions is identical to the corresponding QED contribution to a_{μ} of the preceding order in the electromagnetic coupling.
- By making use of the derived relations the explicit expression for the NLO spacelike kernel function $K_{\Pi}^{(3a)}(Q^2)$ is obtained and the spacelike kernel functions $K_D^{(3a)}(Q^2)$ and $K_D^{(3b)}(Q^2)$ are calculated numerically.
- The obtained results can be employed in the assessments of a_{μ}^{HVP} within the spacelike methods, such as MUonE project, lattice studies, and others.

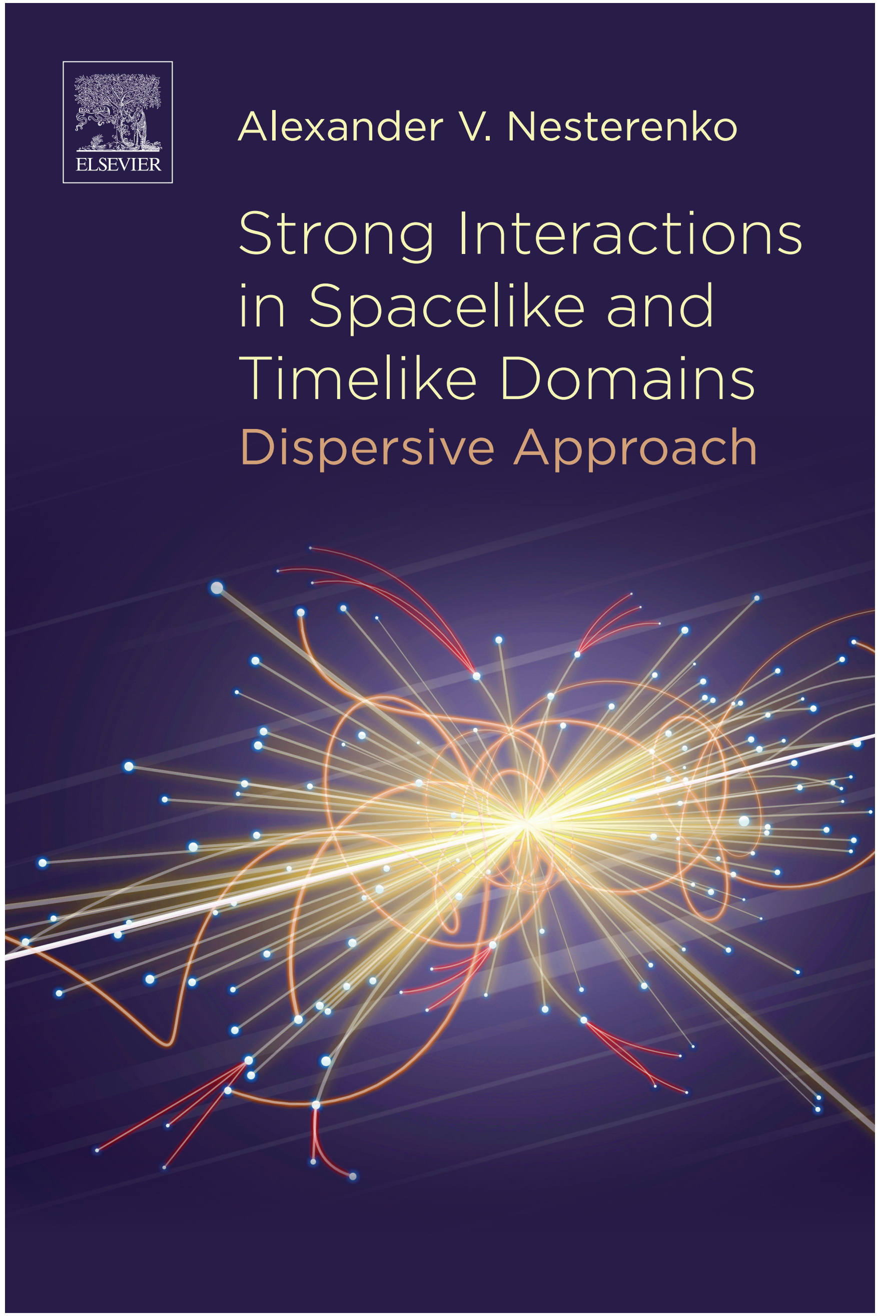
- The integral representations for $\Pi(q^2)$, $R(s)$, and $D(Q^2)$ are obtained in the framework of Dispersively improved perturbation theory (DPT).

- These representations merge, in a self–consistent way, the corresponding perturbative input with intrinsically nonperturbative constraints, which originate in the respective dispersion relations and play a substantial role in the studies of the strong interaction processes at low energies.

- The explicit expression for the perturbative spectral function valid at an arbitrary loop level is obtained, that substantially facilitates the practical calculations within DPT.

- The leading–order HVP contribution to the muon anomalous magnetic moment evaluated within DPT agrees with its recent assessments.

- The DPT hadronic vacuum polarization function can be employed as a supplementing infrared input for the MUonE project and lattice studies.



The detailed discussion of the interplay between hadron dynamics in spacelike and timelike domains, the essentials of the Dispersively improved perturbation theory, and many other closely related topics can be found in

A.V. Nesterenko

Strong interactions in spacelike and timelike domains: Dispersive approach

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