QED and QCD Form Factors At Three Loops

Matteo Fael – CERN

MUonE MITP, 16 Nov. 2022

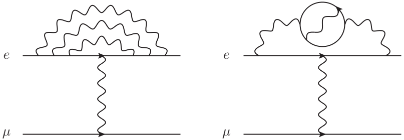
in collaboration with F. Lange, K. Schönwald, M. Steinhauser

MARIE CURIE ACTIONS

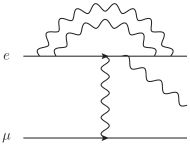
Funded by the European Union

Towards (dominant) N^{3}LO corrections for $\mu e \to \mu e$

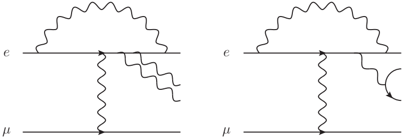
- All virtual (three loops)



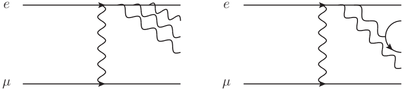
- Single real emission (two loops)



- Double real emission (one loops)



- All real



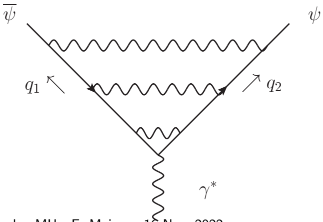
QED and QCD Form Factors

Vector current $j^\mu(x) = \overline{\psi}\gamma^\mu\psi(x)$:

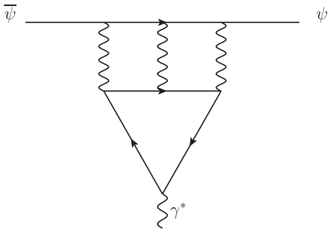
$$\Gamma^\mu(q_1, q_2) = \gamma^\mu F_1(s) - \frac{i}{2m}\sigma^{\mu\nu}q_\nu F_2(s)$$

with $q^\mu = q_1^\mu + q_2^\mu$, $q^2 = s$ and $q_1^2 = q_2^2 = m^2$.

- Non-singlet



- Singlet



Massive vector form factors for QED and QCD

- QED at two loops.
  Mastrolia, Remiddi, Nucl. Phys. B 664 (2003); Bonciani, Mastrolia, Remiddi, Nucl. Phys. B 676 (2004), 399.
- QCD at two loops.
  Bernreuther, Bonciani, Gehrmann, Heinesch, Leineweber, Mastrolia, Remiddi, Nucl. Phys. B 706 (2005), 245-324;
  Gluza, Mitov, Moch, Riemann, JHEP 07 (2009), 001.
  Ahmed, Henn, Steinhauser, JHEP 06 (2017), 125. Ablinger, Behring, Blümlein, Falcioni, De Freitas, Marquard, Rana, Schneider, Phys.Rev. D 97 (2018), 094022.
- QED/QCD at three loop planar (large-N_c limit).
  Henn, Smirnov, Smirnov, Steinhauser, JHEP 01 (2017), 074.
  Ablinger, Blümlein, Marquard, Rana, Schneider, Phys. Lett. B 782 (2018), 528.
- Contribution with massless and massive fermion loops.
  Lee, Smirnov, Smirnov and M. Steinhauser, JHEP 03 (2018), 136.

  Blümlein, Marquard, Rana, Schneider, Nucl. Phys. B 949 (2019), 114751
- **NEW**: non-singlet and n_h-singlet contributions.

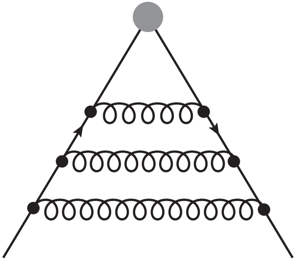
  Lange, MF, Schönwald, Steinhauser, Phys.Rev.Lett. 128 (2022) 17200, PRD 106 (2022) 034029
- **TO DO**: n_l-singlet contributions.
- **ALSO:** 4-loops for massless quarks

