NNLO QED Virtual corrections for e-mu elastic scattering

(and related processes)

Jonathan Ronca

In collaboration with:
Bonciani, Broggio, Di Vita, Ferroglia, Laporta, Mastrolia, Mandal, Mattiazzi, Passera, Primo, Schubert, Torres Bobadilla, and Tramontano

The Evaluation of the Leading Hadronic Contribution to the Muon g-2: Toward the MUonE Experiment

14 – 18 November 2022

20 Sept. 2022

Motivation :: Why NNLO QED?

Aim: extraction of HVP from mu-e elastic scattering data

[On the experimental status:
Umberto Marconi, Giovanni Abbiendi and Riccardo Pilato's talk]

To extract $\Delta\alpha_{had}(t)$ from MUonE's measurement, the ratio of the SM cross sections in the signal and normalisation regions must be known at ≲ 10ppm! [M. Passera, EPFL 2017]

Precision goal: **10ppm**

[On the theory status: Fulvio Piccinini's talk
+ Ettore Budassi and Clara Lavinia del Pio's tlks]

[Carloni Calame, Passera, Trentadue, Venanzoni (2015)]
[Abbiendi, Carloni Calame, Marconi, Matteuzzi, Montagna, Nicrosini, Passera, Piccinini, Tenchini, Trentadue, Venanzoni (2017)]

Theory for muon-electron scattering @ 10 ppm★
A report of the MUonE theory initiative

P. Banerjee[1], C. M. Carloni Calame[2], M. Chiesa[3], S. Di Vita[4], T. Engel[1,5], M. Fael[6], S. Laporta[7,8], P. Mastrolia[7,8], G. Montagna[9,2], O. Nicrosini[2], G. Ossola[10], M. Passera[8], F. Piccinini[2], A. Primo[5], J. Ronca[11], A. Signer[a,1,5], W. J. Torres Bobadilla[11], L. Trentadue[12,13], Y. Ulrich[a,1,5], G. Venanzoni[14]

Large **QED Background** $\sigma^{(n)}_{\text{QED}} \sim 10^{-5}$ ⇒ **Full** NNLO + **dominant** N^{3}LO

This talk + Tomorrow: Yannick Ulrich, Tim Engel, Marco Rocco

Tomorrow: Matteo Fael, Yannick Ulrich

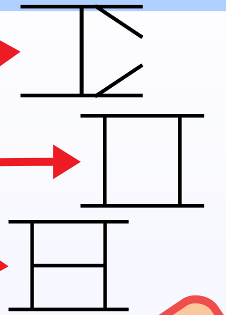
Cross Section and Scattering Amplitudes in pQFT

Target observable: **cross section**

$$\sigma(2\to 2) = \alpha^2 \left[\sum_{j=0}^{n} \left(\frac{\alpha}{\pi}\right)^j \sigma_{\mathrm{N^jLO}} + O(\alpha^{n+1}) \right]$$

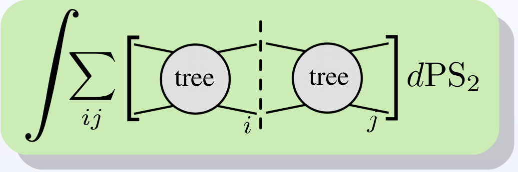
related to the **scattering amplitude**

$$\mathcal{A} = 4\pi\mu^{-2\epsilon} \sum_j \left(\frac{\alpha}{\pi}\right)^j \mathcal{A}_j$$

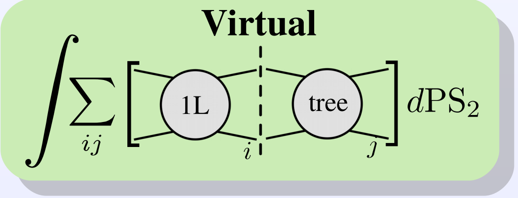


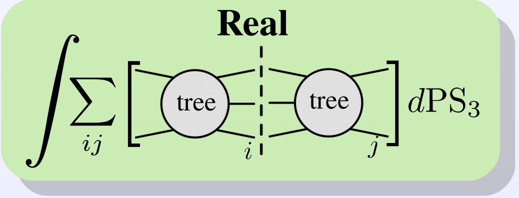
Feynman diagrams

σ_{LO}



σ_{NLO}





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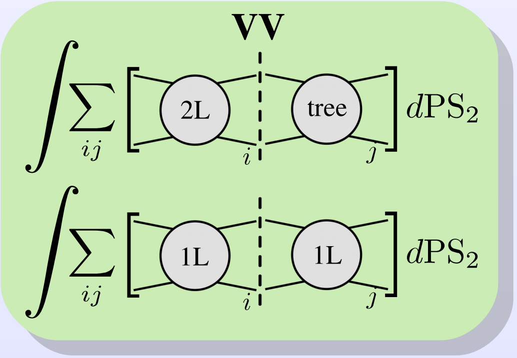

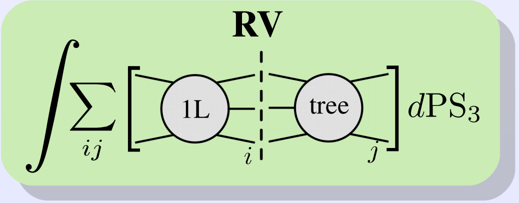

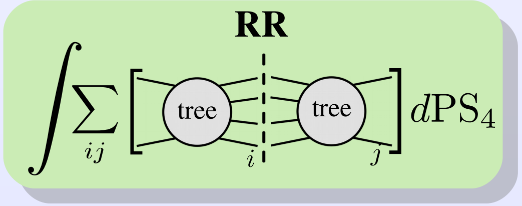

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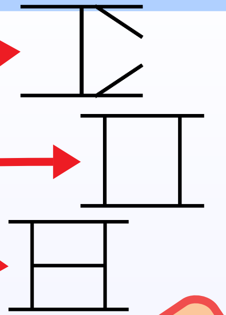

Feynman diagrams

Under control

[Alacevich, Carloni Calame, Chiesa, Montagna, Nicrosini, Piccinini (2019)]

σ_{LO} ⇒ $\int \sum_{ij} \left[\text{tree}_i \,\big|\, \text{tree}_j \right] dPS_2$

σ_{NLO} ⇒

Virtual: $\int \sum_{ij} \left[\text{1L}_i \,\big|\, \text{tree}_j \right] dPS_2$

Real: $\int \sum_{ij} \left[\text{tree}_i \,\big|\, \text{tree}_j \right] dPS_3$

σ_{NNLO} ⇒

VV: $\int \sum_{ij} \left[\text{2L}_i \,\big|\, \text{tree}_j \right] dPS_2$, $\int \sum_{ij} \left[\text{1L}_i \,\big|\, \text{1L}_j \right] dPS_2$

RV: $\int \sum_{ij} \left[\text{1L}_i \,\big|\, \text{tree}_j \right] dPS_3$

RR: $\int \sum_{ij} \left[\text{tree}_i \,\big|\, \text{tree}_j \right] dPS_4$

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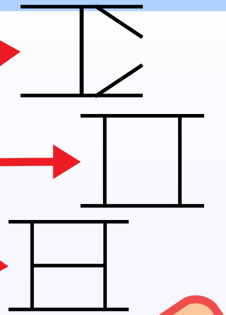
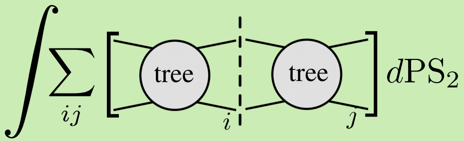


Feynman diagrams

σ_{LO}



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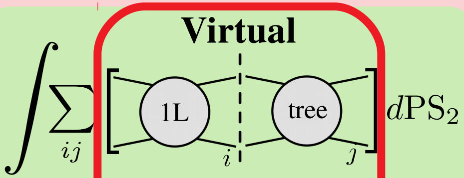
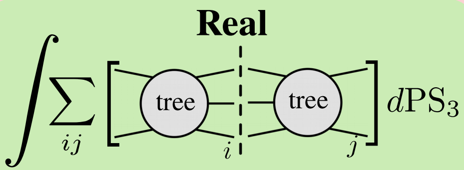
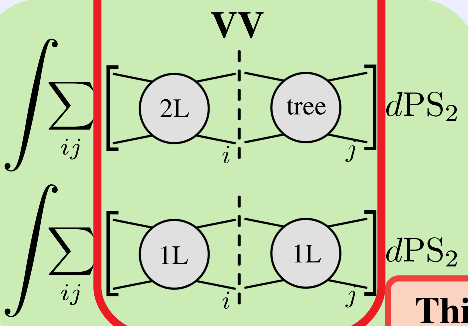
[Alacevich, Carloni Calame, Chiesa, Montagna, Nicrosini, Piccinini (2019)]

