Lattice and Dispersive Models Input for MUonE Analysis

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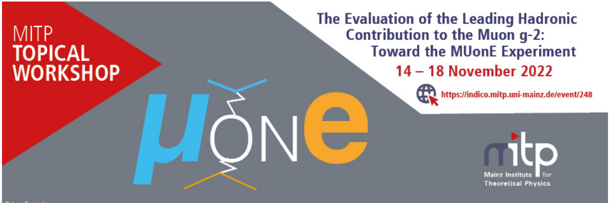
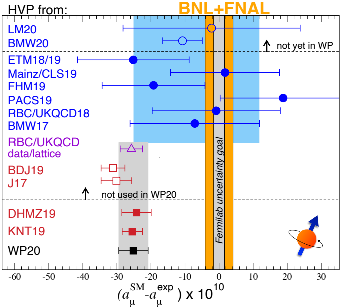
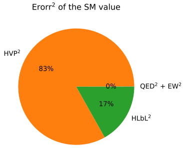




Contribution	value $\times 10^{11}$
QED	$116\,584\,718.931(104)$
Electroweak	$153.6(1.0)$
HVP	$6845(40)$
HLbL	$92(18)$
Total SM value	$116\,591\,810(43)$
Difference: $a_\mu^{\text{exp}} - a_\mu^{\text{SM}}$	$251(59)$

[arXiv:{2104.03281, 2006.04822, 2203.15810}]

- HVP is the main contributor to the total theory uncertainty
- Lots of activities in lattice QCD to improve the precision
- ...
- Lattice QCD, strong isospin breaking & QED:
  - direct effects in HVP of order 1%, yet need to be measured precisely...
  - indirect effects through scale setting quantities
- MUonE experiment as an alternative method



Outline

Data Driven Approach: Dispersive Methods

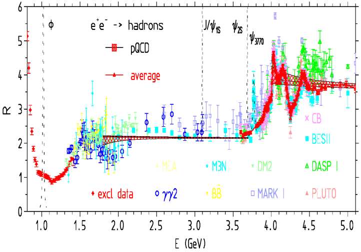
- Relation between the $\mathcal{R}e\,\Pi(Q^2)$ and $\mathcal{I}m\,\Pi(Q^2)$:

$$\boxed{\Pi(Q^2) - \Pi(0) = \frac{Q^2}{\pi}\int_0^\infty ds \frac{\mathcal{I}m\,\Pi(s)}{s(s-Q^2)}}$$

- Imaginary part of $\Pi(s)$ is related to the experimental total cross-section in e+e- annihilation:

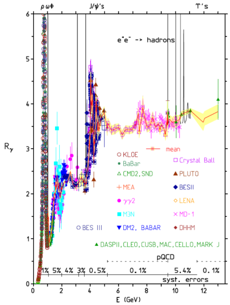
$$\boxed{\mathcal{I}m\,\Pi(s) = \frac{\alpha}{3}R(s)}$$

- Important contributions : $\rho, \omega, \phi, J/\psi$
- **O(1000)** channels
- Model calculations had to be used for some channels
- [Keshavarzi, Nomura, Teubner, Phys.Rev. D97 (2018) no.11]



Time-like vs Space-like Evaluation of a_μ^{HVP}

Time-like

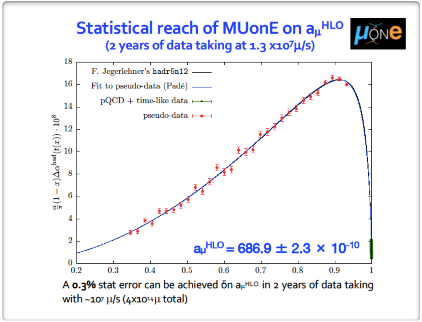


[Credit: F. Jegerlehner]

[Lautrup, Peterman de Rafael '72]

[T. Blum '03]

Space-like



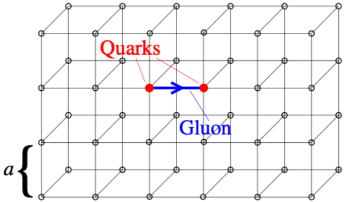
[Credit: G. Venanzoni, G. Abbiendi]