  Lee, von Manteuffel, Schabinger, V. Smirnov, A. Smirnov, PRL 128 (2022) 212002

Setup

generate Feynman diagrams ← `qgraf`

Nogueira, J. Comput. Phys. 105 (1993) 279



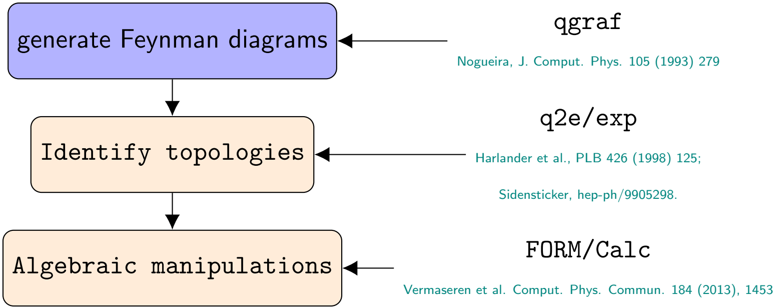
```
diagram              1
pre_factor           (-1)*1

number_propagators   9
number_loops         3
number_legs_in       2
number_legs_out      1

external_leg         q1|1|fq
external_leg         q2|2|fQ
external_leg         q3|3|V

momentum             p1|1,4|fQ,fq
momentum             p2|5,1|g,g
momentum             p3|4,2|fQ,fq
momentum             p4|6,2|g,g
momentum             p5|5,3|fQ,fq
momentum             p6|3,7|fQ,fq
```

Setup


generate Feynman diagrams
Identify topologies
Algebraic manipulations
qgraf
Nogueira, J. Comput. Phys. 105 (1993) 279
q2e/exp
Harlander et al., PLB 426 (1998) 125;
Sidensticker, hep-ph/9905298.
FORM/Calc
Vermaseren et al. Comput. Phys. Commun. 184 (2013), 1453

Projectors

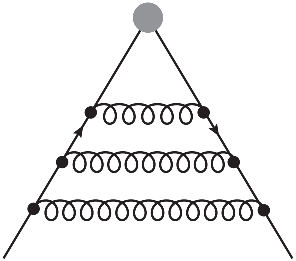
$$\Gamma^\mu(q_1, q_2) = \gamma^\mu F_1(s) - \frac{i}{2m}\sigma^{\mu\nu} q_\nu F_2(s)$$

- Extract form factors with projectors

$$F_i(s) = \mathrm{Tr}\left[(\not{q}_1 + m)\Gamma^\mu(\not{q}_2 - m)\Pi_\mu^{(i)}\right]$$

Integral families

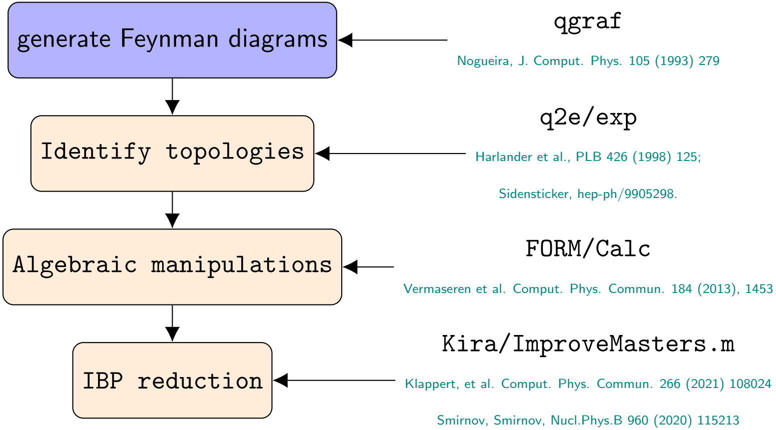
$$I^{(1)}(n_1,\ldots,n_{12}) = \int d^dk_1 d^dk_2 d^dk_3 \frac{N_{10}^{n_{10}}\ldots N_{12}^{n_{12}}}{D_1^{n_1}\ldots D_9^{n_9}}$$



- Three loop momenta k_1, k_2 and k_3.
- Two external momenta q_1 and q_2.
- Express any scalar product $k_i \cdot k_j$ or $q_i \cdot k_j$ as linear combination of $D_1,\ldots,D_9, N_{10},\ldots,N_{12}$.

	non sing	n_h-sing
diagrams	271	66
families	34	17
integrals	302671	106883

Setup



generate Feynman diagrams
Identify topologies
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Nogueira, J. Comput. Phys. 105 (1993) 279
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Sidensticker, hep-ph/9905298.
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Vermaseren et al. Comput. Phys. Commun. 184 (2013), 1453
Kira/ImproveMasters.m
Klappert, et al. Comput. Phys. Commun. 266 (2021) 108024
Smirnov, Smirnov, Nucl.Phys.B 960 (2020) 115213

IBP Reduction

$$I^{(1)}(n_1,\ldots,n_{12}) = \sum_{i=1}^{N} \frac{P_i(s,\epsilon)}{Q_i(s,\epsilon)} f_i^{(1)}$$