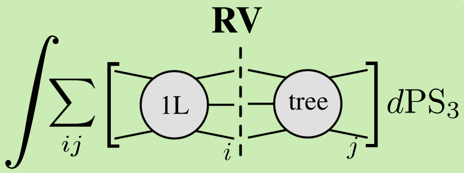
σ_{NLO}

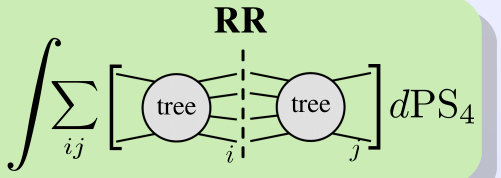




σ_{NNLO}







This talk

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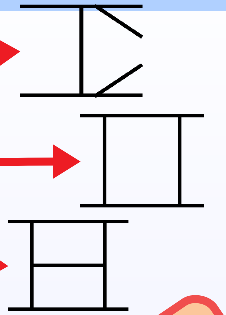


Feynman diagrams

σ_{LO} ⇒ $\int \sum_{ij}$ [tree ¦ tree] dPS_2

Under control

[Alacevich, Carloni Calame, Chiesa, Montagna, Nicrosini, Piccinini (2019)]

σ_{NLO} ⇒

Virtual: $\int \sum_{ij}$ [1L ¦ tree] dPS_2

Real: $\int \sum_{ij}$ [tree ¦ tree] dPS_3

[Tomorrow talks]

McMule

σ_{NNLO} ⇒

VV: $\int \sum_{ij}$ [2L ¦ tree] dPS_2 + $\int \sum_{ij}$ [1L ¦ 1L] dPS_2

RV: $\int \sum_{ij}$ [1L ¦ tree] dPS_3

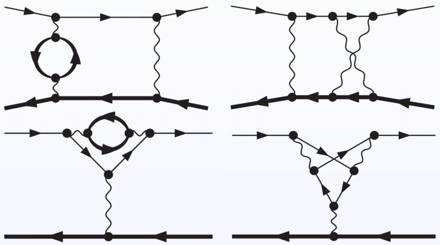
RR: $\int \sum_{ij}$ [tree ¦ tree] dPS_4

This talk

[Carloni Calame, M. Chiesa, Hasan, Montagna, Nicrosini, F. Piccinini (2020)]

Anatomy of $e^- \mu^+ \to e^- \mu^+$ two-loop amplitude

$$\mathcal{M}^{(2)} = \overline{\sum} 2\mathrm{Re}[\mathcal{A}^{(0)*}\mathcal{A}^{(n)}]$$
$$= (S_\epsilon)^2 \int d^d k_1 d^d k_2 \sum_G \frac{\mathcal{N}_G(\mathbf{k},\mathbf{l})}{\prod_{\sigma \in G} D_\sigma(\mathbf{k},\mathbf{l})}$$

 $\in G$

Two-loop Feynman integrals

- 4-point kinematics
- 4 mass-scales variables
  - 2 Mandelstam s, t
  - 2 masses m_e, m_μ

Observation: $\frac{m_e}{m_\mu} \simeq 10^{-5}$

$m_e = 0$
$m_\mu = M$

Simplification of the Dirac trace algebra

Opens the possibility of an *analytical calculation*

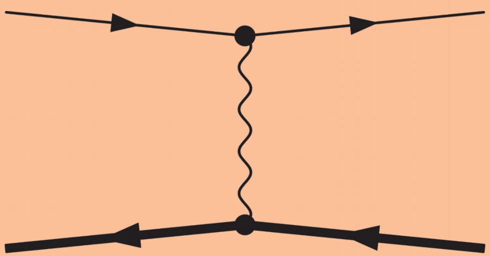
Electron can be "**massified**" afterwards

[Tim Engel's talk]

Feynman integrals are (in general) UV and IR divergent

Using **Dimensional Regularization**: space-time is treated as a free parameter $d = 4 - 2\epsilon$

Crossing: $e^- \mu^+ \to e^- \mu^+$ vs. $e^- e^+ \to \mu^- \mu^+$



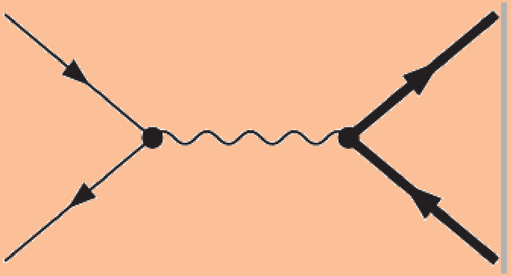
$$e^-(p_1)\,\mu^+(p_2) \to e^-(p_3)\mu^+(p_4)$$

$$p_1^2 = p_3^2 = 0$$
$$p_2^2 = p_4^2 = M^2$$
$$s = (p_1 + p_2)^2$$
$$t = (p_1 - p_3)^2$$
$$u = (p_2 - p_3)^2$$
$$s + t + u = M^2$$

$$\mathcal{M}^{(0)} = \frac{4(s - M^2)^2 + 4st + (d-2)t^2}{t^2}$$

Crossing

$s \to t$
$t \to s$
$u \to u$



$$e^-(p_1)\,e^+(p_2) \to \mu^-(p_3)\mu^+(p_4)$$

$$p_1^2 = p_2^2 = 0$$
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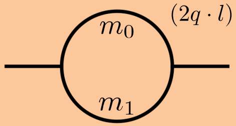
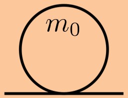
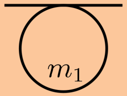
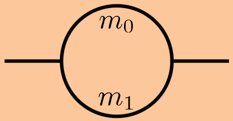
$$\mathcal{M}^{(0)} = \frac{4(t - M^2)^2 + 4st + (d-2)s^2}{s^2}$$

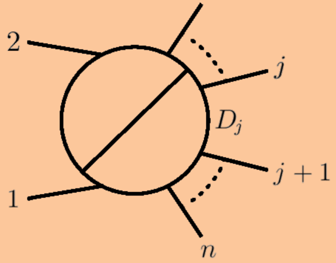
From now on, we consider the cross-related **di-muon production**

Integrand decomposition of Feynman Integrals

$$\mathcal{I}(l) = \int_k \frac{\mathcal{N}(k,l)}{D_1(k,l)\cdots D_n(k,l)}$$

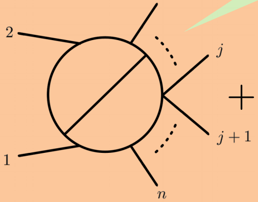
@1L  $=$  $-$  $- \; (l^2 - m_1^2 + m_0^2)$ 

[Ossola, Papadopoulos, PiSau (2006)]
[Ellis, Giele, Kunszt, Melnikov (2007)]
[Mastrolia, Ossola, Papadopoulos, PiSau (2008)]



$$= \frac{\mathcal{N}(k,l)}{D_1 \cdots D_j \cdots D_n}$$

Polynomial division modulo Gröbner basis

$$= \sum_{j=1}^{n}$$  $$+ \frac{\Delta_{\hat{j}}(k,l)}{D_1 \cdots D_j \cdots D_n}$$

Iterating...

$$\vdots$$

$$= \sum_{j=1}^{n} \sum_{i_1 \cdots i_j}^{n} \frac{\Delta_{i_1 \cdots i_j}(k,l)}{D_{i_1} \cdots D_{i_j}}$$

Contain Irreducible Scalar Products

- **Generalizable at n-loop**
- **Works with helicity amplitudes**
- **Possible automation**

[Mastrolia, Ossola (2011)]
[Zhang (2012-2016)]
[Badger, Frellesvig, Zhang (2012-2013)]
[Mastrolia, Mirabella, Ossola, Peraro (2012)]

Adaptive Integrand Decomposition (AID)

$$\mathcal{I}(l) = \int_k \frac{\mathcal{N}(k,l)}{D_1(k,l)\cdots D_n(k,l)}$$

[Collins (1984)]
[van Neerven and Vermaseren (1984)]
[Kreimer (1992)]

Idea

$$d = d_{||} + d_{\perp}$$
$$k = k_{||} + k_{\perp}$$

$\Rightarrow$

$$k_j^{\mu} = k_{||j}^{\mu} + \lambda_j^{\mu}$$
$$k_i \cdot k_j = k_{||i} \cdot k_{||j} + \lambda_{ij}$$

