the hadronic running of α from the lattice

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based on JHEP08(2022)220 [arXiv:2203.08676]
with
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A. Risch, T. San José, J. Wilhelm, H. Wittig

17 November 2022
The Evaluation of the Leading Hadronic Contribution to the Muon $g-2$:
Toward the MUonE Experiment
workshop at MITP Mainz

the running of the electromagnetic coupling α

the QED coupling $\alpha = g^2/(4\pi)$ runs with energy

- in the Thomson limit ($q^2 \to 0$), the fine-structure constant is known at 0.23 ppb
  $\alpha^{-1} = \alpha(0)^{-1} = 137.035\,999\,139(31)$ [PDG 2018, CODATA 2014]
- at the Z pole, $\hat{\alpha}^{(5)}(M_Z)^{-1} = 127.955(10)$ (in the $\overline{\text{MS}}$ scheme)

$$\alpha(q^2) = \frac{\alpha}{1 - \Delta\alpha(q^2)} \qquad \Delta\alpha_{\text{had}}(q^2) = 4\pi\alpha\,\text{Re}\,\bar{\Pi}(q^2), \qquad \bar{\Pi}(q^2) = \Pi(q^2) - \Pi(0)$$

main uncertainty in the Z-pole value: the hadronic contribution to the running,
proportional to the subtracted hadronic vacuum polarization (HVP) function $\bar{\Pi}(q^2)$

- extracted from the exp. R-ratio data via dispersive integral (data-driven method) [Erler 1999; Davier *et al.* 2017; PDG 2018]

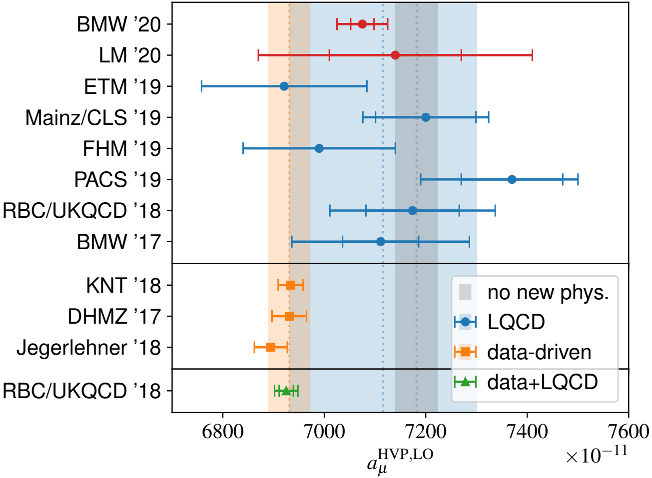
$$\text{Re}\,\bar{\Pi}(q^2) = \frac{q^2}{12\pi} P \int_{m_\pi^2}^{\infty} \frac{R(s)}{s(s-q^2)}\,\text{d}s\,, \qquad \text{Im}\,\bar{\Pi}(q^2) = \frac{R(q^2)}{12\pi}, \qquad R(s) = \frac{\sigma_{e^+e^-\to\text{hadrons}}(s)}{4\pi\alpha^2/(3s)}$$

$$\Delta\alpha^{(3)}_{\text{had}}(4\,\text{GeV}^2) = 58.71(50)\times 10^{-4}, \qquad \Delta\alpha^{(5)}_{\text{had}}(M_Z^2) = 0.027\,64(7), \qquad \text{(on-shell scheme)}$$

- or computed on the lattice [Burger *et al.* 2015; Francis *et al.* Lattice 2015; Borsanyi *et al.* 2018; MC *et al.* Lattice 2019; MC *et al.* 2022]

$$\bar{\Pi}(-Q^2) = -\frac{1}{3}\int \text{d}^4x\, \text{e}^{\text{i}Q\cdot x} \langle j^\gamma_\mu(x) j^\gamma_\mu(0)\rangle$$

the leading-order HVP contribution to a_μ



- extracted from the exp. R-ratio data via dispersive integral (data-driven method)

$$a_\mu^{\mathrm{HVP,LO}} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{m_\pi^2}^{\infty} \frac{\hat{K}(s)}{s^2} R(s)\,\mathrm{d}s$$

$$\hat{K}(4m_\pi^2) \approx 0.63, \quad \lim_{s\to\infty} \hat{K}(s) = 1$$

- or computed on the lattice

$$a_\mu^{\mathrm{HVP,LO}} = 4\alpha^2 \int_0^{\infty} \mathrm{d}Q^2\, f(Q^2)\bar{\Pi}(-Q^2)$$

with $f(Q^2) \geq 0$ a known QED kernel

⇒ tension between lattice and data driven

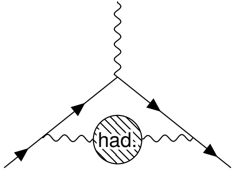
stronger evidence on the intermediate $a_\mu^{\mathrm{HVP,LO}}$ window

[previous talks by G. Colangelo and D. Giusti]

[MC *et al.* (Mainz/CLS) 2022; Alexandrou *et al.* (ETMc) 2022]

[Lehner (RBC/UKQCD) at 5th pl. workshop Edinburg 2022; Gottlieb (Fermilab/HPQCD/MILC) at Benasque 2022]

motivation – the HVP connection



$\Leftrightarrow$

if the lattice QCD confirms the larger value of $a_\mu^{\mathrm{HVP,LO}}$

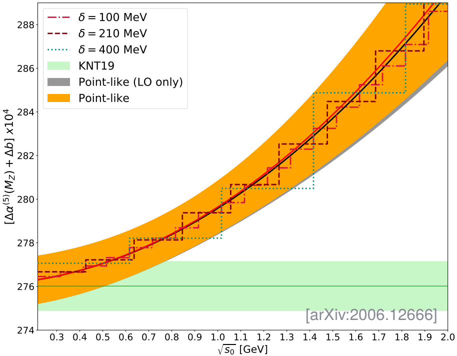
$$a_\mu^{\mathrm{HVP,LO}} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{m_\pi^2}^\infty \mathrm{d}s\, \frac{\hat{K}(s)}{s^2} R(s), \qquad \bar{\Pi}(-Q^2) = \frac{Q^2}{12\pi^2} \int_{m_\pi^2}^\infty \mathrm{d}s\, \frac{R(s)}{s(s+Q^2)}$$

with $0.63 \lesssim \hat{K}(s) < 1 \Rightarrow R(s)$ is larger for some s, $\bar{\Pi}(-Q^2)$ is also larger!