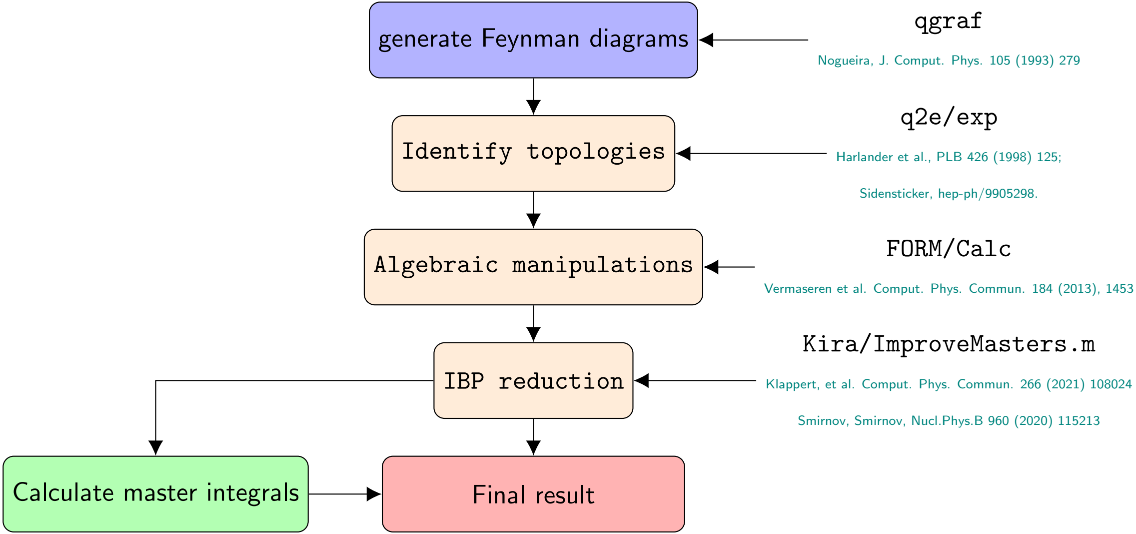
Chetyrkin, Tkachov, Nucl. Phys. B192 (1981) 159
Laporta, Int. J. Mod. Phys. A15 (2000) 5087

- Each integral can be written as linear combination of **master integrals** f_i.
- $P_i(s,\epsilon)$ and $Q_i(s,\epsilon)$ are polynomial in s and $\epsilon = (4-d)/2$.

- We can chose a basis such that ϵ and s in denominators factorizes.
  A. Smirnov, V. Smirnov, Nucl.Phys.B 960 (2020) 115213
  J. Usovitsch, hep-ph/2002.08173.
- `ImproveMasters.m`

	non sing	n_h-sing
diagrams	271	66
families	34	17
integrals	302671	106883
masters	422	316

Setup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Klappert, et al. Comput. Phys. Commun. 266 (2021) 108024
Smirnov, Smirnov, Nucl.Phys.B 960 (2020) 115213
Calculate master integrals
Final result

The method of differential equations

- Differentiate the master integrals w.r.t. $\hat{s} = s/m^2$. Example:

$$\frac{\mathrm{d}}{\mathrm{ds}} f(0,1,1) = \frac{\mathrm{d}}{\mathrm{ds}} \int d^d k \frac{1}{k^2[(k+q_1-q_2)^2 - m^2]}$$
$$= \frac{f(-1,2,1)}{s-4} - \frac{f(0,1,1)}{s-4} + \frac{2f(0,1,1)}{(s-4)s} - \frac{2f(0,2,0)}{(s-4)s}$$
$$\stackrel{IBP}{=} \frac{(ds-4s+4)f(0,1,1)}{2(s-4)s} - \frac{(d-2)f(0,0,1)}{(s-4)s}$$

- The derivative is rewritten as linear combination of the master integrals via IBP relations

The method of differential equations

$$
\begin{aligned}
\frac{\mathrm{d}f_1}{\mathrm{d}s} &= r_{1,1}(s,\epsilon)f_1(s,\epsilon) \qquad + \cdots + r_{1,422}(s,\epsilon)f_{422}(s,\epsilon) \\
&\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\
\frac{\mathrm{d}f_{422}}{\mathrm{d}s} &= r_{422,1}(s,\epsilon)f_1(s,\epsilon) \qquad + \cdots + r_{422,422}(s,\epsilon)f_{422}(s,\epsilon)
\end{aligned}
$$

The Canonical Way

$$\frac{\mathrm{d}\vec{f}}{\mathrm{d}s} = \epsilon\,\hat{R}(s)\,\vec{f}(s,\epsilon)$$

Henn, Phys.Rev.Lett. 110 (2013) 251601

- Automate the construction of the solution.
- Method can be applied to the form factors at 1- and 2-loops.