Enforcing the polynomial division: $k_i \cdot l_j = \sum_{m=1}^{n} c_m D_m(k,l) + \text{ISPs}$

$$\mathcal{I}(l) = \int d^{d_{||}} k_{||} d^{d_{\perp}} k_{\perp} \frac{\mathcal{N}(k,l)}{D_1(k,l)\cdots D_n(k,l)}$$
$$= \int d^{d_{||}} k_{||} \int \prod_{ij} G(\lambda_{ij}) d\lambda_{ij} k_{\perp} d\Theta_{\perp} \frac{\mathcal{N}(k_{||},\lambda_{ij},\Theta_{\perp})}{D_1(k_{||},\lambda_{ij})\cdots D_n(k_{||},\lambda_{ij})}$$

Transverse direction can be integrated out

$$\int d\Theta_{\perp} = \int_{-1}^{1} \prod_{i=1}^{4-d_{||}} \prod_{j=1}^{L} d\cos\theta_{i+j-1,j} (\sin\theta_{i+j-1,j})^{d_{\perp}-i-j-1}$$

$\Rightarrow$

$$\int_{-1}^{1} dx (1-x^2)^{\alpha-\frac{1}{2}} C_n^{(\alpha)}(x) C_m^{(\alpha)}(x) = \delta_{nm} \frac{2^{1-2\alpha}\pi\Gamma(n+2\alpha)}{n!(n+\alpha)\Gamma^2(\alpha)}$$

Gegenbauer polynomials

[Mastrolia, Peraro, Primo (2016)]
[Mastrolia, Peraro, Primo, Torres Bobadilla (2016)]

Divide, Integrate, Divide again

$$\int dk \frac{\mathcal{N}(k,l)}{D_1(k,l)\cdots D_n(k,l)}$$

Divide

$$\sum_{j=1}^{n}\sum_{i_1\cdots i_j}^{n}\int dk_{||}d\lambda d\Theta_\perp \frac{\Delta_{i_1\dots i_j}(k_{||},\lambda,\Theta_\perp)}{D_{i_1}(k_{||},\lambda)\cdots D_{i_j}(k_{||},\lambda)}$$

Integrate

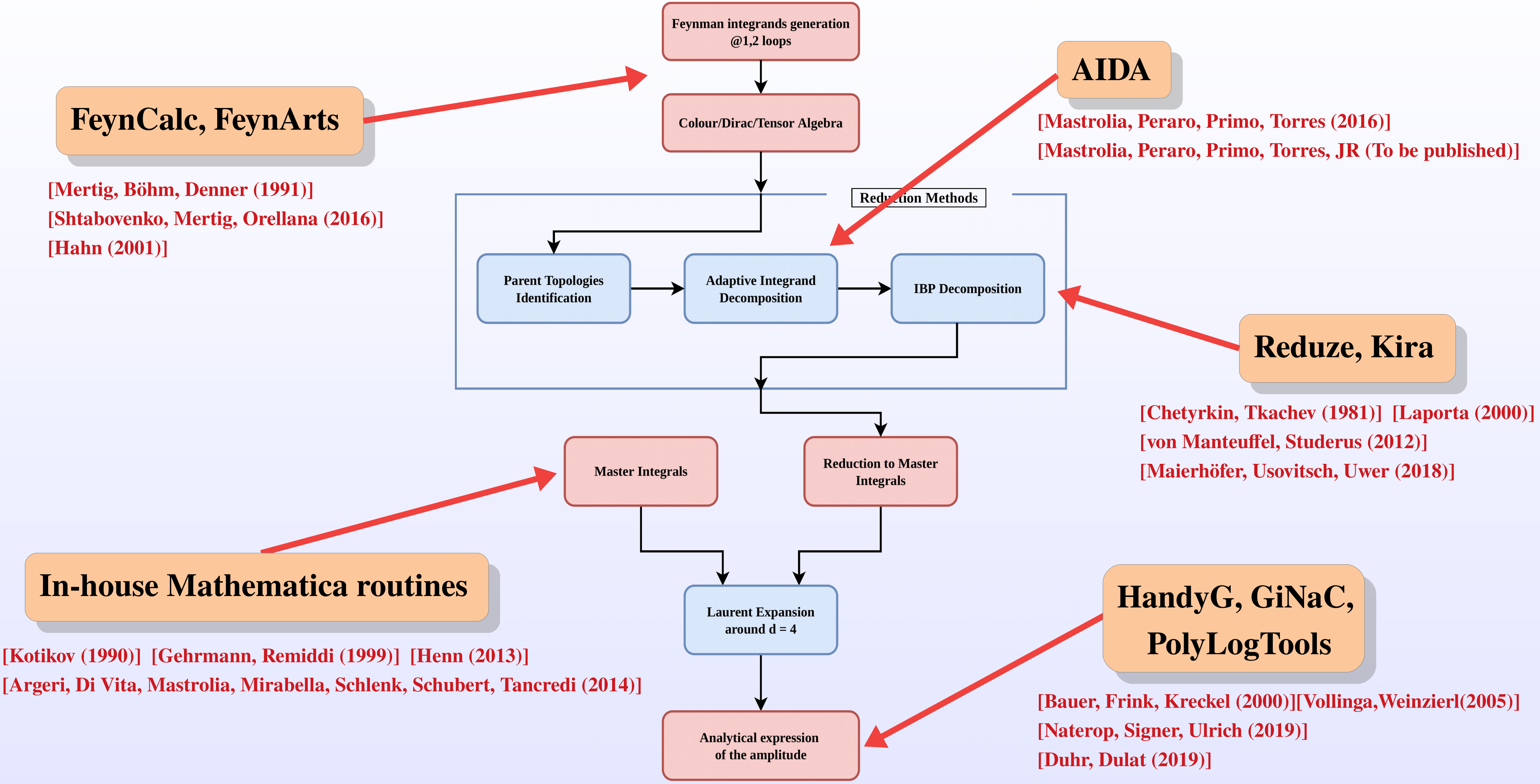
$$\sum_{j=1}^{n}\sum_{i_1\cdots i_j}^{n}\int dk_{||}d\lambda \frac{\Delta^{\text{int}}_{i_1\dots i_j}(k_{||},\lambda)}{D_{i_1}(k_{||},\lambda)\cdots D_{i_j}(k_{||},\lambda)}$$

Divide

$$\sum_{j=1}^{n}\sum_{i_1\cdots i_j}^{n}\int dk_{||}d\lambda \frac{\Delta'_{i_1\dots i_j}(k_{||})}{D_{i_1}(k_{||},\lambda)\cdots D_{i_j}(k_{||},\lambda)}$$

- **Numerator reduced in terms of ISPs**
- **No need for integral identities at 1L**
- **At 2L, amplitude ready for IBPs**

