NNLO hadronic vacuum polarization contributions to the muon g-2 in the space-like region

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The Evaluation of the Leading Hadronic Contribution to the Muon g-2: Toward the MUonE Experiment

Mainz

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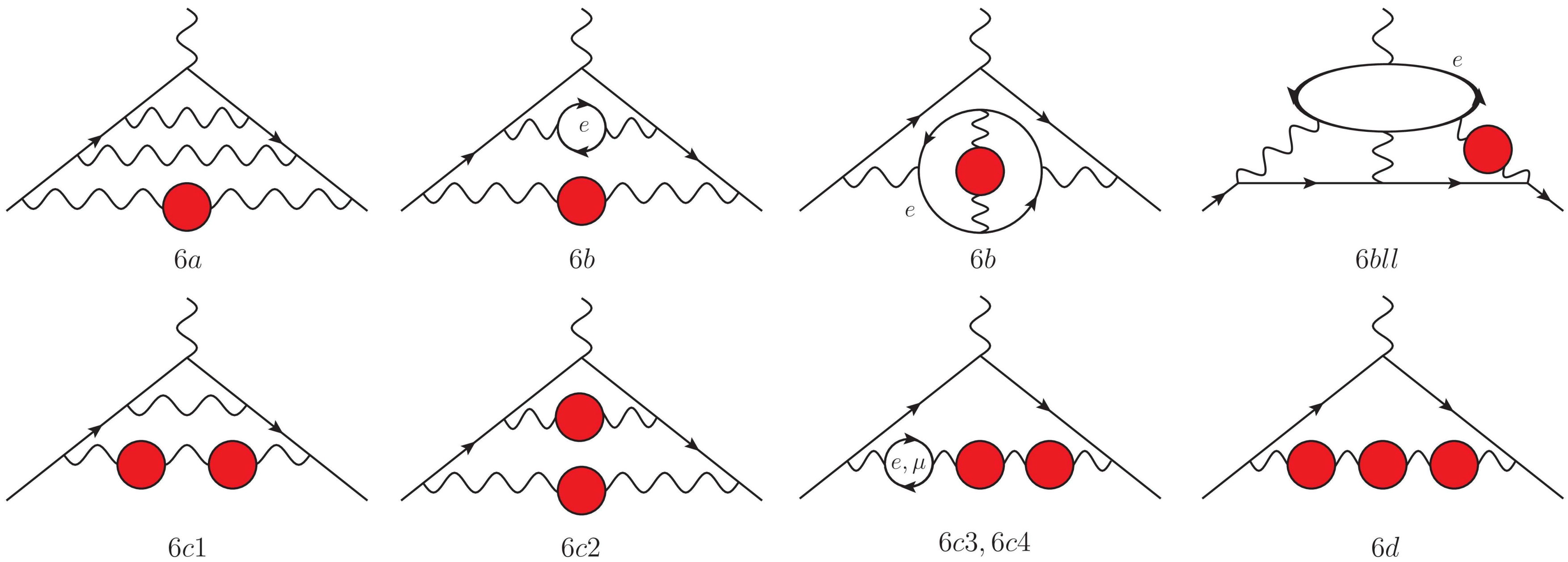
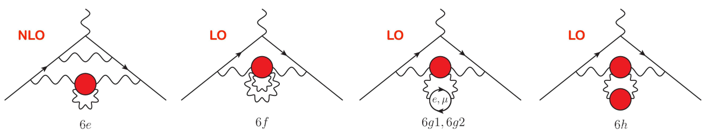
Work in Collaboration with E. Balzani and M. Passera, *Phys. Lett.* B 834 (2022) 137462