- Landau variable

$$\frac{s}{m^2} = -\frac{(1-x)^2}{x}$$

- Result written in terms of special functions, e.g. Harmonic Polylogarithms, GPL

$$H(1,0,-1;x) = \int_0^x \frac{dt_1}{1-t_1}\int_0^{t_1}\frac{dt_2}{t_2}\int_0^{t_2}\frac{dt_3}{1+t_3}$$

Semi-analytic solution of one-parameter integrals

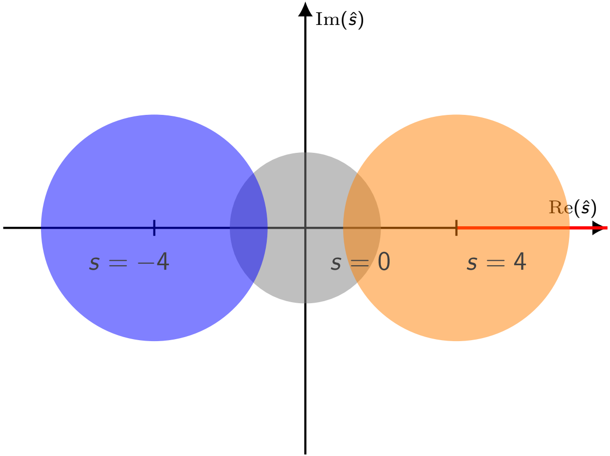
- One dimensionless parameter $\hat{s} = s/m^2$.
- Boundary condition from simple kinematic limit

$$s \to 0$$

- Perform an analytic continuation

  MF, Lange, Schönwald, Steinhauser, JHEP 09 (2021) 152

- Other approaches
  - `SYS`
    Laporta, Int.J.Mod.Phys.A 15 (2000) 5087.
  - `DESS`
    Lee, Smirnov, Smirnov, JHEP 03 (2018) 008.
  - `DiffExp`
    Hidding, Comput.Phys.Commun. 269 (2021) 108125.



Step I: Expansion Arount $s = 0$

- Taylor expansion around $s = 0$:

$$f_n(s,\epsilon) = \sum_{i=-3}^{n_\epsilon} \sum_{j=0}^{50} c_{ij}^{(n)} \epsilon^i s^j$$

- Insert the Ansatz in the differential equations

$$\frac{\mathrm{d}\vec{f}}{\mathrm{d}s} = R(s,\epsilon)\,\vec{f}(s,\epsilon) \qquad \longrightarrow \qquad A\,\vec{c} = \vec{b}$$

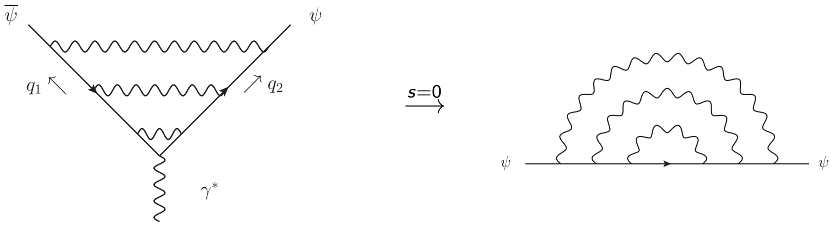
- Order by order in ϵ and s we obtain linear relations for $c_{ij}^{(n)}$.
- Solve $O(10^6)$ equations with `Kira` and `FireFly`.

  Klappert, et al. Comput. Phys. Commun. 266 (2021) 108024

- The solution depends on a minimal set of undetermined coefficients, about $O(10^3)$.

Boundary conditions

- Well known analytic results for massive two-point functions:
  Melnikov, van Ritbergen, Phys.Lett.B 482 (2000) 99;
  Lee, Smirnov, JHEP 02 (2011) 102;



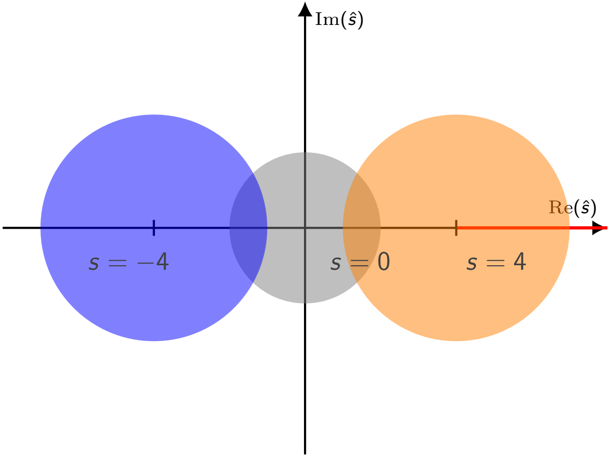
- However: we need higher orders in ϵ, integrals up to weight 9.
- Use SummerTime.m and PSLQ.

