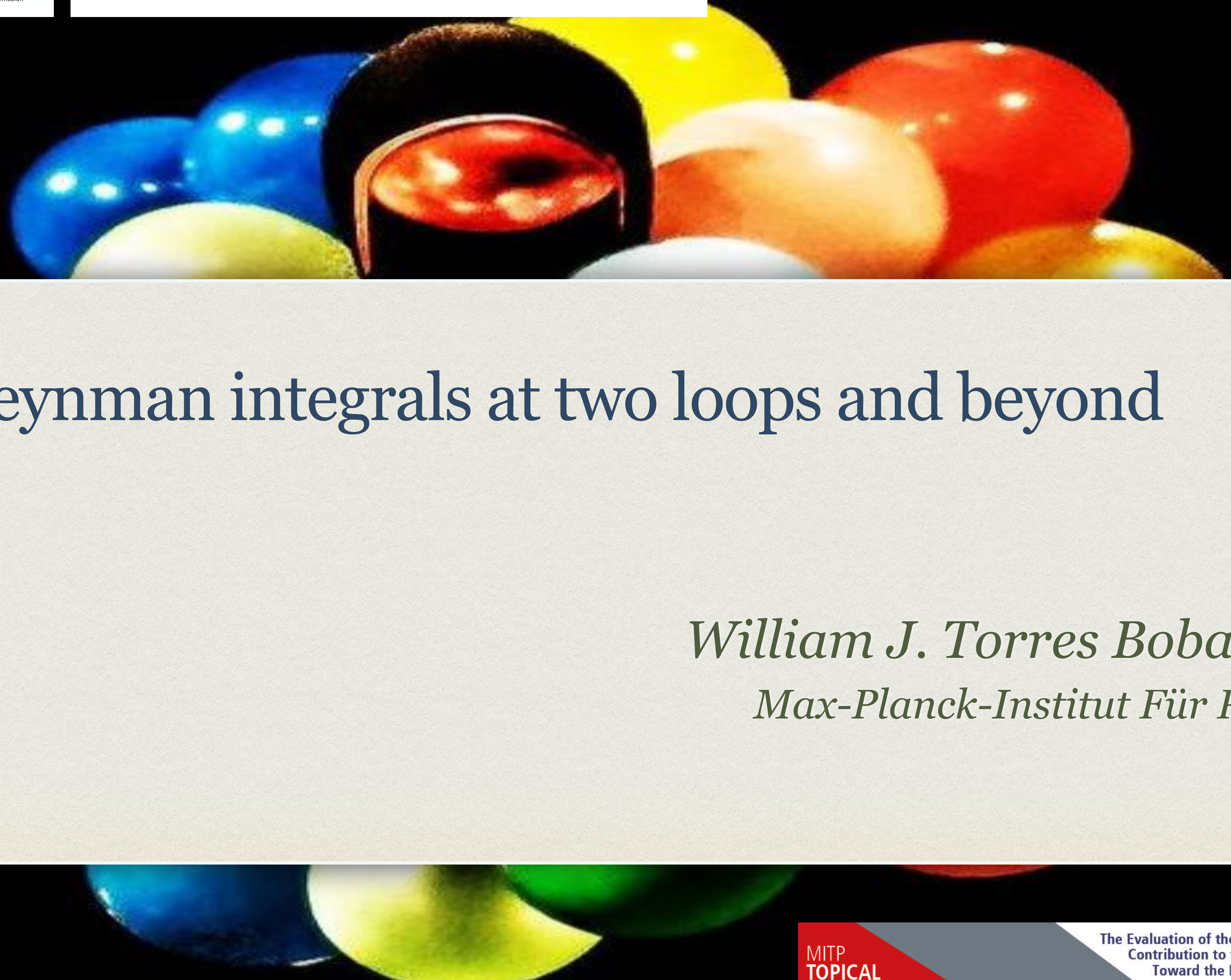

# On the Feynman integrals at two loops and beyond

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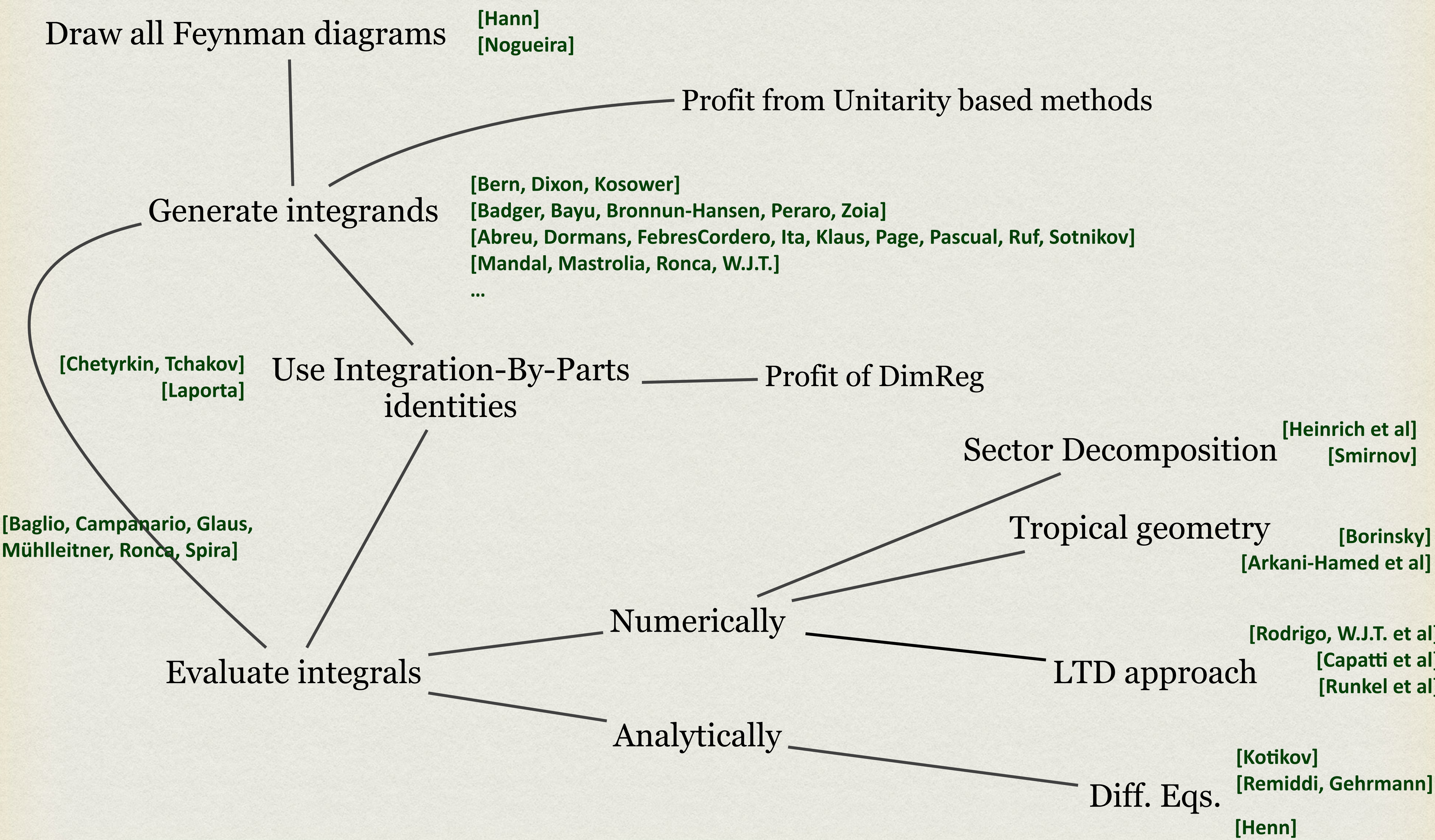
*Max-Planck-Institut Für Physik*

MITP TOPICAL WORKSHOP
The Evaluation of the Leading Hadronic Contribution to the Muon g-2: Toward the MUonE Experiment
14 – 18 November 2022
https://indico.mitp.uni-mainz.de/event/248

# *Standard approach @multi-loop level*



# *Standard approach @multi-loop level*

Complete automation @ NNLO ?

Thresholds

UV

IR

Eur. Phys. J. C (2021) 81:250
https://doi.org/10.1140/epjc/s10052-021-08996-y

THE EUROPEAN PHYSICAL JOURNAL C

Check for updates

Review

**May the four be with you: novel IR-subtraction methods to tackle NNLO calculations**

**W. J. Torres