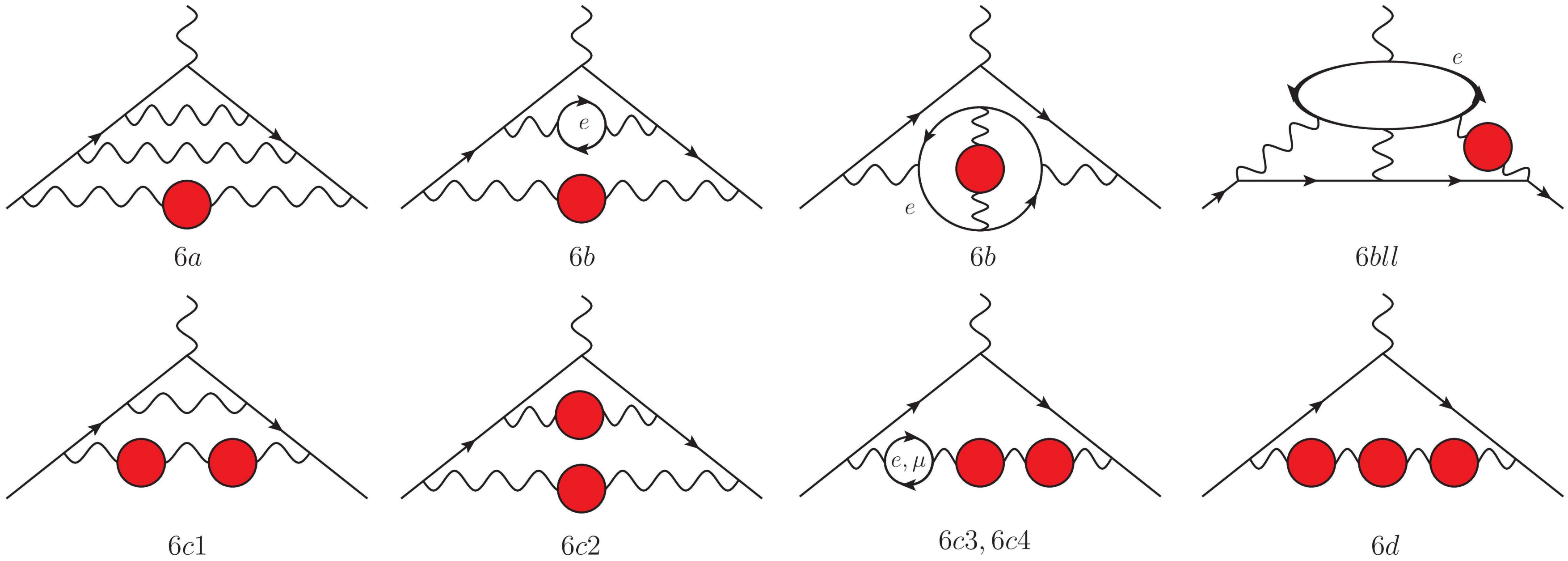


sample NNLO diagrams

- set 6a contains also diagrams with muon loops (like 6b 6bll)
- HVP with internal corrections already incorporated in NLO and LO



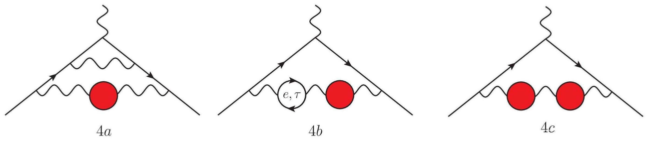
- $a_\mu^{\mathrm{HVP}}(\mathrm{NNLO}; total) = +12.4(1) \times 10^{-11}$ Kurz Liu Marquard Steinhauser 2014



At NLO level

$$a_\mu^{\rm HVP}({\rm NLO};4a) = \frac{\alpha}{\pi^2}\int_{m_\mu^2}^{\infty}\frac{ds}{s}\,2K^{(4)}(s/m^2)\,{\rm Im}\Pi(s) = -\frac{\alpha}{\pi^2}\int_{-\infty}^{0}\frac{dt}{t}\Pi(t)\,{\rm Im}2K^{(4)}(t/m^2)$$

- At NLO, $K^{(4)}(s/m^2)$ and Im$K^{(4)}(s/m^2)$ are known exactly   see Elisa's talk
- At NNLO $K^{(6)}(s/m^2)$ is NOT known analytically.
- Only a few terms of the asymptotic expansion for large s are known.
- We need an example to use as a testbed: NLO