$\Rightarrow$ depending on the energy bin of the increase, also $\Delta\alpha_{\mathrm{had}}(M_Z^2)$ is affected

[Passera, Marciano, Sirlin 2008; Crivellin, Hoferichter, Manzari, Montull 2020; Keshavarzi, Passera, Marciano, Sirlin 2020; Colangelo, Hoferichter, Stoffer 2021]

motivation – global Standard Model fits



if the hadronic cross-section increases [Keshavarzi, Passera, Marciano, Sirlin 2020]

- at higher $\sqrt{s}$:
  the increase in $\Delta\alpha^{(5)}(M_Z)$ is in tension with global SM fits
  $\Rightarrow M_W, \sin^2\theta^{\mathrm{lep}}_{\mathrm{eff}}, M_H$ exclude shifts for $\sqrt{s} > 0.7\,\mathrm{GeV}$ at $95\,\%$ C.L.
- below $0.7\,\mathrm{GeV}$ (ρ-resonance region):
  no significant change in $\Delta\alpha_{\mathrm{had}}$, no tension in global SM fits,
  a $9\,\%$ increase of the integrated cross-section would solve the $(g-2)_\mu$ discrepancy
  $\Rightarrow$ tension with the experimental hadronic cross-section data!

can we use lattice computations to say something more about this?

yes, by studying the running with energy of the electromagnetic coupling!

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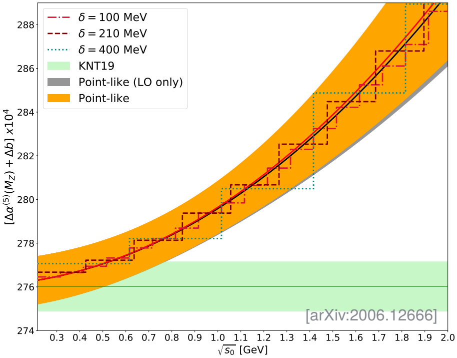

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motivation – t-channel scattering

the leading hadronic contribution to $(g-2)_\mu$ from the running of α [Lautrup, Peterman, de Rafael 1972]

$$a_\mu^{\mathrm{HVP,LO}} = \frac{\alpha}{\pi}\int_0^1 \mathrm{d}x\,(1-x)\Delta\alpha_{\mathrm{had}}(-Q^2), \qquad Q^2 = \frac{x^2 m_\mu^2}{1-x},$$

with the integrand peaked at $x \approx 0.914$, $Q^2 \approx 0.108\,\mathrm{GeV}^2$. [Carloni Calame *et al.*2015]

the MUonE experiment @ CERN: measure the energy dependence of α at space-like Q^2 [Abbiendi *et al.*2017]

- independent determination of $a_\mu^{\mathrm{HVP,LO}}$
- kinematic range $0 < x < 0.932$, corresponding to $Q^2 \lesssim 0.14\,\mathrm{GeV}^2$
- $0.932 < x < 1$ or $Q^2 \gtrsim 0.14\,\mathrm{GeV}^2$ accounts for $13\,\%$ of $a_\mu^{\mathrm{HVP,LO}}$

lattice computation in intermediate region $Q^2 = 0.14 - 4\,\mathrm{GeV}^2$ is complementary to MUonE kinematics

the time-momentum representation (TMR) method

uses the zero-momentum-projected Euclidean-time correlator

$$G(t) = -\frac{1}{3}\int \mathrm{d}^3x \sum_{k=1}^{3}\langle j_k(x) j_k(0)\rangle,$$

and known kernel functions $K(t,Q^2)$ and $w(t)$ [Bernecker, Meyer 2011; Francis *et al.* 2013]

$$\bar{\Pi}(-Q^2) = \int_0^\infty \mathrm{d}t\, K(t,Q^2) G(t), \qquad K(t,Q^2) = t^2 - \frac{4}{Q^2}\sin^2\left(\frac{Qt}{2}\right),$$

$$a_\mu^{\mathrm{HVP,LO}} = \int_0^\infty \mathrm{d}t\, w(t) G(t), \qquad w(t) = 4\pi^2 \int_0^\infty \mathrm{d}Q^2\, f(Q^2) K(t,Q^2)$$

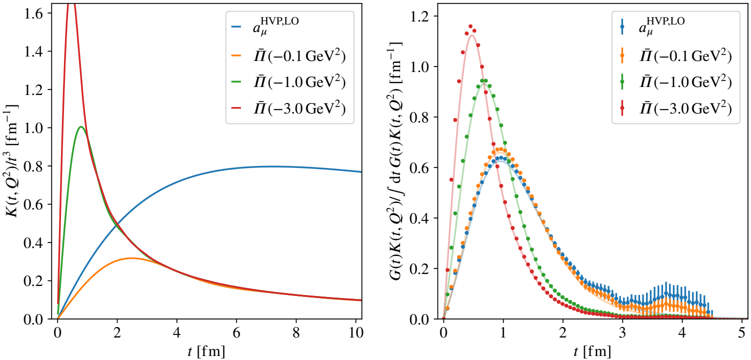
w.r.t. the traditional approach

- same statistical power
- better understanding of the systematics
  - finite-size effects correction • (improved) bounding method
- in principle, $\bar{\Pi}(-Q^2)$ can be computed for any Q^2

similar alternative approach: time moments [Chakraborty 2014]

the TMR method – the kernel

- the $a_\mu^{\mathrm{HVP,LO}}$ kernel is very long range
- the $\bar{\Pi}(-Q^2)$ kernel has a shorter range depending on Q^2



$\bar{\Pi}(-Q^2)$: $K(t,Q^2) = t^2 - \dfrac{4}{Q^2}\sin^2\left(\dfrac{Qt}{2}\right)$, $\qquad a_\mu^{\mathrm{HVP,LO}}$: $w(t) = 4\pi^2 \displaystyle\int_0^\infty \mathrm{d}Q^2\, f(Q^2) K(t,Q^2)$

systematic effects

controlling the tail of the correlator ⇒ main source of statistical uncertainty

- the bounding method is used [Lehner LGT2016; Gérardin, MC *et al.* 2019]
- crucial to control the statistical error on $a_\mu^{\mathrm{HVP,LO}}$, less critical for $\bar{\Pi}(-Q^2)$

simulations of finite-size lattices ⇒ correction of finite-size effects

- computing FSE on the zero-momentum correlator $G(t)$ [Lüscher 1991; Lellouch, Lüscher 2000; Meyer 2011]
- with $m_\pi L \approx 4$ ($L \approx 6\,\mathrm{fm}$ at the physical point), about $1\,\%$ shift fully under control

extrapolation to the continuum limit, extra- or interpolation to physical meson masses

- ensembles around physical meson masses are used by almost all collaborations
- continuum-limit extrapolation with $\sim a^2$ and $\sim a^3$ effects at $Q^2 \gtrsim 2.5\,\mathrm{GeV}^2$