Bobadilla[1,2,a], G. F. R. Sborlini[3], P. Banerjee[4], S. Catani[5], A. L. Cherchiglia[6], L. Cieri[5], P. K. Dhani[5,7], F. Driencourt-Mangin[2], T. Engel[4,8], G. Ferrera[9], C. Gnendiger[4], R. J. Hernández-Pinto[10], B. Hiller[11], G. Pelliccioli[12], J. Pires[13], R. Pittau[14], M. Rocco[15], G. Rodrigo[2], M. Sampaio[6], A. Signer[4,8], C. Signorile-Signorile[16,17], D. Stöckinger[18], F. Tramontano[19], Y. Ulrich[4,8,20]**

*<<Yannick's*

- FDH/FDR —> transition rules both ren. schemes @NNLO
- FDU —> preliminary mappings between VV & VR contributions
- IReg —> Towards full renormalisation @ 2L
- Torino Scheme —> general subtraction method for massless & final states QCD
- qt-subtraction —> benefits from any existing calculation for "F+jet"
- Antenna subtraction —> subtraction term at tree (RR) and one-loop (R) level

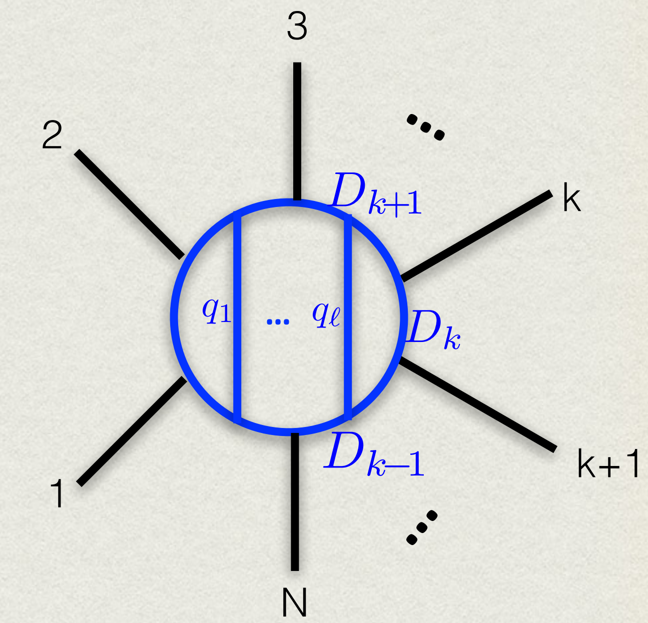
- many more subs. schemes ...