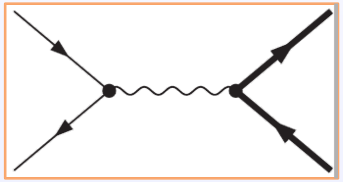
The AIDA framework

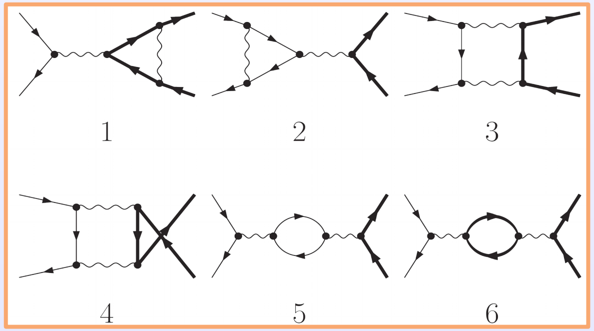


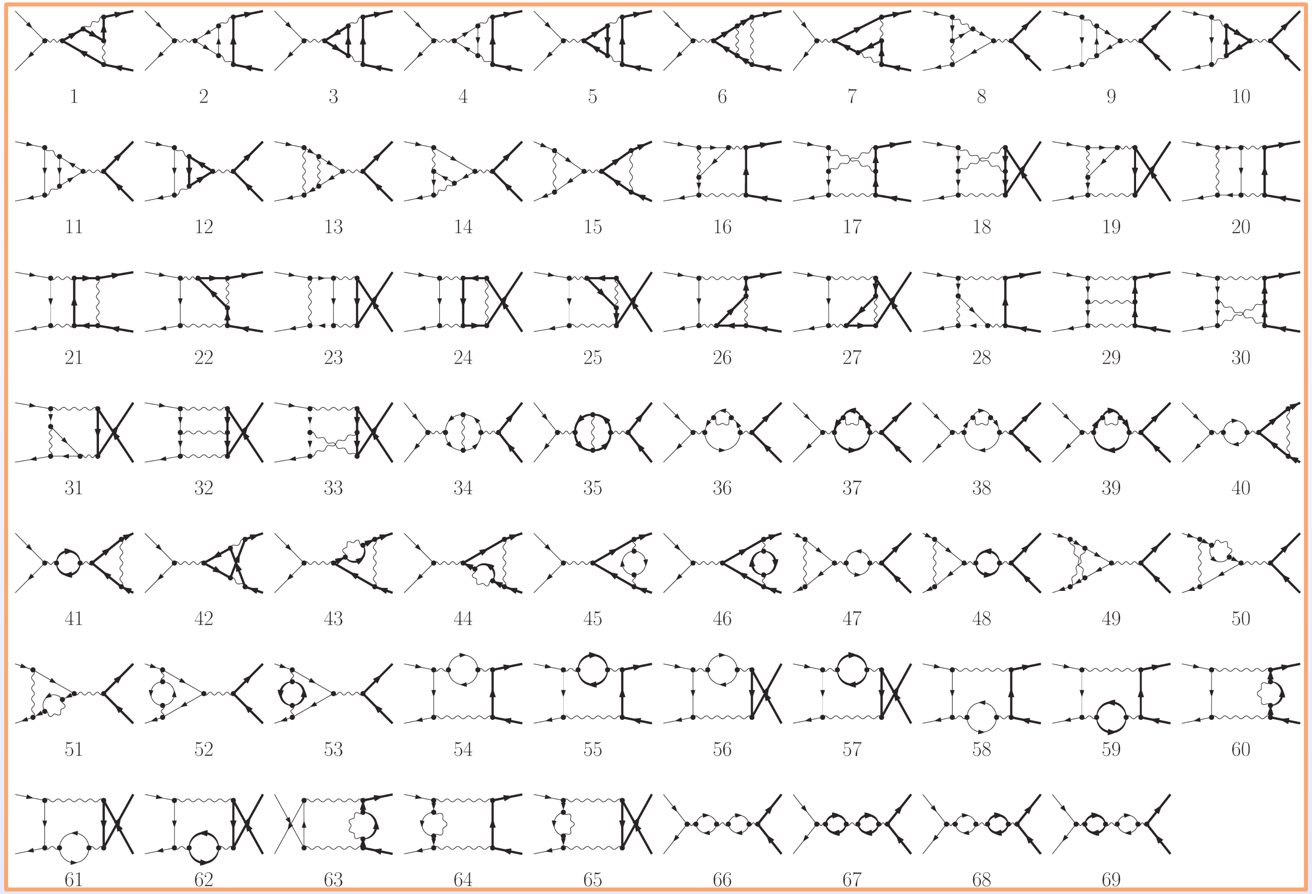
Di-muon production in QED: Feynman Diagrams

$$\mathcal{M}^{(1)} = A^{(1)} + n_l B_l^{(1)} + n_h C_h^{(1)}$$

$$\mathcal{M}^{(2)} = A^{(2)} + n_l B_l^{(2)} + n_h C_h^{(2)} + n_l^2 D_l^{(2)} + n_l n_h D_{lh}^{(2)} + n_h^2 F_h^{(2)}$$

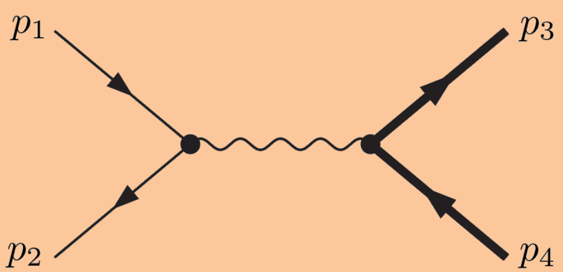






[Bonciani, Broggio, Di Vita, Ferroglia, Mandal, Mastrolia, Mattiazzi, Primo, Schubert, Torres Bobadilla, Tramontano, JR (2021)]

Di-muon production in QED: Feynman Diagrams



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$O(10^4)$ terms
max rank $= 4$

AID

$$\mathcal{M}^{(2)} = \sum_{j=1}^{n} \sum_{i_1 \cdots i_j}^{n} (S_\epsilon)^2 \int d^d k_1 d^d k_2 \frac{\Delta'_{i_1 \cdots i_j}(\mathbf{k}, \mathbf{l})}{D_{i_1} \cdots D_{i_j}}$$

$O(10^4)$ terms
max rank $= 2$

IBPs

$$\mathcal{M}^{(2)} = \sum_{j=1}^{N_{\mathrm{MI}}} c_j(s, t, M; d) I_j^{(2)}(s, t, M; d)$$

$O(10^2)$ terms
max rank $= 2$

Master Integrals

Master Integrals for $\mu e \to \mu e$ are **known in literature**.

How to calculate Masters Integrals? William J. Torres Bobadilla's talk

$$\mathbf{I}^{(2)}_{\mu e \to \mu e} = \mathbf{I}^{(2)}_{ee \to \mu\mu}\Big|_{s \leftrightarrow t}$$

Representation through **Generalized PolyLogarithms**

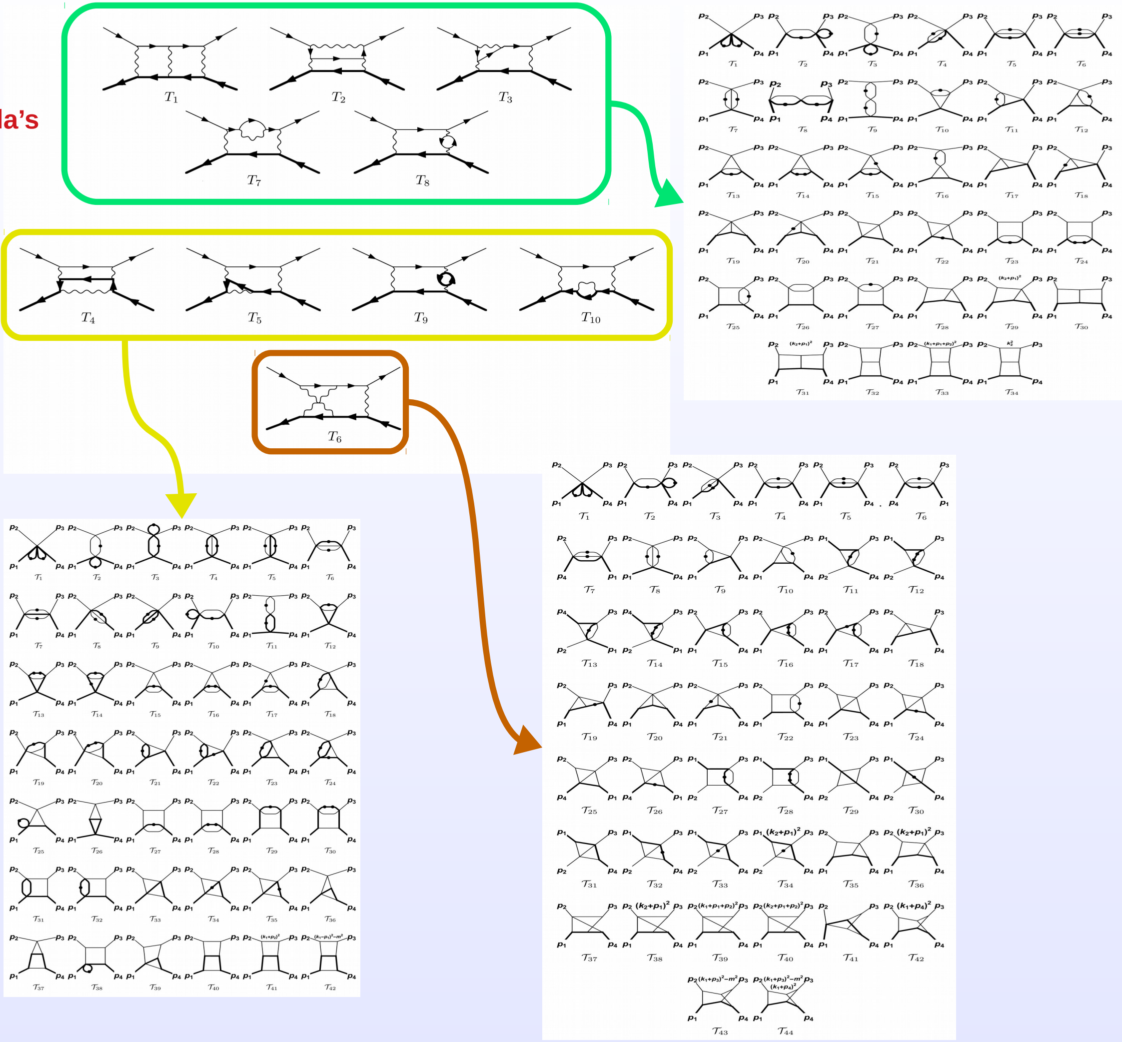
$$I_j^{(2)}(s,t,M;d) = \sum_i C_i(\{w\}_{ji},d)\,G(\{w\}_{ji},1)$$

where

$$G(w_n,\ldots,w_1;\tau) = \int_0^\tau \frac{dt}{t - w_n} G(w_{n-1},\ldots,w_1;t)$$