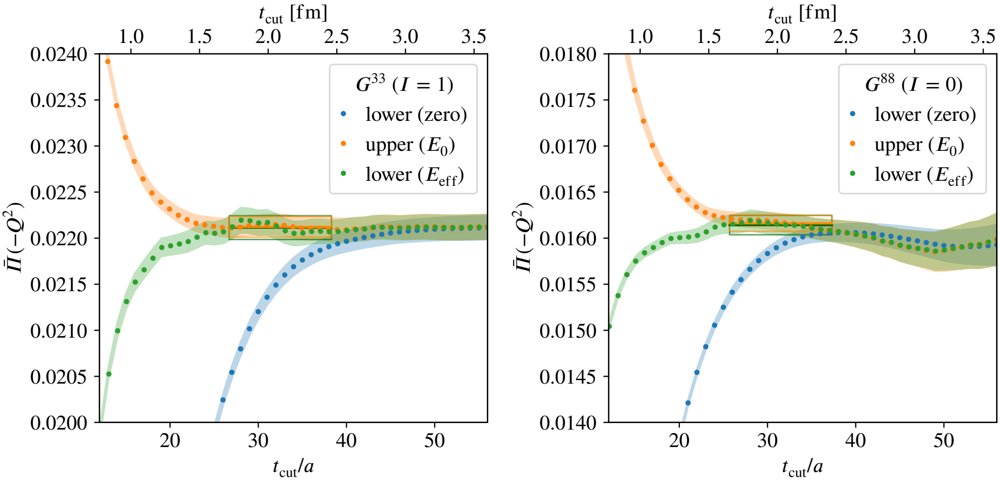
QED and strong isospin breaking corrections

- only available on a few ensembles, work in progress ⇒ included as systematics [Risch, Wittig Lattice 2019; Lattice 2021]
- $\approx 0.3\,\%$, small contribution to the total error

scale setting systematics

- a $1\,\%$ uncertainty on the scale is a $\approx 2\,\%$ systematic error on $a_\mu^{\mathrm{HVP,LO}}$, $\approx 1\,\%$ on $\bar{\Pi}(-1\,\mathrm{GeV}^2)$ (Q^2 dependent)
  ⇒ a per-mille level scale determination is needed

bounding method – example



$$0 \le G(t_{\text{cut}})e^{-E_{\text{eff}}(t_{\text{cut}})(t-t_{\text{cut}})} \le G(t) \le G(t_{\text{cut}})e^{-E_0(t-t_{\text{cut}})}, \qquad \text{for } t \ge t_{\text{cut}},$$

with $aE_{\text{eff}}(t) = \log(G(t)/G(t+a))$ and E_0 ground state in the channel

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correction of finite-size effects

added to the $I = 1$ correlator $G^{33}(t)$, with two different strategies and $t_i = (m_\pi L/4)^2/m_\pi$ [Gérardin, MC *et al.* 2019]

$t < t_i$: HP method with $\vec{n}^2 \leq 3$, with the size of the $\vec{n}^2 = 3$ level included as a systematic error

$t > t_i$: average of MLL-GS and HP methods, with the half-difference included as an extra systematic error

⇒ a model is used only for the small correction $G^{33}(t, L) - G^{33}(t, \infty)$

explicit check of FSE with two pair of ensembles at different volume and otherwise identical simulation parameters

- H105 and N101 at $m_\pi \approx 280\,\mathrm{MeV}$
- H200 and N202 at the $\mathrm{SU}(3)$-symmetric point

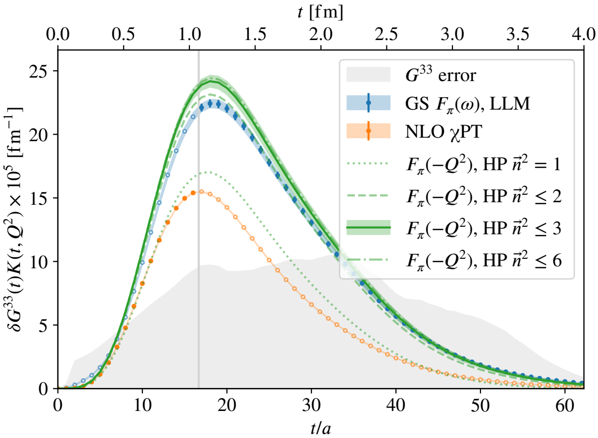
we observe a good agreement between the MLL-GS and HP methods, especially for $t \gtrsim 2\,\mathrm{fm}$,
with the two methods relying on very different input ⇒ robustness of the evaluation of finite size effects

correction of finite-size effects – Hansen-Patella (HP) method

series expansion in $\exp\{-|\vec{n}|m_\pi L\}$, $\vec{n}^2 = 1, 2, 3, 6, \ldots$ [Hansen, Patella 2019; 2020]

- implementation by K. Miura
- neglects $\exp\{-1.93 m_\pi L\}$ contributions
- fast convergence at short and medium distances
- input: forward Compton amplitude, $F_\pi(Q^2)$ monopole ansatz with $M^2 = 0.517(23)\,\mathrm{GeV}^2 + 0.647(30) m_\pi^2$

[Brommel *et al.* 2007]



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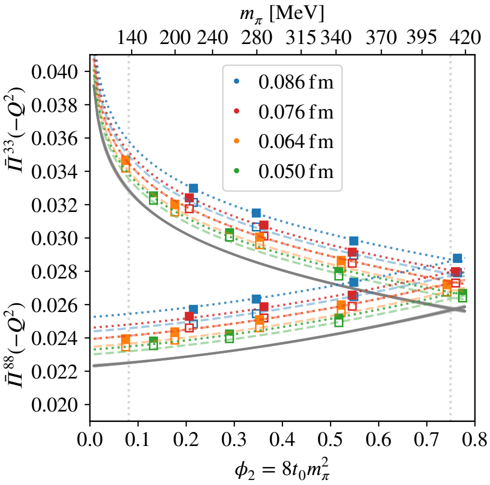
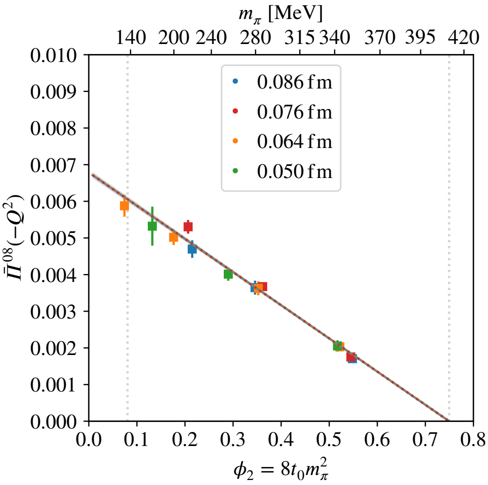
extrapolation results