# *Outline*

- Analytic evaluation
- Numerical evaluation
- Conclusions & future questions

# *Preliminary*

In loop calculations, one finds




$$J_N^{(L),D}(1,\ldots,n;n+1,\ldots,m) = \int \prod_{i=1}^{L} \frac{d^D\ell_i}{\imath\,\pi^{D/2}} \frac{\prod_{k=n+1}^{m} D_k^{-\nu_k}}{\prod_{j=1}^{n} D_j^{\nu_j}}$$

$$D_i = q_i^2 - m_i^2 + \imath 0$$

What to do?

- Evaluate them?
- Analyse them?

First principles

Get mathematical insights

*Everything is connected!*

Profit from mathematical properties

Keep into account behaviour dictated by physics

Investigate further mathematical formalism

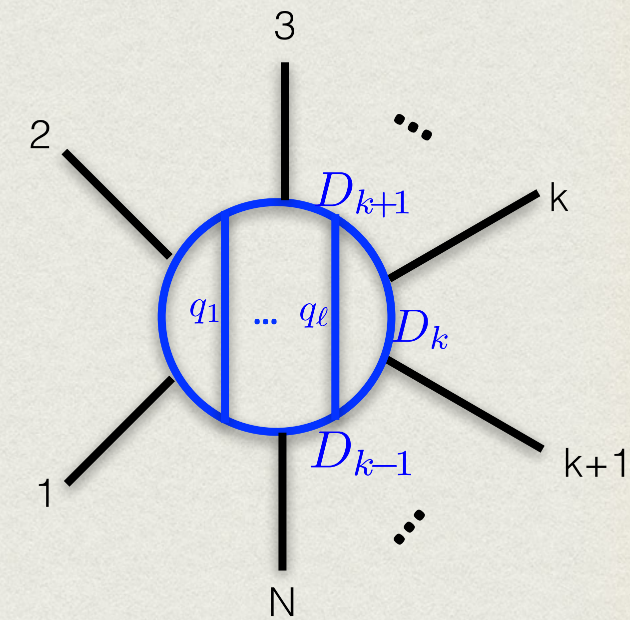
# Analytic evaluation

# *Analytic evaluations*

In loop calculations, one finds

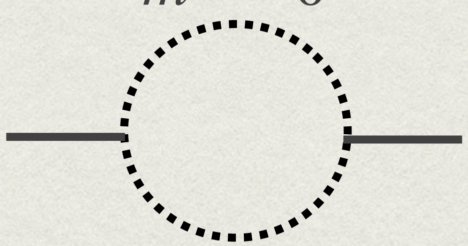
$$J_N^{(L),D}(1,\ldots,n;n+1,\ldots,m) = \int \prod_{i=1}^{L} \frac{d^D\ell_i}{\imath\,\pi^{D/2}} \frac{\prod_{k=n+1}^{m} D_k^{-\nu_k}}{\prod_{j=1}^{n} D_j^{\nu_j}}$$

$$D_i = q_i^2 - m_i^2 + \imath 0$$



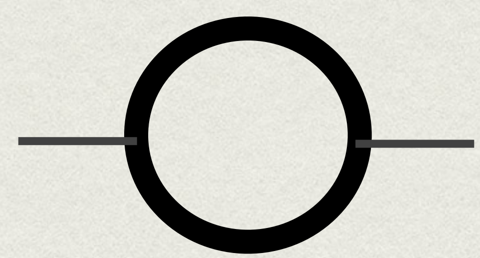

Complexity easily increases:

$m^2 = 0$



$$\frac{1}{\epsilon}(-p^2)^{-1-\epsilon}\left(-2 + \frac{\pi^2}{6}\epsilon^2 + \frac{14}{3}\zeta_3\epsilon^3 + \mathcal{O}\left(\epsilon^4\right)\right)$$

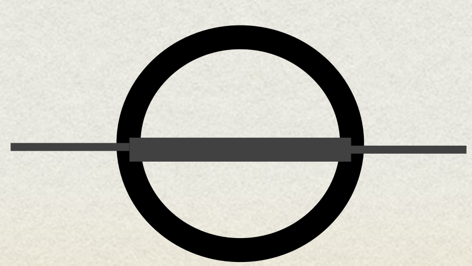
$m^2 \neq 0$



$$\frac{2}{\sqrt{(-p^2)(4m^2 - p^2)}} \log\left(\frac{\sqrt{1 - 4m^2/p^2} + 1}{\sqrt{1 - 4m^2/p^2} - 1}\right) + \mathcal{O}\left(\epsilon\right)$$

*—> squared roots*

$m^2 \neq 0$



$$-\frac{4\mathrm{K}\left(\lambda\right)}{\left(p^2+m^2\right)\sqrt{a_{13}a_{24}}}\left[2\mathcal{E}_4\left(\begin{matrix}0 & -1\\ 0 & \infty\end{matrix};1,\vec{a}\right) + \mathcal{E}_4\left(\begin{matrix}0 & -1\\ 0 & 0\end{matrix};1,\vec{a}\right) + \mathcal{E}_4\left(\begin{matrix}0 & -1\\ 0 & 1\end{matrix};1,\vec{a}\right)\right]$$

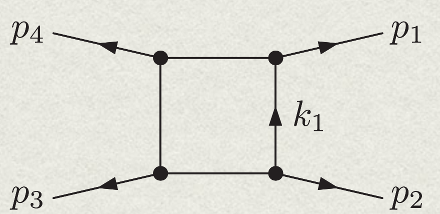
$$\mathrm{K}\left(\lambda\right) = \int_0^1 \frac{dx}{\sqrt{\left(1-x^2\right)\left(1-\lambda x^2\right)}} \qquad \mathcal{E}_4\left(\begin{matrix}n_1 & \ldots & n_k\\ c_1 & \ldots & c_k\end{matrix};t,\vec{a}\right) = \int_0^x dt\,\Psi_{n_1}\left(c_1,t,\vec{a}\right)\mathcal{E}_4\left(\begin{matrix}n_2 & \ldots & n_k\\ c_2 & \ldots & c_k\end{matrix};t,\vec{a}\right)$$