Letters
$w_j = w_j(s,t,M)$

- O(4000) GPLs
- GPLs up to weight 4
- 18 letters



[Kotikov (1990)] [Gehrmann, Remiddi (1999)] [Henn (2013)]
[Argeri, Di Vita, Mastrolia, Mirabella, Schlenk, Schubert, Tancredi (2014)]
[Mastrolia, Passera, Primo, Schubert (2017)]
[Di Vita, Laporta, Mastrolia, Primo, Schubert (2018)]

Checks

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Evaluating the interferences with **HandyG** and **GiNaC**

$$\frac{s}{M^2} = 5, \quad \frac{t}{M^2} = -\frac{5}{4}, \quad \mu = M$$

	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}	ϵ
$\mathcal{M}^{(0)}$	-	-	-	-	$\frac{181}{100}$	-2
$A^{(1)}$	-	-	$-\frac{181}{100}$	1.99877525	22.0079572	−11.7311017
$B_l^{(1)}$	-	-	-	-	−0.069056030	4.94328573
$C_h^{(1)}$	-	-	-	-	−2.24934027	2.54943566
$A^{(2)}$	$\frac{181}{400}$	−0.499387626	−35.4922919	19.4997261	49.0559119	-
$B_l^{(2)}$	-	$-\frac{181}{400}$	0.785712779	−16.1576674	−3.75247701	-
$C_h^{(2)}$	-	-	1.12467013	−9.50785825	−25.8771503	-
$D_l^{(2)}$	-	-	-	-	−3.96845688	-
$E_{hl}^{(2)}$	-	-	-	-	−4.88512563	-
$F_h^{(2)}$	-	-	-	-	−0.158490810	-

[Bonciani, Broggio, Di Vita, Ferroglia, Mandal, Mastrolia, Mattiazzi, Primo, Schubert, Torres Bobadilla, Tramontano, JR (2021)]

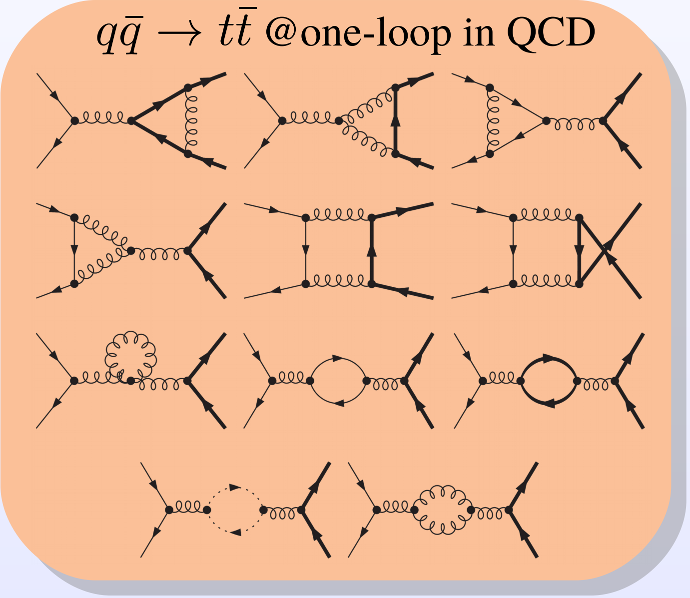
How do we check these terms?

Checks :: Literature

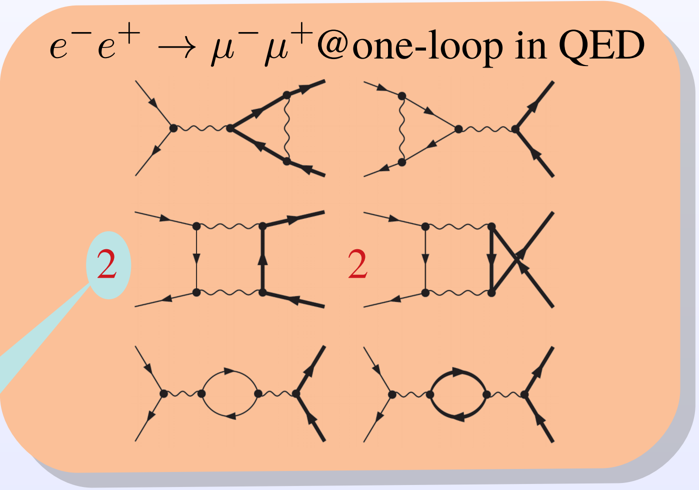
$q\bar{q} \to t\bar{t}$ **in QCD** → $e^- e^+ \to \mu^- \mu^+$ **in QED**

- Top-pair production admit a color decomposition
- 1-loop and 2-loop corrections already known in literature
- Abelian part get contributions from *QED-like diagrams* only

[Czakon(2008)]
[Bonciani, Ferroglia, Gehrmann, Maitre, Studerus (2008)]
[Bärnreuther, Czakon, Fiedler (2014)]



Abelian, Color-stripped



Remainder of color factors

Full agreement with the abelian part of top-pair production

Checks :: IR structure

Two-loop IR poles from **one-loop and tree** (renormalised) **contributions**

$$\mathcal{M}^{(1)}\Big|_{\text{poles}} = \frac{1}{2} Z_1^{\text{IR}}\, \mathcal{M}^{(0)}\Big|_{\text{poles}}$$

$$\mathcal{M}^{(2)}\Big|_{\text{poles}} = \frac{1}{8}\left[\left(Z_2^{\text{IR}} - \left(Z_1^{\text{IR}}\right)^2\right)\mathcal{M}^{(0)} + 2\, Z_1^{\text{IR}}\, \mathcal{M}^{(1)}\right]\Big|_{\text{poles}}$$

[Becher, Neubert (2009)]
[Hill (2017)]

IR Renormalisation Factor

Beta function

$$\ln Z_{\text{IR}} = \frac{\alpha}{4\pi}\left(\frac{\Gamma_0'}{4\epsilon^2} + \frac{\Gamma_0}{2\epsilon}\right) + \left(\frac{\alpha}{4\pi}\right)^2\left(-\frac{3\beta_0\Gamma_0'}{16\epsilon^3} + \frac{\Gamma_1' - 4\beta_0\Gamma_0}{16\epsilon^2} + \frac{\Gamma_1}{4\epsilon}\right) + \mathcal{O}\left(\alpha^3\right)$$

$$\gamma_i = \sum_{j=0}^{n}\left(\frac{\alpha}{\pi}\right)^{j+1}\gamma_i^{(j)} + O\left(\alpha^{n+1}\right)$$

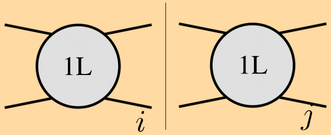
Anomalous dimension

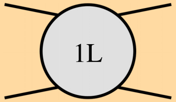
$$\Gamma = \gamma_{\text{cusp}}(\alpha)\ln\left(-\frac{s}{\mu^2}\right) + 2\gamma_{\text{cusp}}(\alpha)\ln\left(\frac{t-M^2}{u-M^2}\right) + \gamma_{\text{cusp,M}}(\alpha, s) + 2\gamma_h(\alpha) + 2\gamma_\psi(\alpha)$$

☑ **Full agreement** with the direct calculation of the two-loop contribution

NNLO QED one-loop squared amplitude

Missing double-virtual contribution:

$\mathcal{M}^{(1,1)} = \sum_{ij}$ 



$\sim \bar{u}(p_3)\Gamma^{\bar{\mu}}_{\text{el}}(\bar{p},l)u(p_1)P_{\bar{\mu}\bar{\nu}}(\bar{p},l)\bar{v}(p_2)\Gamma^{\bar{\nu}}_{\text{mu}}(\bar{p},l)v(p_4)$

1L tensor integrals

Tensor reduction at **amplitude** level (AIDA, TID...)

$$\sum_j \tilde{P}_j(p_i,l)[\bar{u}(p_3)\tilde{\Gamma}^{\bar{\mu}}_{\text{el,j}}(p_i)u(p_1)][\bar{v}(p_2)\tilde{\Gamma}_{\text{mu,j},\bar{\mu}}(p_i)v(p_4)]$$

1L Scalar integrals

IBPs

$$\sum_{ji} c_{ji}(s,t,M^2;d)I_i(s,t,M^2;d)[\bar{u}(p_3)\tilde{\Gamma}^{\bar{\mu}}_{\text{el,j}}(p_i)u(p_1)][\bar{v}(p_2)\tilde{\Gamma}_{\text{mu,j},\bar{\mu}}(p_i)v(p_4)]$$