at $Q^2 = 1.0\,\mathrm{GeV}^2$





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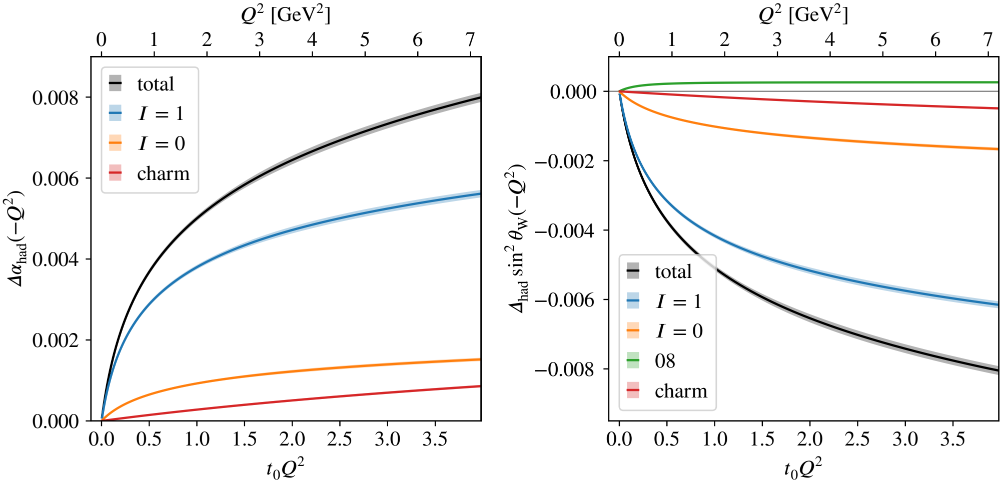
running with energy – results



running with energy – rational approximation

the running is obtained varying Q^2 in the TMR kernel ⇒ each Q^2 choice is a different fit

we present the results

- tabulated for selected values of Q^2 up to $7\,\mathrm{GeV}^2$
- with a rational approximation that interpolates to other values of Q^2

$$\bar{\Pi}(-Q^2) \approx R^N_M(Q^2) = \frac{\sum_{j=0}^{M} a_j Q^{2j}}{1 + \sum_{k=1}^{M} b_k Q^{2k}}$$

with $M = N = 3$ and $a_0 = 0$, by solving the over-constrained system via a least-squares fit for $0.1 < x < 7$

$$\bar{\Pi}(-Q^2) \approx \frac{0.109\,4(23)\,x + 0.093(15)\,x^2 + 0.003\,9(6)\,x^3}{1 + 2.85(22)\,x + 1.03(19)\,x^2 + 0.016\,6(12)\,x^3}\,, \qquad x = \frac{Q^2}{\mathrm{GeV}^2}$$

- the correlation between coefficients is provided
- the rational approximations are provided also for the derivatives w.r.t. the meson masses

running with energy – lepton-inspired two-parameter approximation

for $Q^2 = -t > 0$,

$$\bar{\Pi}(-Q^2) \approx \frac{k}{4\pi\alpha}\left\{-\frac{5}{9} + \frac{4M}{3Q^2} + \left(\frac{4M^2}{3Q^4} - \frac{3M}{Q^2} - \frac{1}{6}\right)\frac{2}{\sqrt{1+4M/Q^2}}\log\left|\frac{1-\sqrt{1+4M/Q^2}}{1+\sqrt{1+4M/Q^2}}\right|\right\}$$

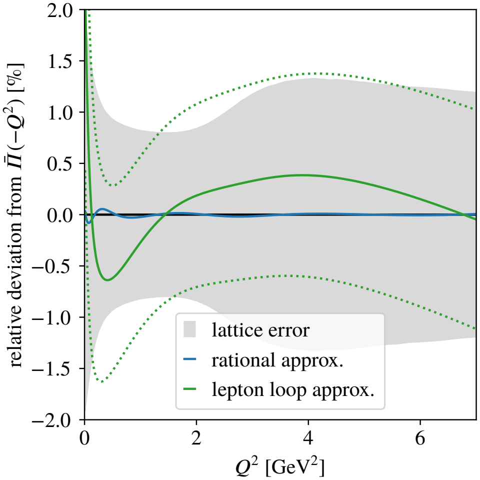
PRELIMINARY!

uncorrelated fit describes the data within better than $\pm 1\,\%$ for $0.1\,\mathrm{GeV}^2 < Q^2 < 7\,\mathrm{GeV}^2$ with

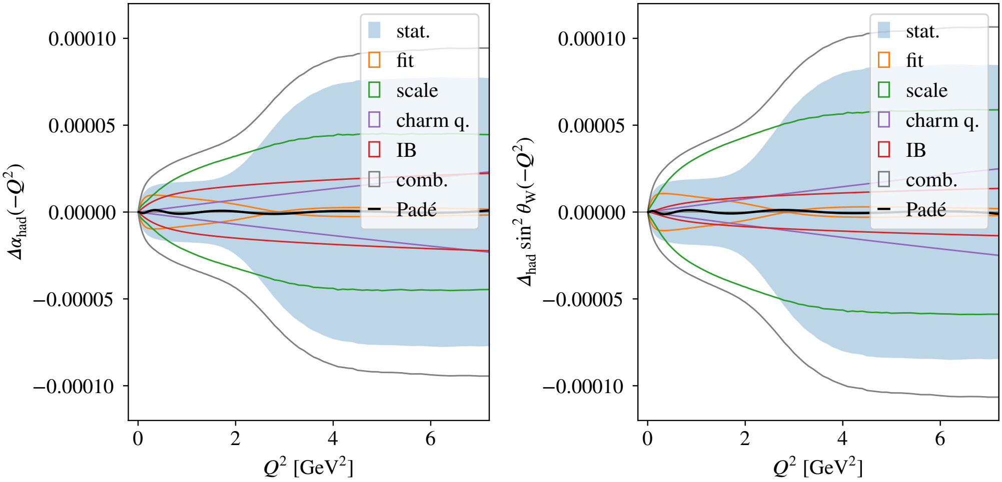
$$k = 0.006\,96(12)$$

$$M = [0.212\,2(35)\,\mathrm{GeV}]^2$$

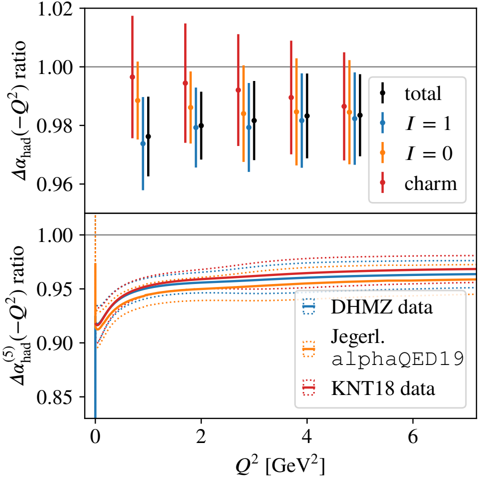
rational approximation with $N = M = 3$ (six parameters) is more precise