*—> elliptic integrals*

# *Analytic evaluations*

- DLOG representation of Feynman integrals

Four-point integral family

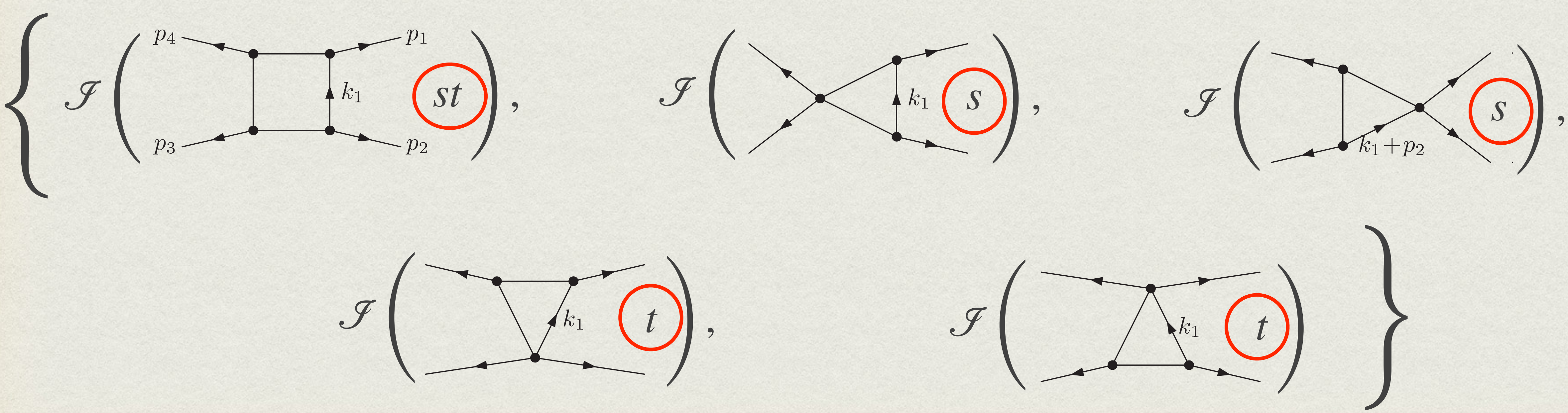



$$\mathcal{I}\left( \cdots \ \mathcal{N} \right) = \frac{d^4 k_1 \, \mathcal{N}}{\left(k_1 - p_1\right)^2 k_1^2 \left(k_1 + p_2\right)^2 \left(k_1 + p_2 + p_3\right)^2} \overset{?}{=} d \log \tau_1 \, \ldots d \log \tau_4$$

$$s = (p_1 + p_2)^2 \,, \; t = (p_2 + p_3)^2$$

- Obtained with the aid of **[Wasser '18], [Henn++ '20]**

Leading Logarithmic singularities



# *Analytic evaluations*

- Landau singularities

  Feynman integral are many-valued analytic function whose singularities lie on some algebraic varieties — Landau Varieties

- Landau equations

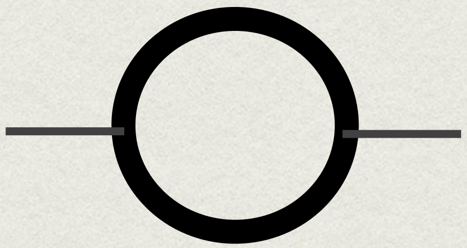
$$q_i^2 - m_i^2 = 0 \text{ or } \alpha_i = 0$$

$$\sum \alpha_i \frac{\partial D_i}{\partial k_j} = \sum \alpha_i q_i = 0$$

*Connection between leading and Landau singularities?*

- Landau singularity of a one-loop scalar bubble

$m^2 \neq 0$



$$\rightarrow -\frac{1}{4}p^2(4m^2 - p^2)$$

# Leading & Landau singularities

## Connection in one-loop Feynman integrals

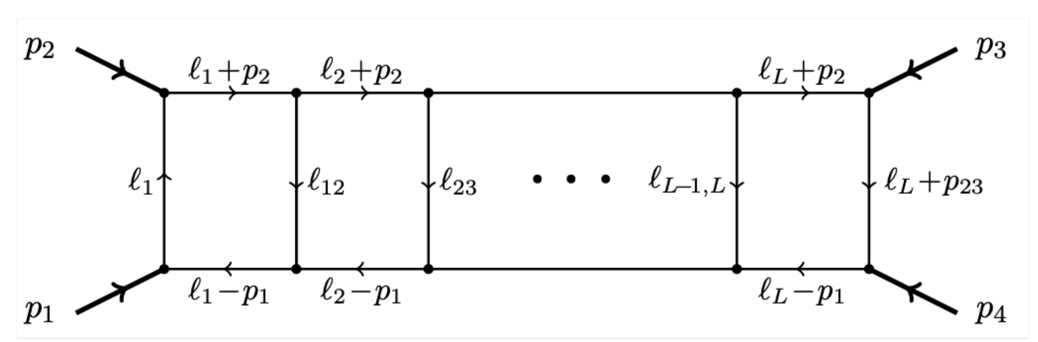
[Flieger , WJT (2022)]

**Theorem 4.1.** *The leading singularity of an n-point one-loop Feynman integral in* $D = n+1$ *space-time dimensions is equal to* $\pm 1/\left(2^n\sqrt{-\det\left(p_i \cdot p_j\right)}\right)$, *with* $i, j \leq n-1$.