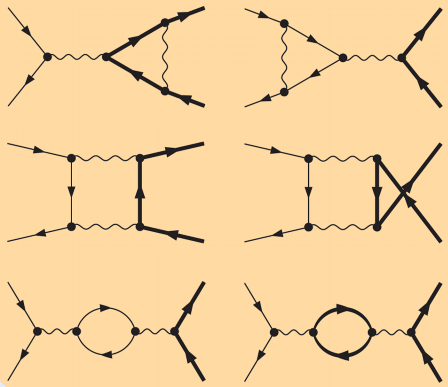
1L Master integrals

Master integrals at order ϵ^2

Interferences Sums

$$\mathcal{M}^{(1,1)} = \sum_{i=-4}^{0} \mathcal{M}^{(1,1)}_i \epsilon^i + O(\epsilon)$$

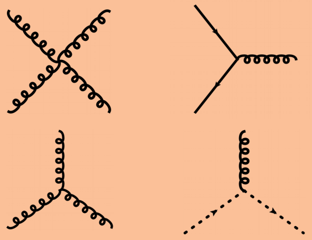
One-loop diagrams



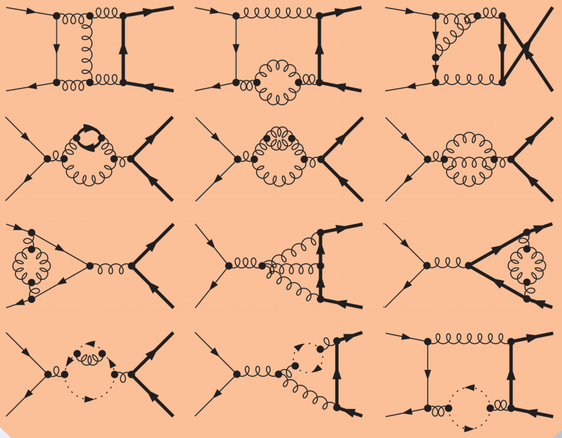
Extension to two-loop top-pair production @NNLO QCD

Full two-loop $q\bar{q} \to t\bar{t}$ @NNLO QCD calculation available only **numerical**

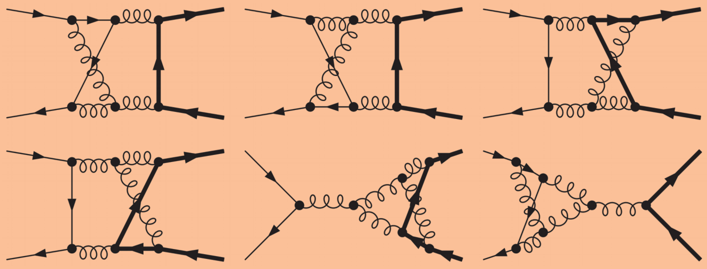
More particles and interactions



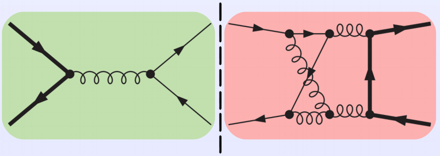
More diagrams (~220)



New topologies occur



BUT



$$\propto T^a_{ij} T^a_{lk} f^{abc} (T^d T^c)_{kl} (T^b T^d T^a)_{ji} = 0 \;!$$

No additional Master Integrals are required

Extension to two-loop top-pair production @NNLO QCD

$$\mathcal{M}^{(1)} = 2(N_c^2-1)\Big(N_c A^{(1)} + B^{(1)} + n_l C_l^{(1)} + n_h C_h^{(1)}\Big)$$

$$\mathcal{M}^{(2)} = 2(N_c^2-1)\Big(N_c^2 A^{(2)} + B^{(2)} + \frac{C^{(2)}}{N_c^2} + n_l N_c D_l^{(2)} + n_h N_c D_h^{(2)} + n_l \frac{E_l^{(2)}}{N_c} + n_h \frac{E_h^{(2)}}{N_c} + n_l^2 F_l^{(2)} + n_l n_h F_{lh}^{(2)} + n_h^2 F_h^{(2)}\Big)$$

Evaluating the interferences with **HandyG** and **GiNaC**

$$\frac{s}{M^2} = 5, \quad \frac{t}{M^2} = -\frac{5}{4}, \quad \mu = M$$

	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}	ϵ^{1}
$A^{(0)}$	-	-	-	-	$\frac{181}{100}$	-2
$A^{(1)}$	-	-	$-\frac{181}{400}$	0.1026418456757775	1.356145770566065	2.230403451742140
$B^{(1)}$	-	-	$\frac{181}{400}$	-0.3180868339485723	-5.763132746701004	2.913169881363488
$C_l^{(1)}$	-	-	0	0	-0.01726400752682416	1.235821434465827
$C_h^{(1)}$	-	-	0	0	-0.5623350683773134	0.6373589172648111
$A^{(2)}$	$\frac{181}{800}$	1.391733154324222	-2.298174307221209	-4.145752448999165	17.37136598564062	-
$B^{(2)}$	$-\frac{181}{400}$	-1.323646320375650	8.507455541210568	6.035611156200398	-35.12861106350758	-
$C^{(2)}$	$\frac{181}{800}$	-0.06808683394857230	-18.00716652035224	6.302454931016090	3.524044912826756	-
$D_l^{(2)}$	0	$-\frac{181}{800}$	0.2605057338631945	-0.7250180282219092	-1.935417246635768	-
$D_h^{(2)}$	0	0	0.5623350683773134	0.1045606449242690	-1.704747997587188	-
$E_l^{(2)}$	0	$\frac{181}{800}$	-0.3323207299541260	7.904121951420471	2.848697836597635	-
$E_h^{(2)}$	0	0	-0.5623350683773134	4.528240788258799	12.73232424278180	-
$F_l^{(2)}$	0	0	0	0	-1.984228442234312	-
$F_{lh}^{(2)}$	0	0	0	0	-2.442562819239786	-
$F_h^{(2)}$	0	0	0	0	-0.07924540546146283	-

[Mandal, Mastrolia, Torres Bobadilla, JR (2022)]

First fully-analytical calculation

Full agreement with the literature

[Czakon(2008)]
[Bonciani, Ferroglia, Gehrmann, Maitre, Studerus (2008)]
[Bärnreuther, Czakon, Fiedler (2014)]
[Fael, Passera (2019)]
[Fael (2018)]

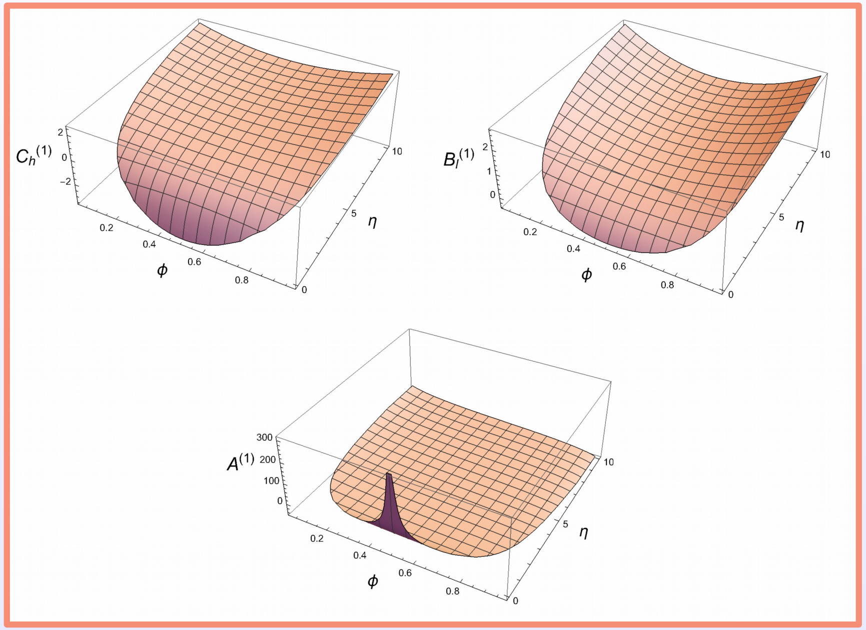
Results: One-loop di-muon production @NLO QED