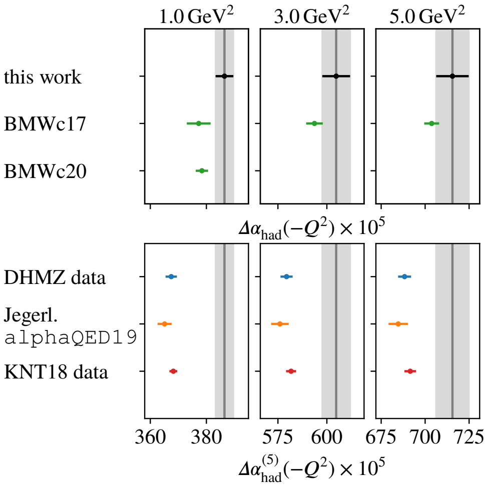


running with energy – summary of systematics



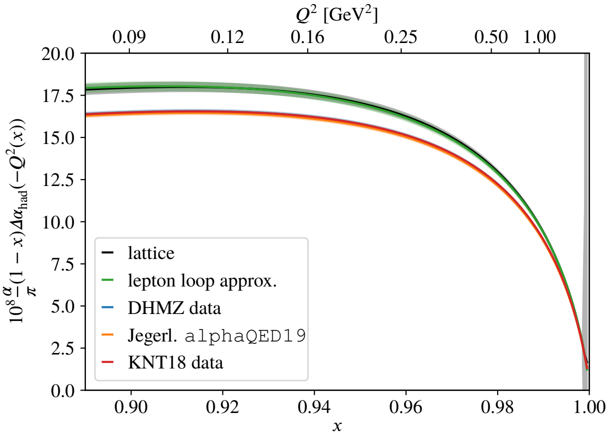
running with energy – comparison





running with energy – as function of x

where $Q^2 = \frac{x^2 m_\mu^2}{1-x}$



running to M_Z

we use the Euclidean split technique (or Adler function approach) [Chetyrkin *et al.* 1996; Eidelman *et al.* 1999, Jegerlehner 2008]

$$\Delta\alpha^{(5)}_{\mathrm{had}}(M_Z^2) = \Delta\alpha^{(5)}_{\mathrm{had}}(-Q_0^2) + \left[\Delta\alpha^{(5)}_{\mathrm{had}}(-M_Z^2) - \Delta\alpha^{(5)}_{\mathrm{had}}(-Q_0^2)\right] + \left[\Delta\alpha^{(5)}_{\mathrm{had}}(M_Z^2) - \Delta\alpha^{(5)}_{\mathrm{had}}(-M_Z^2)\right]_{\mathrm{pQCD}}$$

- $\Delta\alpha^{(5)}_{\mathrm{had}}(-Q_0^2)$ with Q_0^2 between 3 and 7 GeV^2 is evaluated on the lattice

$$\Delta\alpha^{(5)}_{\mathrm{had}}(-5\,\mathrm{GeV}^2) = 0.007\,16(9)$$

- $[\Delta\alpha^{(5)}_{\mathrm{had}}(-M_Z^2) - \Delta\alpha^{(5)}_{\mathrm{had}}(-Q_0^2)]$ from either pQCD or R-ratio data (KNT18)

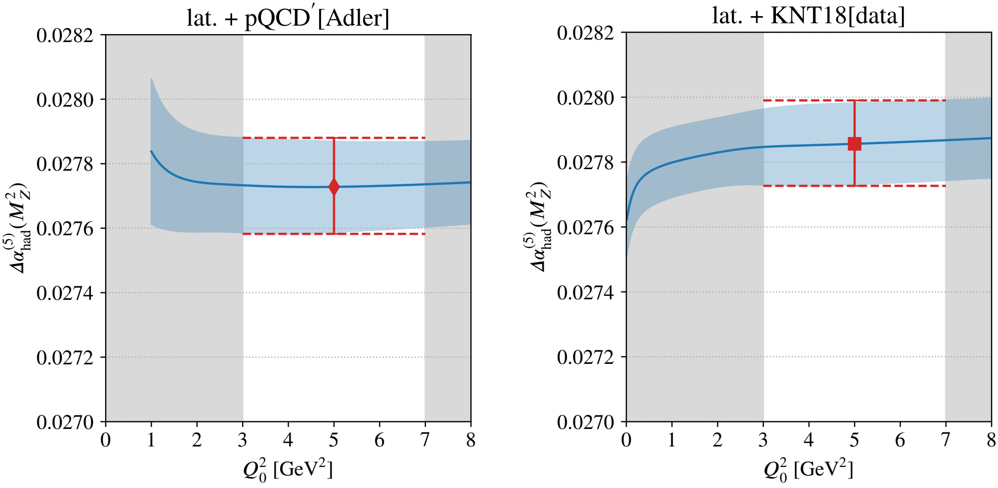
$$\left[\Delta\alpha^{(5)}_{\mathrm{had}}(-M_Z^2) - \Delta\alpha^{(5)}_{\mathrm{had}}(-5\,\mathrm{GeV}^2)\right] = 0.020\,53(11) \quad \text{or} \quad 0.020\,66(9)$$

- $[\Delta\alpha^{(5)}_{\mathrm{had}}(M_Z^2) - \Delta\alpha^{(5)}_{\mathrm{had}}(-M_Z^2)]_{\mathrm{pQCD}} = 0.000\,045(2)$ has a negligible error [Jegerlehner 1986, 2020]

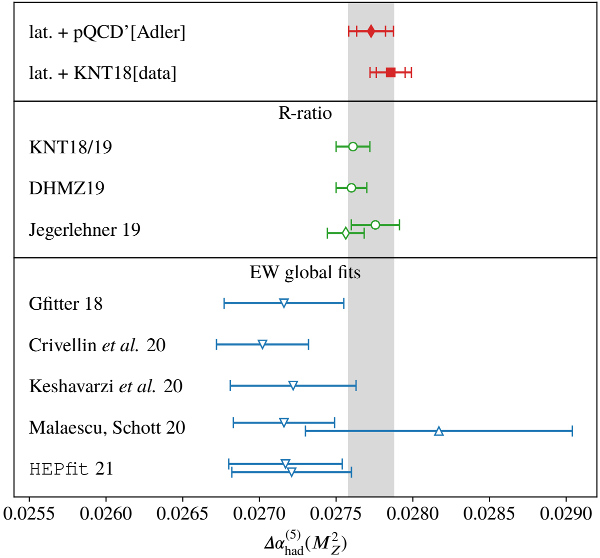
using pQCD ⇒ result independent from R-ratio input

$$\Delta\alpha^{(5)}_{\mathrm{had}}(M_Z^2) = 0.027\,73(15)$$

running to M_Z – results



running to M_Z – results and comparison



conclusions

we computed on the lattice the HVP contribution to the running of α

- with sub-percent statistical errors
- bounding method helps at small Q^2 • correction for finite-size effects is essential

$$\Delta\alpha_{\mathrm{had}}^{(5)}(-Q^2) = 0.007\,16(9) \qquad \text{at } Q^2 = 5\,\mathrm{GeV}^2$$