**Theorem 4.2.** *The leading singularity of an n-point one-loop Feynman integral in* $D = n$ *space-time dimensions is equal to* $\pm 1/\left(2^n\sqrt{(-1)^{D-1}\mathrm{LanS}}\right)$

## Extension at multi-loop level

**Theorem 5.2.** *The leading singularity of the four-point L-loop ladder Feynman integral of Fig. 4(b) in four space-time dimensions with off-shell external momenta* ($p_i^2 \neq 0$ *for* $i = 1,2,3,4$) *and massless propagators is equal to* $\left(s^L t\sqrt{\lambda_K\left(1, \frac{p_1^2 p_3^2}{st}, \frac{p_2^2 p_4^2}{st}\right)}\right)^{-1}$, *with* $s = (p_1+p_2)^2$ *and* $t = (p_2+p_3)^2$.



# *Analytic evaluations*

Standard approach

- DEQ :: Feynman integrals are not independent

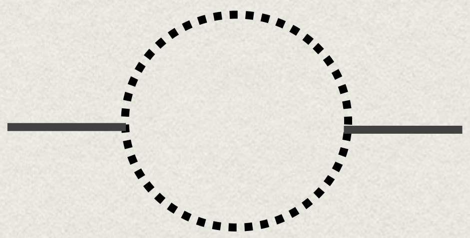
$$\partial_x \overrightarrow{J}(x) = A_i(x,\epsilon)\overrightarrow{J}(x)$$

*<<Fael's*

*Canonical form*

Conjecture: there exist a basis of uniform transcendental weight functions (UT basis)

$m^2 = 0$



$$\frac{1}{\epsilon}(-p^2)^{-1-\epsilon}\left(-2+\frac{\pi^2}{6}\epsilon^2+\frac{14}{3}\zeta_3\epsilon^3+\mathcal{O}\left(\epsilon^4\right)\right)$$

Uniform weight function

$$\partial_x \overrightarrow{g}(x) = \epsilon B(x)\overrightarrow{g}(x) \longrightarrow d\overrightarrow{g}(x,\epsilon) = \epsilon\left(d\tilde{B}\right)\overrightarrow{g}(x;\epsilon) \longleftarrow g(x,\epsilon) = \frac{1}{\epsilon^{2L}}\sum_{k>0}\epsilon^k g^{(k)}(x)$$

$$\tilde{B} = \sum_k B_k \log \alpha_k(x)$$

$g^{(k)}(x)$ has weight $k$

**[Henn (2013)]**

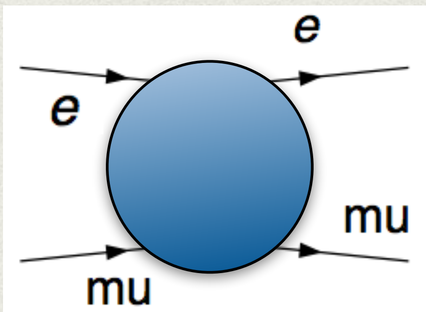
- Solution in terms of iterated integrals :: HPL/GPL (PolyLogs)

$$\mathcal{G}\left(a_1,\ldots,a_n;x\right) = \int_0^x dt\frac{1}{t-a_n}\mathcal{G}\left(a_1,\ldots,a_{n-1};t\right)$$

Numerical implementations:
`GinaC`, `HandyG`, `FastGPL`, ...

# *Analytic evaluations*

The simplest application :: $e\mu \to e\mu$

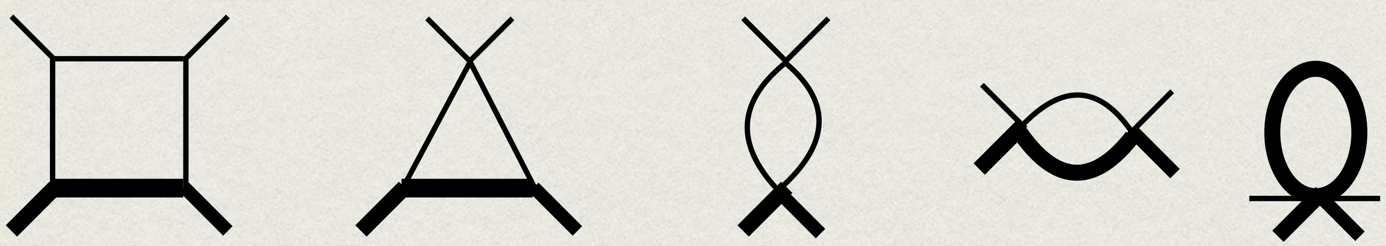



$$s = (p_1 + p_2)^2\,,$$
$$t = (p_2 - p_3)^2$$
$$u = (p_2 - p_3)^2 = 2m_\mu^2 - s - t$$

$$\mu(p_1) + e(p_2) \to e(p_3) + \mu(p_4)$$

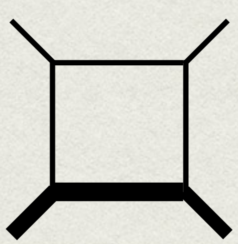
- @1L —> reduces to a set of 5 master integrals

*<<Jonathan's*

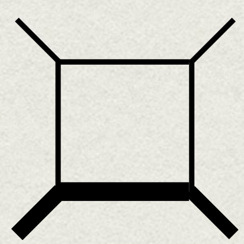


$\longrightarrow$ $\partial_x \vec{I} = A_x(d, s, t, m_\mu^2)\vec{I}$

- Calculate integrals in $d = 4 - 2\epsilon$ and redefine MIs



$\longrightarrow$ $(s - m^2)t$



$\longrightarrow$ $\partial_x \overrightarrow{g} = \epsilon\, A_x(s, t, m_\mu^2)\,\overrightarrow{g}$