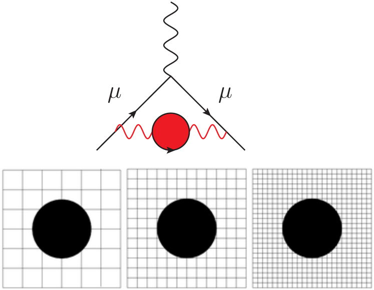
$$a_\mu^{\mathsf{HVP}} = \frac{\alpha}{\pi^2}\int_0^\infty \frac{ds}{s}\,K(s)\,\mathrm{Im}\Pi_{\mathsf{had}}(s) \qquad \rightarrow \qquad \frac{\alpha}{\pi}\int_{-\infty}^0 \frac{dt}{t\beta}\left(\frac{\beta-1}{\beta+1}\right)^2 \hat{\Pi}_{\mathsf{had}}(t)$$

- Π_{had} is the hadronic part of the photon vacuum polarization
- $\mathrm{Im}\Pi_{\mathsf{had}}(s)$ is related to the experimental total cross-section in e^+e^- annihilation

Space-like Evaluation of a_μ^{HVP}: Lattice QCD

(1) Formulate QCD on a (finite-size) lattice in Euclidean time
(2) Generate ensembles of field configuration with MC simulations
(3) Compute correlation function of fields as a function of time/momentum
(4) Average over configurations
(5) Extrapolate to continuum, infinite volume, and physical quark masses



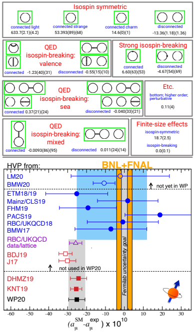


Space-like Evaluation of a_μ^{HVP}: Lattice QCD

- In WP20, lattice results ($<$ Mar/2020) were averaged; uncertainty 2.6%
- BMW20 reported first lattice result with sub-percent uncertainty:
  - reduced tension with experiment: $\sim 1.5\sigma$,
  - some tension with the R-ratio method (WP20); $\sim 2.1\sigma$

	value $\times 10^{10}$	error $\%$
$a_\mu^{\mathrm{HVP,\,LO}}$ (R-ratio, WP20)	$693.1(4.0)$	0.6%
$a_\mu^{\mathrm{HVP,\,LO}}$ (lattice, WP20)	$711.6(18.4)$	2.6%
$a_\mu^{\mathrm{HVP,\,LO}}$ (lattice, BMW20)	$707.5(5.5)$	0.8%

[WP20: arXiv:2006.04822, BMW20: arXiv:2002.12347]



- Intermediate Euclidean-time window values of a_μ are introduced for further investigations

Space-like Evaluation of a_μ^{HVP}: Lattice QCD

Dominant sources of error:

1) Determination of signal at small Q^2
   - The integrand peaks at Q^2 about $m_\mu^2/4$

$$\left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2 f(Q^2)\hat{\Pi}_{\mathsf{had}}(Q^2)$$

   - These very low momenta cannot be directly accessed on current lattices ($L \approx 10$ fm required)
   - Analytic functions (like Padé) in combination of the method of time moments have been suggested & used to describe $\hat{\Pi}(Q^2)$ over small values of Q^2; increase of statistical error at higher moments
   - In alternative, time-momentum representation the problem with small Q^2 shows itself as exponential growth of the relative statistical error at large time

2) Continuum extrapolation, scale setting errors, finite volume effects, disconnected diagrams, $\cdots$

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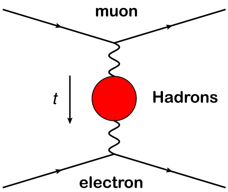
   - In alternative, time-momentum representation the problem with small Q^2 shows itself as exponential growth of the relative statistical error at large time
   - A hybrid method (with MUonE) can be used to circumvent this problem
2) Continuum extrapolation, scale setting errors, finite volume effects, disconnected diagrams, $\cdots$

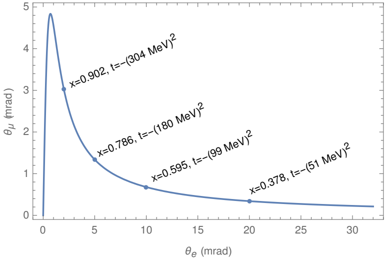
Space-like Evaluation of a_μ^{HVP}: MUonE

- HVP contributes to the running of QED fine structure coupling

$$\alpha(Q^2) = \frac{\alpha_0}{1 - \hat{\Pi}(Q^2)}$$

- Comparing experimental data & perturbative calculations yields HVP through its contribution to $\alpha(Q^2)$
- MUonE extracts $\Delta\alpha_{\mathrm{had}}(Q^2)$ from the shape of the differential $\mu - e$ scattering cross section





[arXiv:2004-13663]