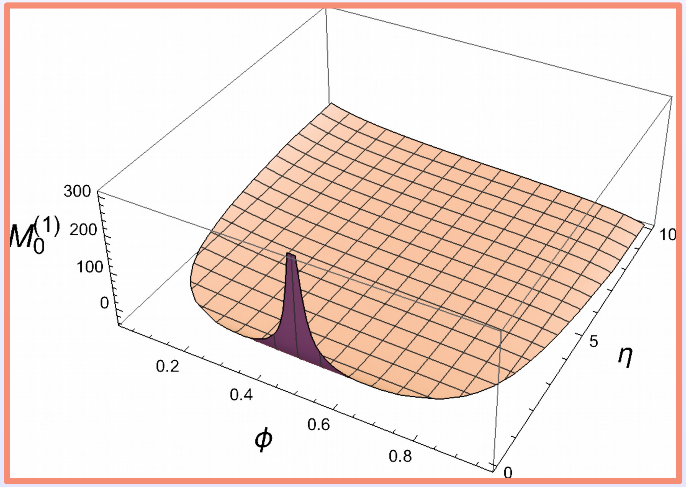
$$\mathcal{M}_0^{(1)} = A^{(1)} + n_l B_l^{(1)} + n_h C_h^{(1)}\Big|_{\text{finite}}$$

$$\eta = \frac{s}{4M^2} - 1, \quad \phi = -\frac{(t - M^2)}{s}$$

Production region

$$\eta > 0, \quad \frac{1}{2}\left(1 - \frac{\eta}{1+\eta}\right) < \phi < \frac{1}{2}\left(1 + \frac{\eta}{1+\eta}\right)$$

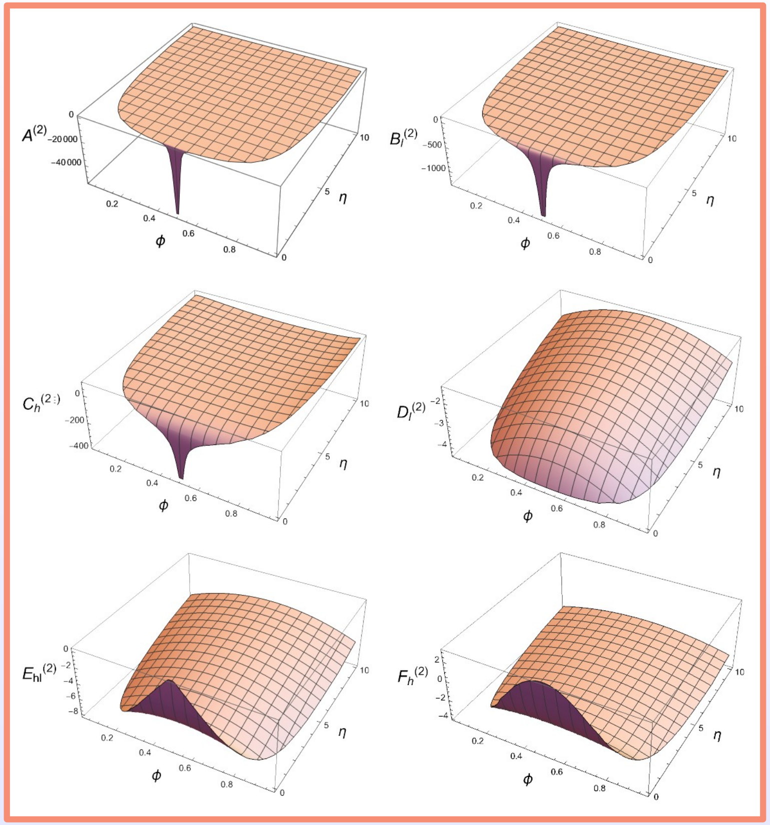




[Bonciani, Broggio, Di Vita, Ferroglia, Mandal, Mastrolia, Mattiazzi, Primo, Schubert, Torres Bobadilla, Tramontano, JR (2021)]

Results: Two-loop di-muon production @NNLO QED

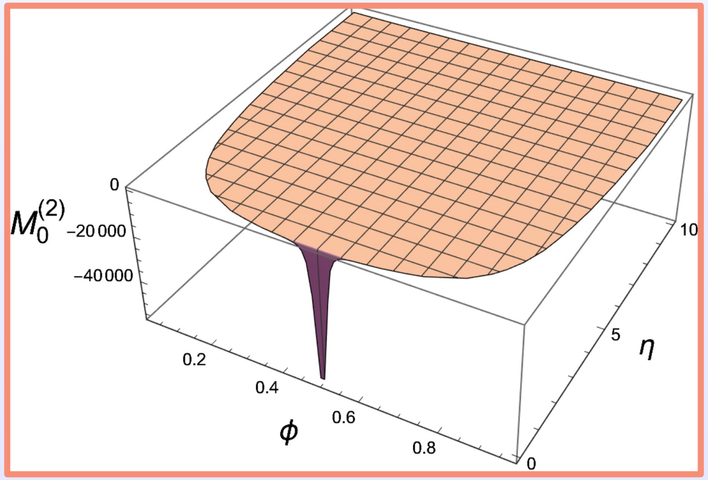
$$\mathcal{M}_0^{(2)} = A^{(2)} + n_l B_l^{(2)} + n_h C_h^{(2)} + n_l^2 D_l^{(2)} + n_l n_h E_{lh}^{(2)} + n_h^2 F_h^{(2)} \Big|_{\text{finite}}$$



$$\eta = \frac{s}{4M^2} - 1, \qquad \phi = -\frac{(t - M^2)}{s}$$

Production region

$$\eta > 0, \qquad \frac{1}{2}\left(1 - \frac{\eta}{1+\eta}\right) < \phi < \frac{1}{2}\left(1 + \frac{\eta}{1+\eta}\right)$$



[Bonciani, Broggio, Di Vita, Ferroglia, Mandal, Mastrolia, Mattiazzi, Primo, Schubert, Torres Bobadilla, Tramontano, JR (2021)]

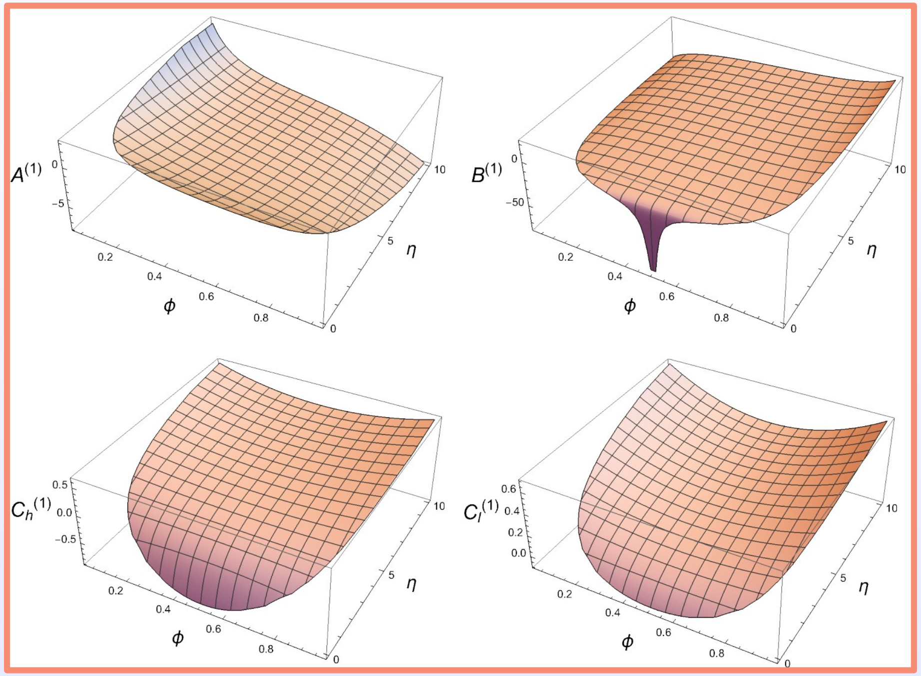
Results: One-loop top-pair production @NLO QCD

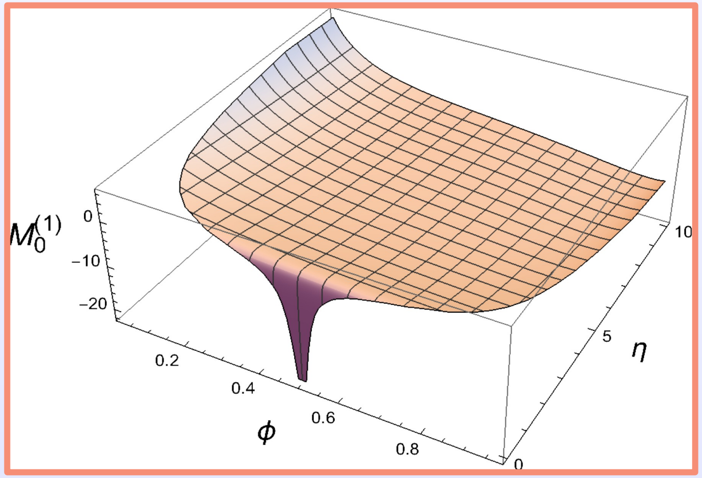
$$\mathcal{M}_0^{(1)} = 2(N_c^2 - 1)\left(N_c A^{(1)} + \frac{B^{(1)}}{N_c} + n_l C_l^{(1)} + n_h C_h^{(1)}\right)\Big|_{\text{finite}}$$

$$\eta = \frac{s}{4M^2} - 1, \qquad \phi = -\frac{(t - M^2)}{s}$$

Production region

$$\eta > 0, \qquad \frac{1}{2}\left(1 - \frac{\eta}{1+\eta}\right) < \phi < \frac{1}{2}\left(1 + \frac{\eta}{1+\eta}\right)$$