$x = s, t, m_\mu^2$

# *State of the art for massless loops*

*courtesy of Antonela Matijašić, Julian Miczajka, and Won Lim*

| # of loops / # of points | 1 | 2 | 3 |
| --- | --- | --- | --- |
| 4 | ✓ | 2-mass planar & non-planar [Henn, Melinkov, Smirnov '14; Gehrmann et al. '14, Caola et al. '14] | Massless planar & non-planar [Henn, Smirnov, Smirnov '13; Henn, Mistlberger, Smirnov, Wasser '20] 1-mass planar [Canko, Syrrakos '22] |
| 5 | ✓ | Massless planar & non-planar [Gehrmann , Henn, Lo Presti '15; Chicherin et al. '18, Chicherin et al. '19] 1-mass planar [Abreu et al. '20] 3 non-planar 1-mass topologies [Abreu et al. '21] | X |
| 6 | $\mathcal{O}(\varepsilon^0)$ [Henn, Drummond, Dixon '11; Spradlin, Volovich '11] $\mathcal{O}(\varepsilon)$ [Henn, Matijašić, Miczajka '22] | Massless, planar on the maximal cut [Henn, Peraro, Xu, Zhang '21] | X |

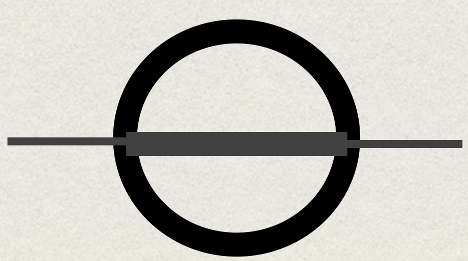
↳ construction of UT basis is known for any # of points [Spradlin, Volovich '11]

# *State of the art for 2—>2 process w/ internal masses*

- Jet production :: [Czakon++ '19], [Chen++ '22]
- $t\bar{t}$ production :: [Bonciani++ '08,'11,'13], [Czakon '08], [DiVita++ '19], [Mastrolia++ 22], [Adams++ '18], [Badger++ '22]
- $pp \to H + jet$ :: [Salvatori++ '20,'22], [Bonetti++ '22]
- $e\mu$ scattering :: [Henn++ '13], [DiVita++ '18], [Duhr++ '21], [Mastrolia++ 21]
- $HH, ZH, ZH$ production :: [Grazzini++ '18], [Chen++ '21]
- Drell-Yan :: [Bonciani '16], [Heller '20], [Armadillo '22]
- ...

Unavoidable appearance of *elliptic integrals*

$m^2 \neq 0$



$$-\frac{4\mathrm{K}(\lambda)}{(p^2+m^2)\sqrt{a_{13}a_{24}}}\left[2\mathcal{E}_4\begin{pmatrix}0 & -1 \\ 0 & \infty\end{pmatrix};1,\vec{a}\bigg) + \mathcal{E}_4\begin{pmatrix}0 & -1 \\ 0 & 0\end{pmatrix};1,\vec{a}\bigg) + \mathcal{E}_4\begin{pmatrix}0 & -1 \\ 0 & 1\end{pmatrix};1,\vec{a}\bigg)\right]$$

$$\mathrm{K}(\lambda) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\lambda x^2)}} \qquad \mathcal{E}_4\left(\begin{matrix}n_1 & \dots & n_k \\ c_1 & \dots & c_k\end{matrix};t,\vec{a}\right) = \int_0^x dt\,\Psi_{n_1}(c_1,t,\vec{a})\,\mathcal{E}_4\left(\begin{matrix}n_2 & \dots & n_k \\ c_2 & \dots & c_k\end{matrix};t,\vec{a}\right)$$

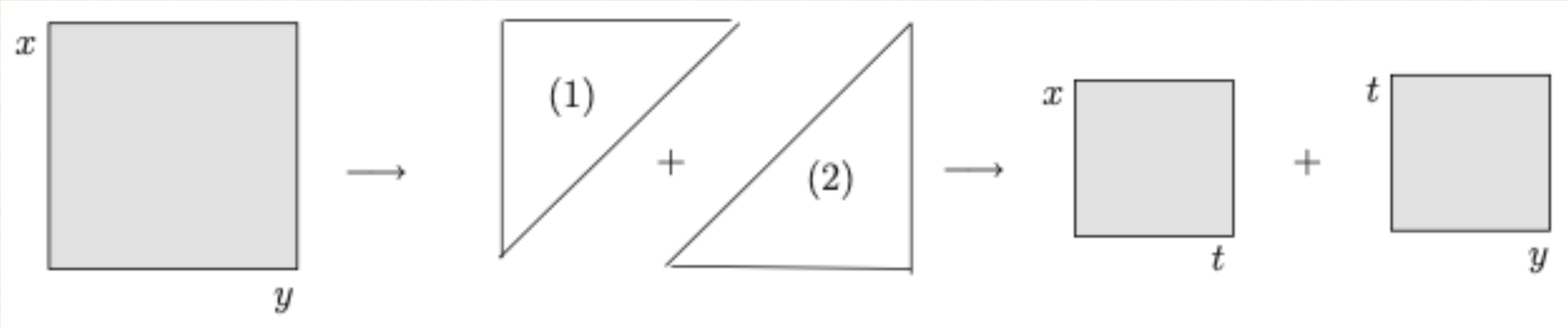
*—> elliptic integrals*

# Numerical evaluation

# *Sector decomposition*

[Binoth, Heinrich (2000)]