- approaching the precision of the data-driven estimate • up to 3.5σ tension with the data-driven estimate
- consistent with the lattice results for the intermediate $a_\mu^{\mathrm{HVP,LO}}$ window
- precision limited by current scale setting on CLS ensembles
- full correction for isospin breaking effects still missing [work in progress: Risch, Wittig Lattice 2019; Lattice 2021]

and the running up to M_Z

$$\Delta\alpha_{\mathrm{had}}^{(5)}(M_Z^2) = 0.027\,73(15)$$

- employing the Euclidean split technique and pQCD for the running at large $Q^2 \Rightarrow$ no R-ratio data dependency
- lattice contributes to $\approx 25\,\%$ of the value and up to $50\,\%$ of the error
- the result agrees with ones based on the R-ratio within the uncertainties
- lattice input does not introduce a tension in global EW fits

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outlook

implement changes already used for the muon $g-2$ window result [MC *et al.* (Mainz/CLS) 2022]

- mostly the same dataset
- two extra ensembles at the finer lattice spacing ($a \approx 0.039\,\mathrm{fm}$)
- alternative set of renormalization and improvement coefficients [Heitger, Joswig 2021; Fritzsch 2018]
- variety of fits combined with the Akaike Information Criterion (AIC)

and add full isospin breaking corrections, [work in progress: Risch, Wittig Lattice 2019; Lattice 2021]
improve the scale setting [e.g. Bali *et al.* (RQCD) arXiv:2211.03744]

computing higher Q^2 (up to M_Z^2) directly on the lattice?
using the discrete Adler function $\Delta_2(Q^2) = \Pi(-Q^2) - \Pi(-Q^2/4)$ [MC, Harris, Meyer, Toniato, Török 2021; Harris Lattice 2021]

- naively, $\Lambda \ll |Q| \ll a^{-1}$
- thermal effects are $\sim (\pi T/Q)^4 \Rightarrow$ using finite-temperature ensembles at $T = Q/8\pi$
- such that only $T \ll |Q| \ll a^{-1}$ needs to be satisfied
- different from step scaling with a finite-volume scheme $\Rightarrow$ the volume is parametrically large, *e.g.* $L \approx 4/T$
- similar proposal from BMWc [Frech Lattice 2021; Lattice 2022]

thanks
for your attention!

questions?

backup slides

the running of the electroweak mixing angle

the electroweak mixing (Weinberg) angle θ_W parametrizes the mixing between the $\mathrm{SU}(2)_L$ and $\mathrm{U}(1)_Y$ sectors of the Standard Model. At tree level,

$$\sin^2\hat{\theta}_\mathrm{W} = \frac{g'^2}{g^2+g'^2}, \quad \text{or} \quad \sin^2\theta_\mathrm{W} = 1 - \frac{M_W^2}{M_Z^2},$$

where g and g' are the $\mathrm{SU}(2)_L$ and $\mathrm{U}(1)_Y$ coupling respectively

- Z vector coupling $v_f = T_f - 2Q_f \sin^2\theta_f^\mathrm{eff}$
- weak charge of the proton $Q_W(p) \sim 1 - 4\sin^2\theta_\mathrm{W}(0)$

like the couplings, the mixing angle is renormalization scheme and energy dependent

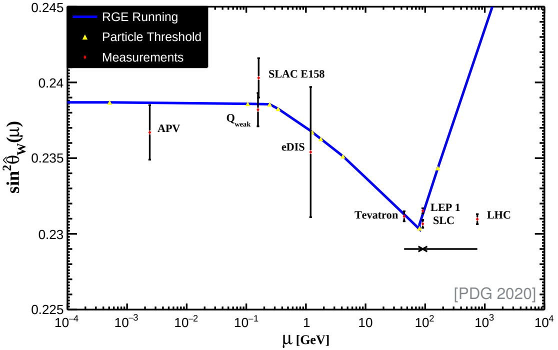
$$\sin^2\theta_\mathrm{W}(Q) = \sin^2\theta_\mathrm{W}(0)\big[1 + \Delta\sin^2\theta_\mathrm{W}(Q^2)\big],$$

and the leading hadronic contribution to the running [Jegerlehner 1986; 2011]

$$\Delta_\mathrm{had}\sin^2\theta_\mathrm{W}(Q^2) = -\frac{4\pi\alpha}{\sin^2\theta_\mathrm{W}}\bar{\Pi}^{Z\gamma}(Q^2), \qquad \bar{\Pi}^{Z\gamma}(Q^2) = \Pi^{Z\gamma}(Q^2) - \Pi^{Z\gamma}(0),$$

is proportional to the subtracted $Z\gamma$-mixing HVP function

the running of the electroweak mixing angle



experimental values measured at colliders enter global SM fits [PDG 2022]

$$\sin^2\hat\theta_W(M_Z) = 0.231\,22(4)$$

upcoming experiments at low Q^2

- P2 @ MESA, Mainz [Becker *et al.* 2018]
  $0.13\,\%$ target precision at $Q^2 = 0.005\,\mathrm{GeV}^2$
- MOLLER @ JLab [Benesch *et al.* 2014]

the running to the Thomson limit is affected by non-perturbative QCD physics that

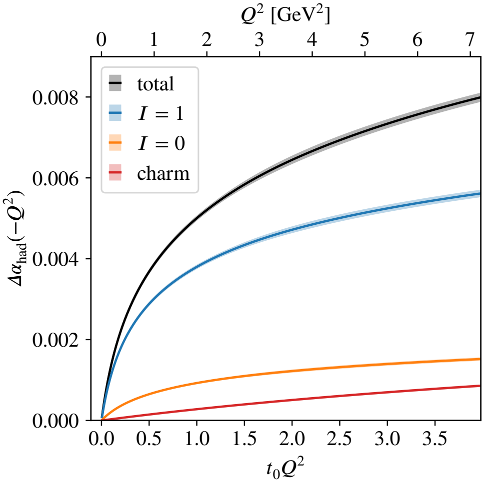
- can be extracted from hadronic cross-section data [Erler, Ferro-Hernández 2017]