  Lee, Mingulov, Comput.Phys.Commun. 203 (2016) 255

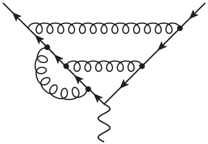
- The n_h-singlet needs a proper asymptotic expansion around $s = 0$. Use Asy.m.

  Janzten, Smirnov, Smirnov, EPJ.C 72 (2012) 2139

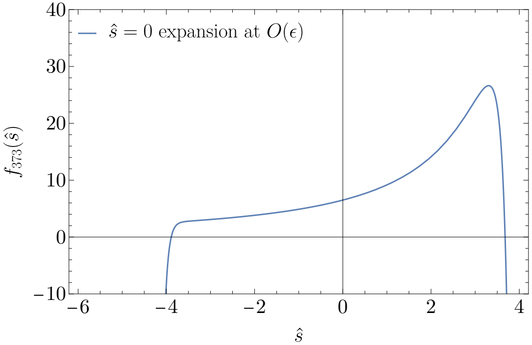
Expand and match

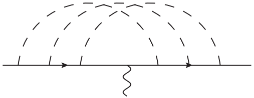


- Build new expansion around new point $s = s_0$.
- Fix unknown constants using the previous expansion at an intermediate point.
- Singular points:
  - $s = 4$: two-particle threshold
  - $s = \infty$: high energy limit
  - $s = 16$: four-particle threshold (NEW!)



Example



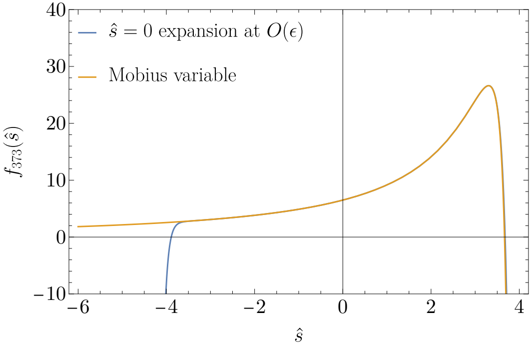


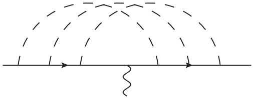
- Improve convergence
- Möbius transformation

$$s \to \frac{8w}{1+w}$$

- Singularities are equally distant in w plane $s = \{-\infty, 0, 4\} \to w = \{-1, 0, 1\}$.

Example





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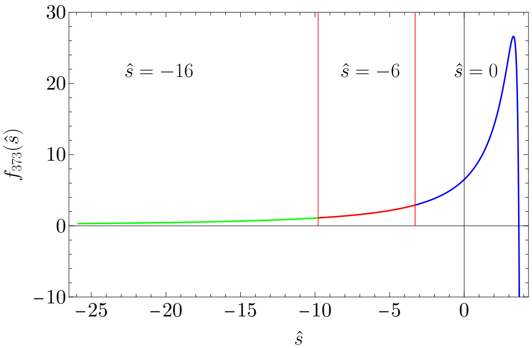
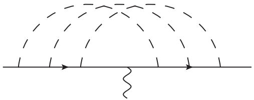
Expand and Match





- Taylor expansion around $s = -16$

$$f_n(s,\epsilon) = \sum_{i=-3}^{n_\epsilon} \sum_{j=0}^{50} c_{ij}^{(n)} \epsilon^i (s+6)^j$$

- Solve linear system
- Use $s = 0$ expansion to fix the boundary values.
- Iterate the procedure until all values of s are covered.

Special Expansion Points

- $s = 4$: two-particle threshold.

$$f_n(s,\epsilon) = \sum_{i=-3}^{\infty} \sum_{j=-s_{\min}}^{50} \sum_{k=0}^{i+3} c_{ijk}^{(n)} \epsilon^i \left[\sqrt{4-s}\right]^j \log^k(4-s)$$

- $s = 16$: four-particle threshold.

$$f_n(s,\epsilon) = \sum_{i=-3}^{\infty} \sum_{j=-s_{\min}}^{50} \sum_{k=0}^{i+3} c_{ijk}^{(n)} \epsilon^i \left[\sqrt{16-s}\right]^j \log^k(16-s)$$

- $s = \infty$: high-energy/massless limit.

$$f_n(s,\epsilon) = \sum_{i=-3}^{\infty} \sum_{j=-s_{\min}}^{50} \sum_{k=0}^{i+6} c_{ijk}^{(n)} \left(\frac{1}{s}\right)^j \log^k(-s)$$