Schematically




$$I = \int_0^1 dxdy\, f(x,y) \quad = \int_0^1 dxdy\, f(x,y)(\Theta(\mathbf{x}-\mathbf{y}) + \Theta(\mathbf{y}-\mathbf{x})) \; = \int_0^1 dxdy\, f(xy,y) + \int_0^1 dxdy\, f(x,xy)$$

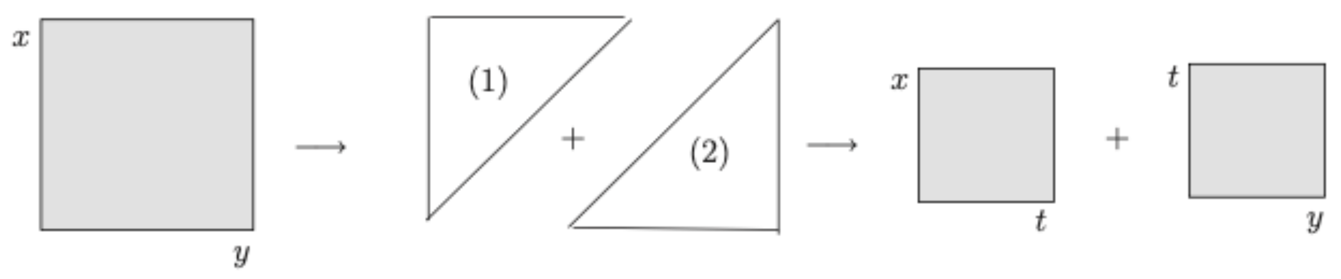
*Apply it to Feynman integrals*

$$J_N \sim \Gamma(n - LD/2) \int_{x_j \geq 0} \left[ d^{n-1} x_j \right] \frac{\mathcal{U}^{n-(L+1)D/2}}{\mathcal{F}^{n-LD/2}}$$

*"Symanzik form" of Feynman integrals*

- Subtraction of poles & expansion in $\epsilon$
- Numerical evaluation of finite integrals at each order of $\epsilon$
- Evaluation of integrals in unphysical regions :: easy
- Evaluation of integrals in physical regions :: thresholds :: contour deformation

# *Sector decomposition*




*Recent developments*

$$I = \int_{\mathbb{R}_+^d} d^d X \prod_{i=1}^{d} X_i^{a_i - 1} \prod_{j=1} P_j(X)^{-c_j} \qquad \text{with} \qquad P_j(X) = \sum_k c_k^{(j)} \prod_{i=1}^{d} X_i^{v_i^{(j)}}$$

**[Arkani-Hamed, Hillman, Mizera (2022)]**

- Tropicalisation :: $X_i = \mathrm{e}^{x_i}$
- Study of asymptotic regions in the integration domain (e.g. $x_i$ scales to infinity)
- Leading asymptotic behaviour of the integral *along a given ray* :: $\mathrm{e}^{\mathrm{Trop}}$

$$\mathrm{Trop}(x) = a \cdot x - \sum_j c_j \max\left(x \cdot v^{(j)}\right)$$

- Problem of leading divergence is reduced to determining the values of Trop along the extremal rays of the divergent cones.
- Case of one polynomial :: Newton polytope $\mathrm{Newt}\left(P_j\right)$

*more details in Giulio's talk*

# *Loop-tree duality*

**[Rodrigo, W.J.T. *et al* (2020)]**

- In multi-loop Feynman integral, re-write Feynman propagators:

$$G_F(q_{i_S}) = \frac{1}{q_{i_S}^2 - m_{i_S}^2 + \imath 0} = \frac{1}{q_{i_S,0}^2 - (q_{i_S,0}^{(+)})^2}$$

In terms of spatial components

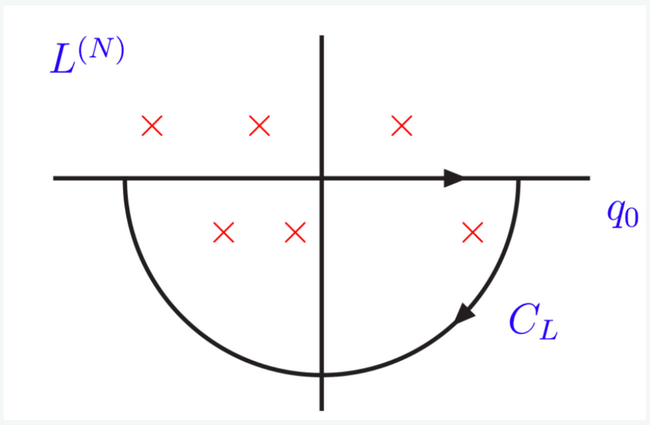
$$q_{i_S,0}^{(+)} = +\sqrt{\mathbf{q}_{i_S}^2 + m_{i_S}^2 - \imath 0}$$

Pull out full dependence of the energy components of loop momenta

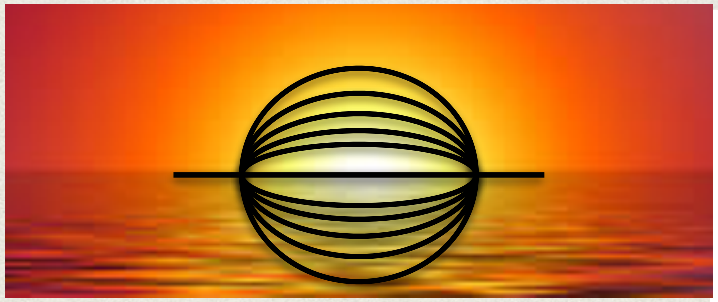
usual Feynman $\imath 0$ prescription!