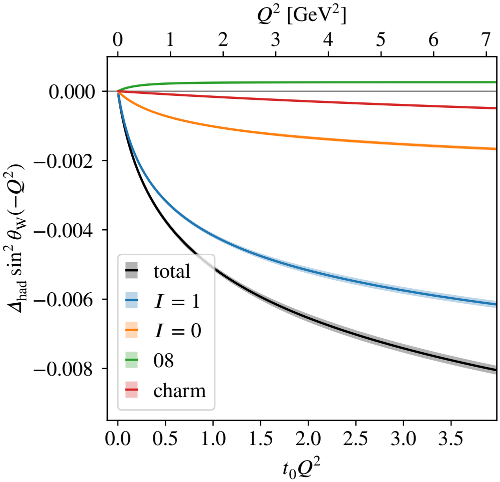
$$\sin^2\hat\theta_W(0) = 0.238\,68(5)(2), \qquad (\overline{\text{MS}}\text{ scheme})$$

  with additional input for flavor separation

- or can be computed on the lattice [Burger *et al.* 2015; Francis *et al.* 2015; Gülpers *et al.* 2015]
  ⇒ lattice easily provides flavour separation

he running of the electroweak mixing angle – results





the running of the electroweak mixing angle – conclusions

we present a result for the $Z\gamma$-mixing HVP contribution to the running of $\sin^2\theta_{\mathrm{W}}$

$$\bar{\Pi}^{08}(-Q^2) = \frac{0.021\,7(11)x + 0.015\,1(12)x^2}{1 + 2.93(8)x + 2.15(12)x^2}, \qquad x = \frac{Q^2}{\mathrm{GeV}^2}$$

that has a finite limit for large Q^2

$$\bar{\Pi}^{08}(-Q^2) = 0.007\,04(17) \qquad \text{for } Q^2 \gtrsim 7\,\mathrm{GeV}^2$$

- using flavor separation on the lattice
- most precise determination to date

ensembles

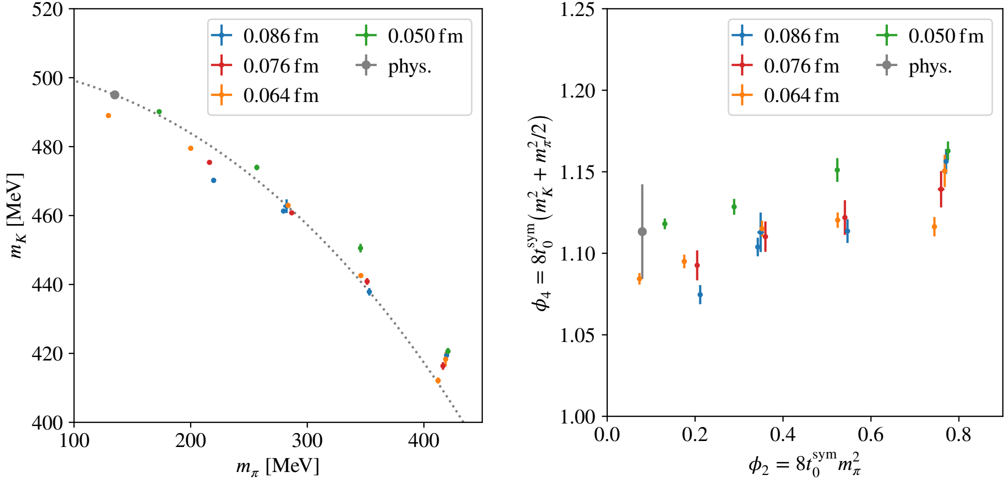
from the CLS initiative

[Bruno *et al.* 2015, Bruno, Korzec, Schaefer 2017]

tree-level Lüscher-Weisz gauge action, non-perturbatively $\mathcal{O}(a)$-improved Wilson fermions,
open BCs in time, except B450, N451, D450, and E250 that have periodic BCs in time,

	T/a	L/a	t_0^{sym}/a^2	a [fm]	L [fm]	m_π, m_K [MeV]		$m_\pi L$	#cnfg (con., dis.)	
H101	96	32	2.860	0.086	2.8	415		5.8	2 000	-
H102	96	32			2.8	355	440	5.0	1 900	1 900
H105	96	32			2.8	280	460	3.9	1 000	1 000
N101	128	48			4.1	280	460	5.8	1 155	1 155
C101	96	48			4.1	220	470	4.6	2 000	2 000
B450	64	32	3.659	0.076	2.4	415		5.1	1 600	-
S400	128	32			2.4	350	440	4.3	1 720	1 720
N451	128	48			3.7	285	460	5.3	1 000	1 000
D450	128	64			4.9	215	475	5.3	500	500
H200	96	32	5.164	0.064	2.1	420		4.4	1 980	-
N202	128	48			3.1	410		6.4	875	-
N203	128	48			3.1	345	440	5.4	1 500	1 500
N200	128	48			3.1	285	465	4.4	1 695	1 695
D200	128	64			4.1	200	480	4.2	2 000	1 000
E250	192	96			6.2	130	490	4.1	485	485
N300	128	48	8.595	0.050	2.4	420		5.1	1 680	-
N302	128	48			2.4	345	460	4.2	2 190	2 190
J303	192	64			3.2	260	475	4.2	1 040	1 040
E300	192	96			4.8	175	490	4.3	600	600

ensemble landscape



lattice correlators

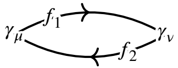
on $N_\mathrm{f} = 2 + 1$ ensembles from the CLS initiative [Bruno *et al.* 2015, Bruno, Korzec, Schaefer 2017]
with $\mathrm{SU}(3)_F$ notation, in the isospin-symmetric limit (light quark ℓ: either u or d):

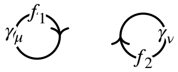
$I = 1$ contribution: $$G^{33}_{\mu\nu}(x) = \frac{1}{2} C^{\ell,\ell}_{\mu\nu}(x),$$

$I = 0$ contribution: $$G^{88}_{\mu\nu}(x) = \frac{1}{6}\Big[C^{\ell,\ell}_{\mu\nu}(x) + 2C^{s,s}_{\mu\nu}(x) + 2D^{\ell-s,\ell-s}_{\mu\nu}(x)\Big],$$