Renormalization and IR subtraction

- UV renormalization in the on-shell scheme. Mass and wave function renormalization.
  Melnikov, van Ritbergen, Phys.Lett.B 482 (2000) 99;
  Chetyrkin, Steinhauser, Nucl.Phys.B 573 (2000) 617-651

- Structure of IR poles is universal

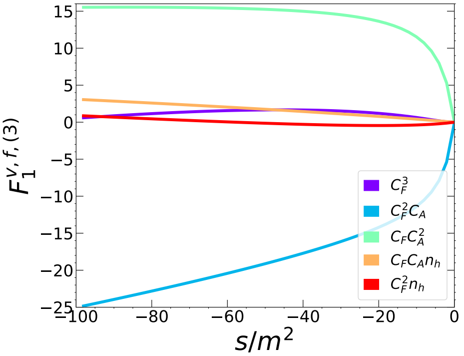
$$F_i^{\mathrm{UV\,ren}}(s) = Z_{\mathrm{IR}} F_i^{\mathrm{f}}(s)$$

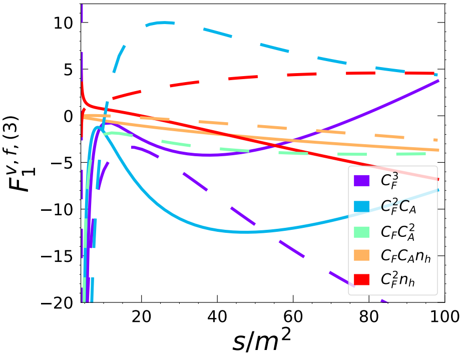
with Z_{IR} given in term of the $\Gamma \equiv \Gamma_{\mathrm{cusp}}$ anomalous dimension

$$\log Z_{\mathrm{IR}} = -\frac{1}{2\epsilon}\left(\frac{\alpha_s}{\pi}\right)\Gamma^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2\left[\frac{\beta_0\Gamma^{(1)}}{16\epsilon^2} - \frac{\Gamma^{(2)}}{4\epsilon}\right]$$
$$+ \left(\frac{\alpha_s}{\pi}\right)^3\left[-\frac{\beta_0^2\Gamma^{(1)}}{96\epsilon^3} + \frac{\beta_1\Gamma^{(1)} + 4\beta_0\Gamma^{(2)}}{96\epsilon^2} - \frac{\Gamma^{(3)}}{6\epsilon}\right] + \dots,$$

Grozin, Henn, Korchemsky, Marquard, Phys.Rev.Lett. 114 (2015) 6, 062006; JHEP 01 (2016) 140.

Results in QCD



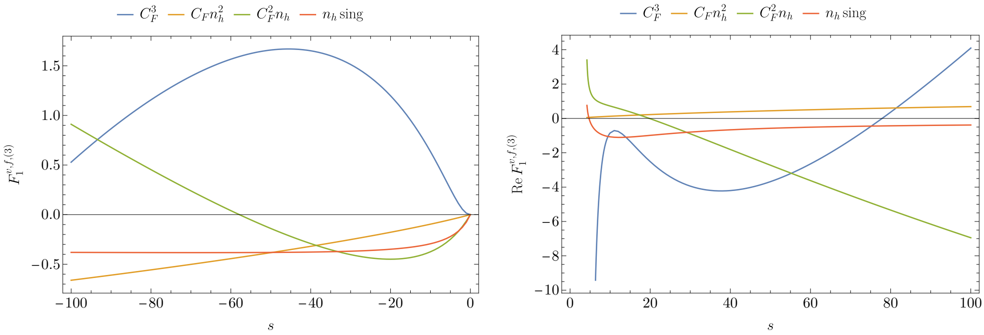


Lange, MF, Schönwald, Steinhauser, Phys.Rev.Lett. 128 (2022) 172003, PRD 106 (2022) 034029.

- Expansion points:

$$\hat{s} \in \{-\infty, -32, -28, -24, -16, -12, -8, -4, 0, 1, 2, 5/2, 3, 7/2, 4, 9/2, 5, 6, 7, 8, 10, 12, 14, 15, 16, 17, 19, 22, 28, 40\}$$

Results for QED



Lange, MF, Schönwald, Steinhauser, Phys.Rev.Lett. 128 (2022) 172003, PRD 106 (2022) 034029.