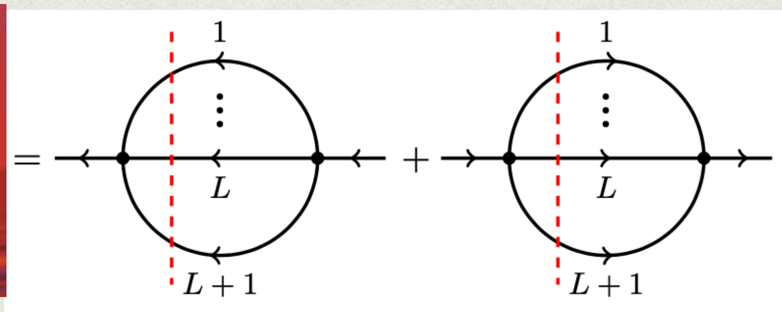
- *Apply the Cauchy residue thm for each "energy" integration*

Cauchy contour is always closed from below the real axis




- Representation of Feynman integrals in terms of only causal thresholds





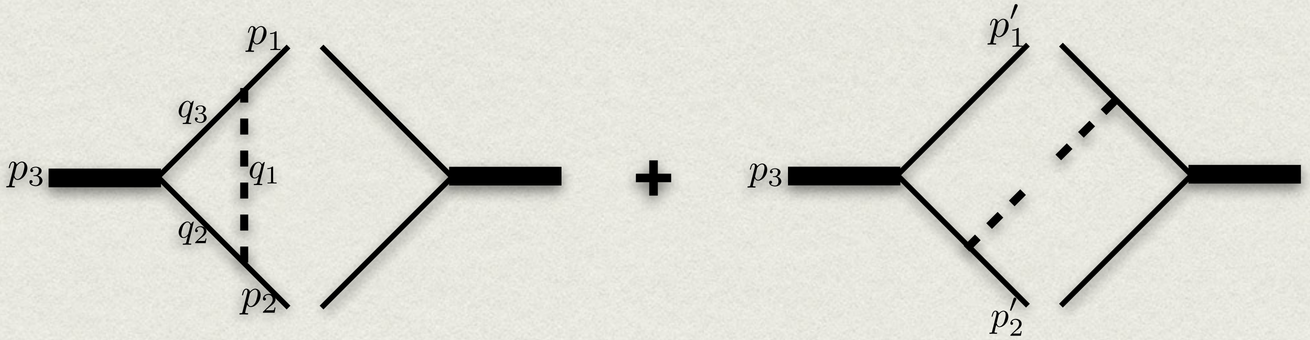

$$= -\int_{\vec{\ell}_1,\cdots,\vec{\ell}_L} \frac{1}{x_{L+1}} \left( \frac{1}{\lambda_1^-} + \frac{1}{\lambda_1^+} \right)$$

# *Loop-tree duality*

- Open loops into connected trees
- Cancel singularities (IR & UV) before integrating

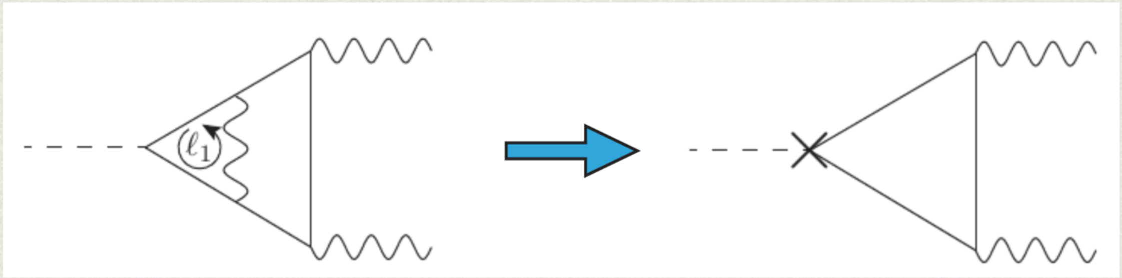
IR singularities @1L

[Kinoshita (1962)]
[Lee and Nauenberg (1964)]




Match real and virtual contribution before integrating

UV singularities




perform UV renormalisation @ integrand level
—> introduce an unintegrated UV counter-term

- Geometry understanding from cosmological polytope

[Benincasa, W.J.T. *et al* (2021)]

- Ultimate goal :: Numerically evaluate Feynman integrals in strictly D=4

# *Numerical evaluations*

- Additional tools/approaches

  — Auxiliary Mass Flow :: DEQ in $x \sim \imath 0$ [Liu++ ’17,’20,’22]
  — Series expansions :: solve DEQ along path [Moriello ’19] —> (`DiffExp`, `SeaSyde`)

# Conclusions

# Conclusions

- Lot of progress in analytic calculations involving squared roots & elliptic integrals!
- relation between Landau and leading singularities for multi-loop integrals
- Include 5pt processes w/ internal masses (e.g. $pp \to t\bar{t}j$)
- Complete missing calculations (e.g. $pp \to Hj@3L$ in the effective theory)
- Automation in the calculation of elliptic integrals?
- Extend and understand canonical form in DEQ for elliptic integrals

- Automated evaluations of (many) Feynman integrals (`SecDec, Fiesta, …`)
- Deal w/ bottlenecks :: evaluation of integrals in the physical region
- Make extensive use of tropical geometry in numerical evaluations of not only leading poles of Feynman integrals or Feynman integrals
- Use LTD @ NNLO :: cancellation of IR & UV before loop integrations