Space-like Evaluation of a_μ^{HVP}: Hybrid Method

Divide & Conquer:

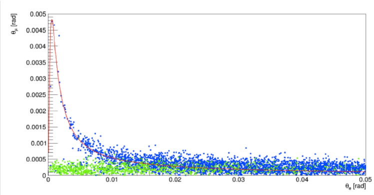
$$
\begin{aligned}
a_\mu^{\mathsf{HVP}} &= I_0 + I_1 + I_2 \\
I_0 &= \left(\frac{\alpha}{\pi}\right)^2 \int_0^{0.14} dQ^2 f(Q^2)\hat{\Pi}_{\mathsf{had}}(Q^2) \\
I_1 &= \left(\frac{\alpha}{\pi}\right)^2 \int_{0.14}^{Q^2_{\mathsf{max}}} dQ^2 f(Q^2)\hat{\Pi}_{\mathsf{had}}(Q^2) \\
I_2 &= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q^2_{\mathsf{max}}}^{\infty} dQ^2 f(Q^2)\hat{\Pi}_{\mathsf{had}}(Q^2)
\end{aligned}
$$

- I_0: contains $\sim$ 84% of the $a_\mu^{\mathsf{had,\ LO}}$ & can be calculated precisely with the MUonE experiment
- I_1: use lattice QCD or R-ratio
- I_2: use perturbation theory

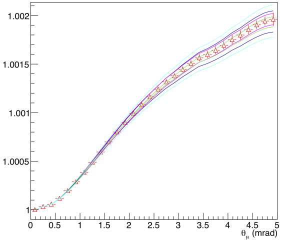
Extraction of HVP from μ-e scattering

- A proposed template fit inspired by contribution of lepton-pairs to the space-like photon vacuum polarization looks very good on test data

$$\Delta\alpha_{\text{had}}(t) = k\left\{ -\frac{5}{9} - \frac{4M}{3t} + \left(\frac{4M^2}{3t^2} + \frac{M}{3t} - \frac{1}{6}\right)\frac{2}{\sqrt{1-\frac{4M}{t}}}\log\left|\frac{1-\sqrt{1-\frac{4M}{t}}}{1+\sqrt{1-\frac{4M}{t}}}\right|\right\}$$



[arXiv:2201.13177, arXiv:2102.11111]



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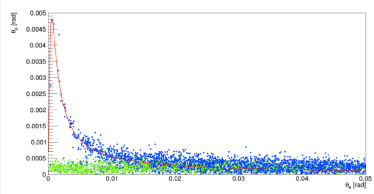

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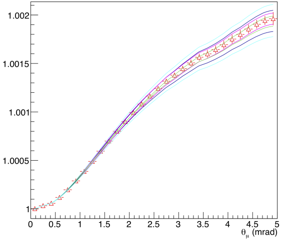

- The above template fit can be potentially problematic with highly precise data
- Alternative methods such as using Padé based fits are suggested and explored

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HVP, Stieltjes Functions, Padé Approximants

In the context of lattice QCD, the use of Padé approximants for low Q^2 regions $\hat{\Pi}(Q^2)$ was suggestion by [Aubin, Blum, Golterman, Peris (2012); Golterman, Maltman, Peris (2013)]

- It was introduced to deal with low signal at small Q^2 in lattice QCD
- They investigated the Padé approximants using mock data from dispersive τ-based $I = 1$ data:

$$\hat{\Pi}^{I=1}(Q^2) = Q^2 \int_{4m_\pi^2}^{\infty} ds \, \frac{\rho^{I=1}(s)}{s(s+Q^2)}$$

- They chose 40 equally-spaced points in Q^2 (from 0.01 to 0.4 GeV^2) from $\hat{\Pi}^{I=1}(Q^2)$, exploited an exaggerated lattice-based covariance matrix, and created a mock data set
- After performing their analysis, they compared their results with the "exact" value obtained directly from the original data set:

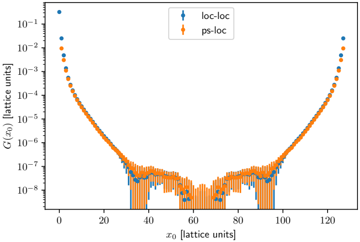
$$\tilde{a}_{\mu,\mathrm{HLO}}\Big|_{Q^2 \leq 1\mathrm{GeV}^2} = 1.204(27) \times 10^{-7}$$