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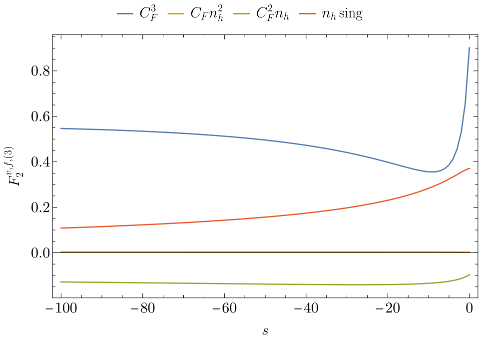

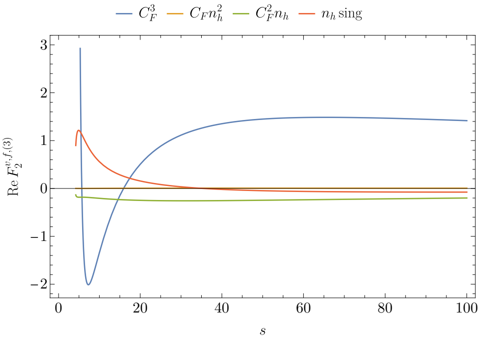

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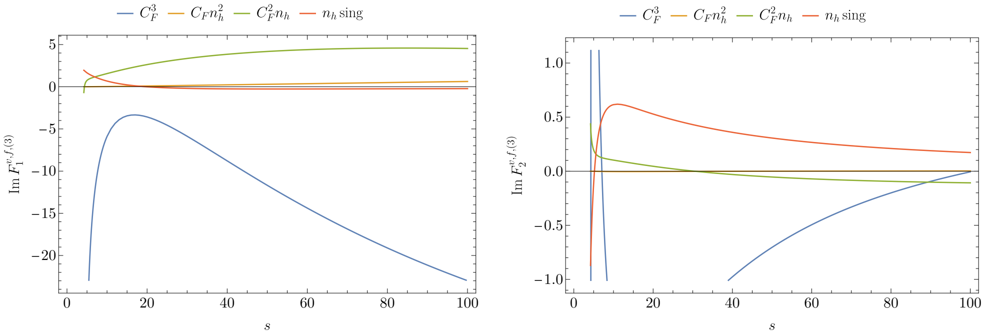


Lange, MF, Schönwald, Steinhauser, Phys.Rev.Lett. 128 (2022) 172003, PRD 106 (2022) 034029.

Towards N^{3}LO Monte Carlo

Experiment	process	F^ℓ	$\sqrt{s}$ [GeV]	Range	
MUonE	$\mu e \to \mu e$	e	0.405	$t \in [0.14, -10^{-3}]\,\mathrm{GeV}^2$	$t/m_e^2 \in [-5 \times 10^5, -4 \times 10^3]$
		μ	0.405	$t \in [0.14, -10^{-3}]\,\mathrm{GeV}^2$	$t/m_\mu^2 \in [-12.5, -0.1]$
BESIII	$ee \to \ell\ell$	e	~ 4 GeV	$s = 16$ GeV^2	$s/m_e^2 \simeq 6 \times 10^7$
		μ	~ 4 GeV	$s = 16$ GeV^2	$s/m_\mu^2 \simeq 1400$
		τ	~ 4 GeV	$s = 16$ GeV^2	$s/m_\tau^2 \simeq 5$
P2	$ep \to ep$	e	1.08 GeV	$t \in [-13, -5] \times 10^{-3}\,\mathrm{GeV}^2$	$t/m_e^2 \in [-5, -2] \times 10^4$

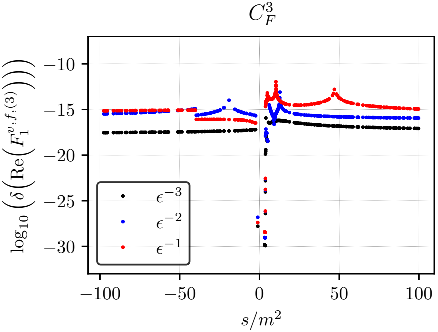
Estimation of the accuracy

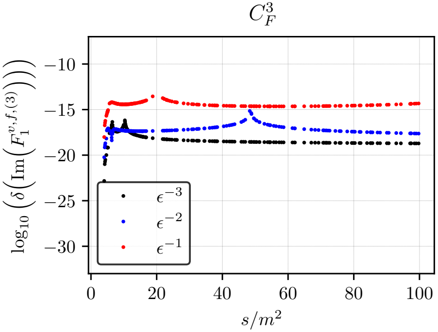
- Exact bare amplitude up to α_s^2 with HPLs (ginac). Exact UV and IR counter terms.
- We estimate the precision from the numerical pole cancellations of the renormalized and infrared-subtracted form factor:

$$\delta\left(F^{f,(3)}\Big|_{\epsilon^i}\right) = \frac{F^{(3)}\Big|_{\epsilon^i} + F^{(\mathrm{CT}+Z)}\Big|_{\epsilon^i}}{F^{(\mathrm{CT}+Z)}\Big|_{\epsilon^i}}$$

- Close to the zeros of the form factors, the relative precision deteriorates.
- Nominal precision by removing all points whose coefficient size is smaller than 2.5 % of the average in each range
  - $s/m^2 < 0$
  - $0 < s/m^2 < 3.95$
  - $4.05 < s/m^2 < 16$
  - $s/m^2 > 16$
- n_h singlet is finite.