[Mandal, Mastrolia, Torres Bobadilla, JR (2022)]

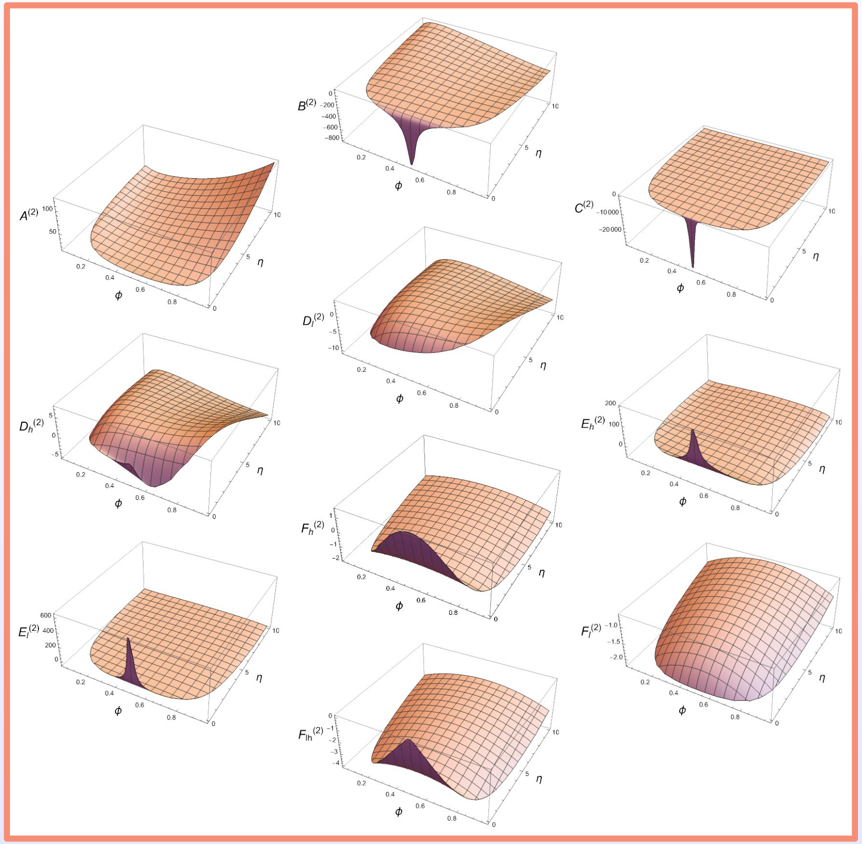
Results: Two-loop top-pair production @NNLO QCD

$$\mathcal{M}_0^{(2)} = 2(N_c^2-1)\left(N_c^2 A^{(2)} + B^{(2)} + \frac{C^{(2)}}{N_c^2} + n_l N_c D_l^{(2)} + n_h N_c D_h^{(2)} + n_l \frac{E_l^{(2)}}{N_c} + n_h \frac{E_h^{(2)}}{N_c} + n_l^2 F_l^{(2)} + n_l n_h F_{lh}^{(2)} + n_h^2 F_h^{(2)}\right)\Big|_{\text{finite}}$$

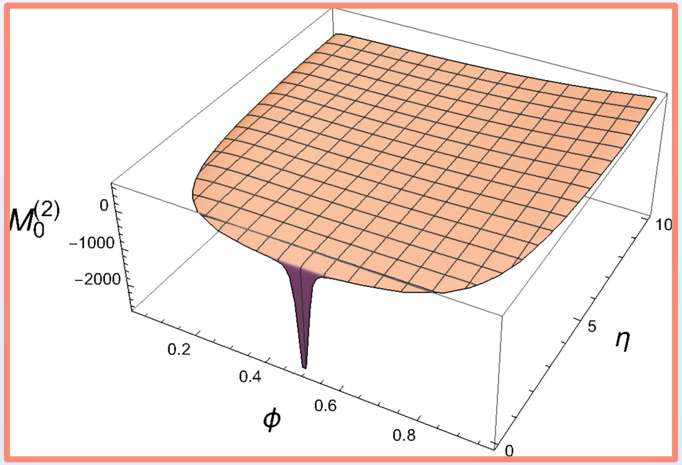
$$\eta = \frac{s}{4M^2} - 1, \quad \phi = -\frac{(t-M^2)}{s}$$

Production region

$$\eta > 0, \quad \frac{1}{2}\left(1 - \frac{\eta}{1+\eta}\right) < \phi < \frac{1}{2}\left(1 + \frac{\eta}{1+\eta}\right)$$



[Mandal, Mastrolia, Torres Bobadilla, JR (2022)]



Conclusions & Outlook

Electron-muon elastic scattering @NNLO QED is a crucial input for the **MUonE experiment**

- ☑ **Crossing:** the two-loop contributions to **di-muon production** @NNLO QED via electron-positron annihilation
  - ☑ First QED **analytical two-loop** calculation for di-muon production process
  - ☑ Complete **automation** through the **AIDA framework**
  - ☑ **Cross-checked** against
    - Independent calculations
    - IR structure cross-checked against the SCET prediction
  - ☑ Grid of 10500 phase-space points has been generated

- ☑ Extension to **two-loop** contributions to **top-pair production via quark-antiquark annihilation** @NNLO QCD
- ☑ **Analytical one-loop squared** contributions to di-muon production @NNLO QED
- ☑ Inclusion of **non-zero** electron mass to electron-muon elastic scattering calculation: **massification**
- ☑ Differential and total cross section [Broggio, Engel, Ferroglia, Mandal, Mastrolia, Passera, Rocco, Signer, Torres Bobadilla, Ulrich, Zoller (to be published)]

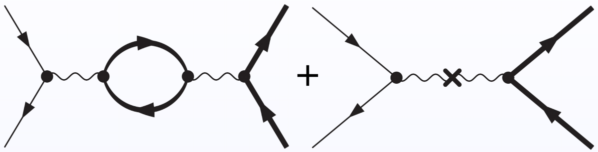
- **Threshold expansion** for both di-muon and top-pair production @NNLO
- NNLO QED double virtual mu-e scattering for **polarized** electrons

On the massification:
[Mitov, Moch (2006)]
[Becher, Melnikov (2007)]
[Engel, Gnendiger, Signer, Ulrich (2019)]
[Heller (2021)]

Thank you

UV Renormalization

$\mathcal{M}_b^{(2)}$ is **UV divergent** $\xrightarrow{\text{Renormalisation}}$ $\mathcal{M}^{(2)}$



$$\mathcal{M} = Z_{2,e} Z_{2,\mu} \mathcal{M}_b(\alpha_b = \alpha_b(\alpha), M_b = M_b(M))$$

where

$$M_b(M) = Z_M M$$

$$\alpha_s S_\epsilon = \alpha(\mu^2)\mu^{2\epsilon} Z_\alpha$$

Renormalisation constants:

$$Z_j = 1 + \left(\frac{\alpha}{\pi}\right) \delta Z_j^{(1)} + \left(\frac{\alpha}{\pi}\right)^2 \delta Z_j^{(2)} + O(\alpha^3)$$

Renormalisation schemes

- **On-shell** renormalisation $Z_{2,e}, Z_{2,\mu}, Z_M$
- $\overline{\text{MS}}$ renormalisation Z_α

Renormalised interferences:

$$\mathcal{M}^{(0)} = \mathcal{M}_b^{(0)}$$

$$\mathcal{M}^{(1)} = \mathcal{M}_b^{(1)} + (\delta Z_{2,\mu}^{(1)} + Z_\alpha^{(1)}) \mathcal{M}_b^{(0)}$$

$$\mathcal{M}^{(2)} = \mathcal{M}_b^{(2)} + (\delta Z_{2,\mu}^{(1)} + Z_\alpha^{(1)}) \mathcal{M}_b^{(1)} + (\delta Z_{2,\mu}^{(2)} + \delta Z_{2,e}^{(2)} + Z_\alpha^{(2)} + \delta Z_{2,\mu}^{(1)} Z_\alpha^{(1)}) \mathcal{M}_b^{(0)} + \delta Z_M^{(1)} \mathcal{M}_{\text{massCT}}^{(1)}$$