Sample Covariance Matrix from Lattice QCD

- We continue their study using a correlation matrix calculated on a lattice ensemble of size $64^3 \times 128$ at $a \approx 0.066$ fm
- We calculate current-current correlator and integrate over spacial dimensions

$$G(x_0) = -\frac{1}{3}\sum_{k=1}^{3}\int dx^3 \langle J_k(x) J_k(0)\rangle$$

- After subtracting associated contact terms, we take FFT to obtain $Q^2\Pi(Q^2)$
- We extract a correlation matrix and use it with chosen 40 equally-spaced points of $\hat{\Pi}^{I=1}(Q^2)$ (curtesy of Kim Maltman)

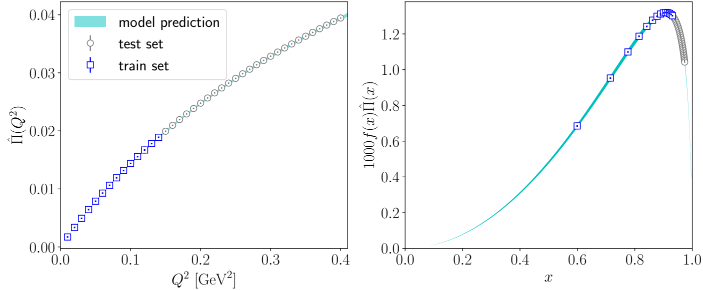


Padé-Based Fit Function

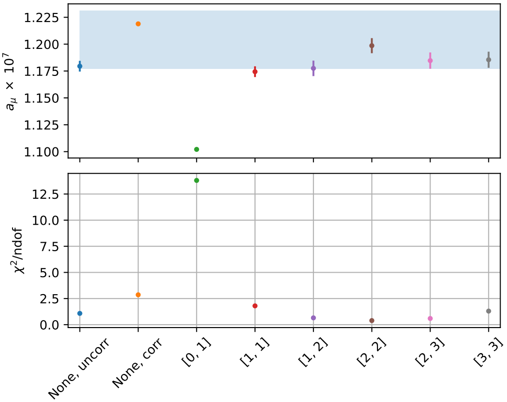
- Writing the HVP in terms of a Stieltjes function warrants an existence of the converging sequence of order $[N-1, N]$ and $[N, N]$ Padé approximants (PAs), defined as

$$\Delta\alpha_{\text{had}}(Q^2) = c_0 + Q^2\left(a_0 + \sum_{i=1}^{N} \frac{a_i}{b_i + Q^2}\right)$$

where $Q^2 = -t$ and $a_0 = 0$ in $[N-1, N]$ PAs



Some Fit Results



Mock data:	20 correlated data are included in the fits
Fit func:	Padé-based fits and a lepton-like ansatz
Challenge:	Small eigenvalue of the covariance matrices makes the fit more challenging; manifested in large χ^2/ndof
Message:	Better fit functions are needed with very precise covariance matrices

Summary & Concluding Remarks

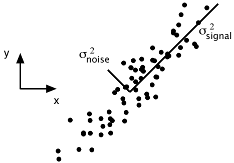
- We performed a test study using a mock data from dispersive τ-based data
- We used Padé based functions and a lepton-like ansatz to investigate Q^2 dependence of the mock data
- Our investigation showed, the lepton-like ansatz works as good as higher order Pade-based fits if we ignore correlations
- With correlated data, however, one has to exploit other functions such as the Padé approximants
- Padé approximants (and similar, analytic functions) can be used for extrapolating both lattice and MUonE data, and comparing them if needed

Thanks for your attention!

Back-up Slides

Principal Component Analysis (PCA)

- PCA is an analysis that used for dimensional reduction
  - identifies the *principal components* in data that show as much variation in data as possible
  - projects data onto only the first few principal axes/directions to obtain lower-dimensional data with most variation



[https://arxiv.org/pdf/1404.1100.pdf]

- Principal axes are eigenvectors of the data's covariance/correlation matrix.
- Principal components are the projection of data on the principal axes

Principal Component Analysis using SVD

- Say C is the covariance matrix of our data
- SVD or eigenvalue decomposition of C (an $n \times n$ matrix)

$$C = \sum_{i=1}^{n} \lambda_i \, |i\rangle\langle i|, \quad (\lambda_i > \lambda_{i+1})$$

- Modified SVD inflates the smallest eigenvalues of C and replaces the inverse of C with

$$\tilde{C}^{-1} = c_0 \sum_{i=1}^{n-k} \frac{1}{\lambda_i} \, |i\rangle\langle i| + \frac{c_1}{\lambda_{\text{cut}}} \sum_{\text{rest}} |i\rangle\langle i|$$