Pole cancellation in QED





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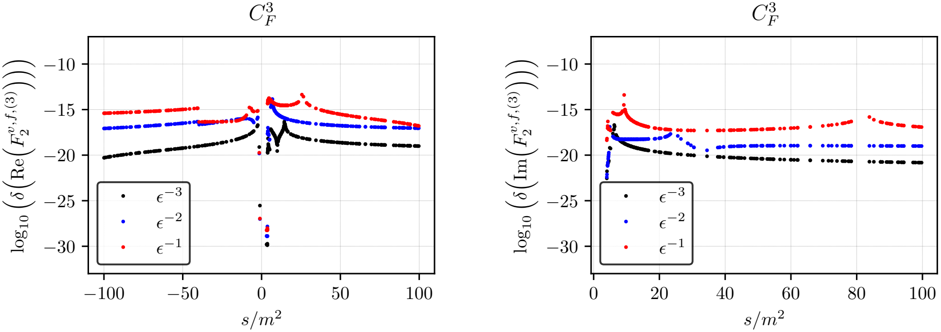
Pole cancellation in QED



Lange, MF, Schönwald, Steinhauser, Phys.Rev.Lett. 128 (2022) 172003, PRD 106 (2022) 034029.

Fortran Library

- `https://gitlab.com/formfactors3l/FF3l`
- What is implemented:
  - UV renormalized form factors (but not IR subtracted).
  - Non-singlet and n_h singlet.
  - QCD and QED, α_s and α in $\overline{\mathrm{MS}}$.
  - Chebyshev interpolation grids for $-40 < s/m^2 < 3.75$ and $4.25 < s/m^2 < 60$.
  - Series expansions around $s = \pm\infty$ and $s = 4$ (also $s = 0$ for n_h singlet).
- Mathematica interface.
- 10^6 random points, single core, for F_1 and F_2 from $1/\epsilon^3$ to ϵ^0.
  - QED: 8s
  - QCD: 21s

```
program example1
  use ff3l
  implicit none
  double complex :: ff
  double precision :: s = 10
  integer :: eporder

  do eporder = -3,0
    ff = ff3l_veF1_qed(s,eporder)
    print *,"F1( s = ",s,", ep = ",eporder," ) = ", ff
  enddo

end program example1
```

```
F1( s = 10. , ep =  -3 ) =  (-0.71409073727949590,0.45217590894700582)
F1( s = 10. , ep =  -2 ) =  (-3.0657921999410798,-3.7071910425182888)
F1( s = 10. , ep =  -1 ) =  (-1.1430546715428169,-0.61128365210649505)
F1( s = 10. , ep =  0 ) =  (-32.761925339732414,-6.1756393171639195)
```

Options

- Switch on/off non-singlet and n_h singlet

```
call ff3l_nonsinglet_on()
call ff3l_nonsinglet_off()
call ff3l_nhsinglet_on()
call ff3l_nhsinglet_off()
```

- Set the values for n_l and n_h

```
call ff3l_set_nl(nl)
call ff3l_set_nh(nh)
```

Mathematica interface

```
In[] := Install["PATH/ff3l"]

In[] := s = 10;
In[] := eporder = 0;
In[] := FF3lveF1[s,eporder]
Out[] := 221.338 - 264.829 I
```

- Switch on/off non-singlet and n_h singlet

```
In[] := FF3lNonSingletOff[]
In[] := FF3lNonSingletOn[]
In[] := FF3lNhSingletOff[]
In[] := FF3lNhSingletOn[]
```

- Set the values for n_l and n_h

```
In[] := FF3lSetNl[nl]
In[] := FF3lSetNh[nh]
```

Conclusions

- Non-singlet and n_h-singlet contributions to the QED/QCD massive form factors at N^3LO.
- Currents: vector, axial-vector, scalar, pseudoscalar.
- Bare form factors are determined as expansions around certain regular and singular kinematic points.
- We provide a Fortran and Mathematica package with interpolation grids and expansions.
- Precision (QED), vector form factors: at least 10 digits.