Z-γ mixing: $$G^{08}_{\mu\nu}(x) = \frac{1}{2\sqrt{3}}\Big[C^{\ell,\ell}_{\mu\nu}(x) - C^{s,s}_{\mu\nu}(x) + D^{2\ell+s,\ell-s}_{\mu\nu}(x)\Big],$$

where the connected and disconnected Wick's contractions are

$C^{f_1,f_2}_{\mu\nu} = -\Big\langle$ 

$\Big\rangle$, $D^{f_1,f_2}_{\mu\nu} = \Big\langle$ 

$\Big\rangle$

and the relevant correlators are given by (note: $G^{\ell}_{\mathrm{con}} = 2G^{33}$ and $G^{s}_{\mathrm{con}} = 3G^{88}_{\mathrm{con}} - G^{33}$)

$$G^{\gamma\gamma} = G^{33} + \frac{1}{3}G^{88} + \frac{4}{9}C^{c,c},$$

$$G^{Z\gamma} = \left(\frac{1}{2} - \sin^2\theta_\mathrm{W}\right)G^{\gamma\gamma} - \frac{1}{6\sqrt{3}}G^{08} + \frac{4}{9}\left(\frac{3}{8} - \sin^2\theta_\mathrm{W}\right)C^{c,c}.$$

renormalization and $\mathcal{O}(a)$ improvement

for the local current [Bhattacharya *et al.* 2006, […], Gérardin, Harris, Meyer 2018]

$$V^3_{\mu,R} = Z_V\big(1 + 3\bar{b}_V a m_q^{\mathrm{av}} + b_V a m_{q,\ell}\big) V_\mu^{3,I} = Z_3 V_\mu^{3,I},$$

$$\begin{pmatrix} V_\mu^8 \\ V_\mu^0 \end{pmatrix}_R = Z_V \begin{pmatrix} 1 + 3\bar{b}_V a m_q^{\mathrm{av}} + b_V \frac{a(m_{q,\ell}+2m_{q,s})}{3} & \left(\frac{b_V}{3} + f_V\right)\frac{2a(m_{q,\ell}-m_{q,s})}{\sqrt{3}} \\ r_V d_V \frac{a(m_{q,\ell}-m_{q,s})}{\sqrt{3}} & r_V 1 + (3\bar{d}_V + d_V) a m_q^{\mathrm{av}} \end{pmatrix} \begin{pmatrix} V_\mu^8 \\ V_\mu^0 \end{pmatrix}^I = \begin{pmatrix} Z_8 & Z_{80} \\ Z_{08} & Z_0 \end{pmatrix} \begin{pmatrix} V_\mu^8 \\ V_\mu^0 \end{pmatrix}^I$$

where

$$V_\mu^{a,I} = V_\mu^a + a c_V \partial_0 T_{0\mu}^a, \qquad V_\mu^{0,I} = V_\mu^0 + a\bar{c}_V \partial_0 T_{0\mu}^0.$$

while for the conserved current

$$V_{\mu,R}^a = V_\mu^a + a c_V^{\mathrm{cs}} \partial_0 T_{0\mu}^a, \qquad V_{\mu,R}^0 = V_\mu^0 + a\bar{c}_V^{\mathrm{cs}} \partial_0 T_{0\mu}^0.$$

⇒ we use only the conserved vector current for the flavor-singlet component, and we set

$$f_V = 0, \qquad \bar{c}_V = c_V \qquad \bar{c}_V^{\mathrm{cs}} = c_V^{\mathrm{cs}}.$$

bounding method

$$G(t) = \sum_{n=0}^{\infty} \frac{Z_n^2}{2E_n} \mathrm{e}^{-E_n t}$$

for a correlator with positive spectral decomposition, and $t > t_\mathrm{c}$

$$0 \leq G(t_\mathrm{c}) \mathrm{e}^{-E_\mathrm{eff}(t_\mathrm{c})(t-t_\mathrm{c})} \leq G(t) \leq G(t_\mathrm{c}) \mathrm{e}^{-E_0(t-t_\mathrm{c})},$$

where $E_\mathrm{eff}(t) = -(1/a) \log G(t+a)/G(t)$ is the effective mass
and E_0 is the ground state in the given channel, depending on the volume L^3 and on m_π
- for $I = 1$, $E_0 = m_\rho$ or $E_{2\pi}$, • for $I = 0$, $E_0 = m_\omega \approx m_\rho$ or $E_{3\pi}$

improved bounding method: [Lehner LGT2016; Gérardin, MC *et al.* 2019]
if $E_0, \ldots E_N$ and $Z_0, \ldots Z_{N-1}$ are available, one can bound the subtracted correlator

$$\tilde{G}(t) = G(t) - \sum_{n=0}^{N-1} \frac{Z_n^2}{2E_n} \mathrm{e}^{-E_n t},$$

that approaches zero faster ⇒ dedicated spectroscopy effort

extrapolation to the physical point

a combined fit of $\bar{\Pi}^{33}$, $\bar{\Pi}^{88}$ and $\bar{\Pi}^{08}$, with two discretization each (one discr. for $\bar{\Pi}^{08}$) is used

$$\bar{\Pi}^{33,X}(a^2/t_0^{\rm sym}, \phi_2, \phi_4) = \bar{\Pi}^{\rm sym} + \delta_2^X a^2/t_0^{\rm sym} + \gamma_1^{33}(\phi_2 - \phi_2^{\rm sym}) + \gamma_{\rm log}^{33} \log \phi_2/\phi_2^{\rm sym} + \eta_1(\phi_4 - \phi_4^{\rm sym}),$$
$$\bar{\Pi}^{88,X}(a^2/t_0^{\rm sym}, \phi_2, \phi_4) = \bar{\Pi}^{\rm sym} + \delta_2^X a^2/t_0^{\rm sym} + \gamma_1^{88}(\phi_2 - \phi_2^{\rm sym}) + \gamma_2^{88}(\phi_2 - \phi_2^{\rm sym})^2 + \eta_1(\phi_4 - \phi_4^{\rm sym}),$$
$$\bar{\Pi}^{08,{\rm CL}}(a^2/t_0^{\rm sym}, \phi_2, \phi_4) = \lambda_1(\phi_4 - 3/2\phi_2),$$

where $X = \mathrm{CL}$ or LL, $\phi_2 = 8t_0 m_\pi^2$, $\phi_4 = 8t_0(m_K^2 + m_\pi^2/2)$.

- we add also a $\delta_3^X a^3/(t_0^{\rm sym})^{3/2}$ ⇒ better fit at large Q^2 ⇒ smooth transition around $Q^2 = 2.5\,\mathrm{GeV}^2$
- $\sim a^2 \log a$ term? [MC, Harris, Meyer, Toniato, Török 2021]
  ⇒ assuming free theory coefficient, up to $0.4\,\%$ downward shift, within the statistical error
- extrapolation of the charm contribution done separately

definition of the isosymmetric QCD world

we set the scale with [Bruno, Korzec, Schaefer 2015]

$$\sqrt{8t_0} = 0.415(4)(2)\,\mathrm{fm}$$

and we define the isospin symmetric point as [discussion at the 4th Muon $g-2$ workshop at KEK (virtual), 2021]

$$m_\pi = m_{\pi^0} = 134.976\,8\,\mathrm{MeV}$$

$$m_K^2 - \frac{m_\pi^2}{2} = \frac{m_{K^\pm}^2 + m_{K^0}^2 - m_{\pi^\pm}^2}{2} \quad \Rightarrow \quad m_K = 495.011\,\mathrm{MeV}$$

and the valence charm quark mass is tuned to reproduce the physical D_s meson mass [Gérardin *et al.* 2019]

we also publish the derivatives w.r.t. ϕ_2 and $\phi_4 \Rightarrow m_\pi, m_K$