Asymptotic expansion for large s of $K^{(4)}(s/m^2)$ in powers of $r = m^2/s$ (Lautrup 1997)

$$K^{(4)}(r) = r\left(\frac{23\ln r}{36} - \frac{\pi^2}{3} + \frac{223}{54}\right) + r^2\left(\frac{19\ln^2 r}{144} + \frac{367\ln r}{216} - \frac{37\pi^2}{48} + \frac{8785}{1152}\right) + r^3\left(\frac{141\ln^2 r}{80} + \frac{10079\ln r}{3600} - \frac{883\pi^2}{240} + \frac{13072841}{432000}\right) + \dots$$

from this expansion we derive *approximated* space-like kernel $\bar{\kappa}^{(4)}(x)$

We use the *modified* ansatz of [Groote Körner Pivovarov 2002] [Chakraborty Davies Kobonen Lepage VandeWater 2018]

$$K^{(4)}(s/m^2) = r\int_0^1 \mathrm{d}\xi\left[\frac{L(\xi)}{\xi + r} + \frac{P(\xi)}{1 + r\xi}\right] \qquad L(\xi) = G(\xi) + H(\xi)\ln\xi$$

G, H, P are polynomials of degree $n-1$, (n arbitrary) $G(\xi) = \sum_{i=0}^{n-1} g_i\xi^i$, $H(\xi) = \sum_{i=0}^{n-1} h_i\xi^i$, $P(\xi) = \sum_{i=0}^{n-1} p_i\xi^i$.
The coefficients g_i, h_i and p_i of the polynomials are found performing the integration over ξ, expanding for small r, and fitting the coefficients of $r^{i+1}\ln r$, $r^{i+1}\ln^2 r$, r^{i+1} with asymptotic expansion.

$$a_\mu^{\mathrm{HVP}}(\mathrm{NLO};4a) = \frac{\alpha}{\pi^2}\int_{m_\mu^2}^{\infty}\frac{ds}{s}\,2K^{(4)}(s/m^2)\,\mathrm{Im}\Pi(s)$$

$$= \frac{\alpha}{\pi^2}m^2\int_0^1 d\xi\left(L(\xi)\int_{m_\mu^2}^{\infty}\frac{ds}{s}\,\frac{\mathrm{Im}\Pi(s)}{s + m^2/\xi} + P(\xi)\int_{m_\mu^2}^{\infty}\frac{ds}{s}\,\frac{\mathrm{Im}\Pi(s)}{s + m^2\xi}\right) \quad \text{denominators linear in s !}$$

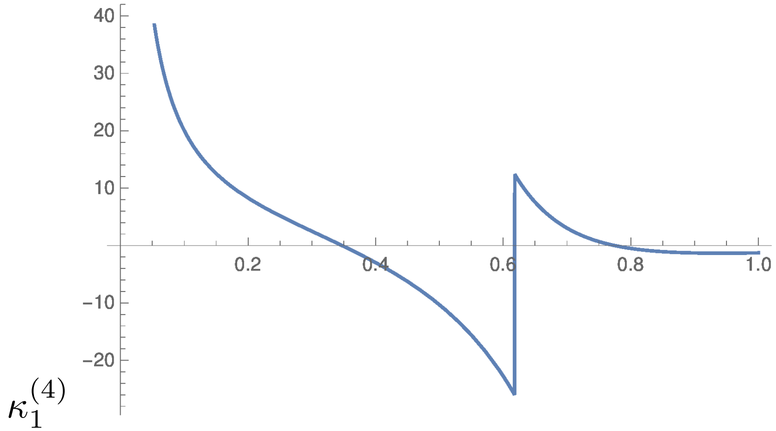
$$= -\frac{\alpha}{\pi^2}\int_0^1 d\xi\left(L(\xi)\Pi\left(-\frac{m^2}{\xi}\right) + \frac{P(\xi)}{\xi}\Pi\left(-m^2\xi\right)\right) \qquad -\frac{m^2}{\xi} \le -m^2 \le -m^2\xi \le 0 \quad \text{whole negative axis}$$

As the arguments of Π do not overlap, we combine the two integrals into one. Define an approximated piecewise space-like kernel $\bar\kappa^{(4)}(x)$

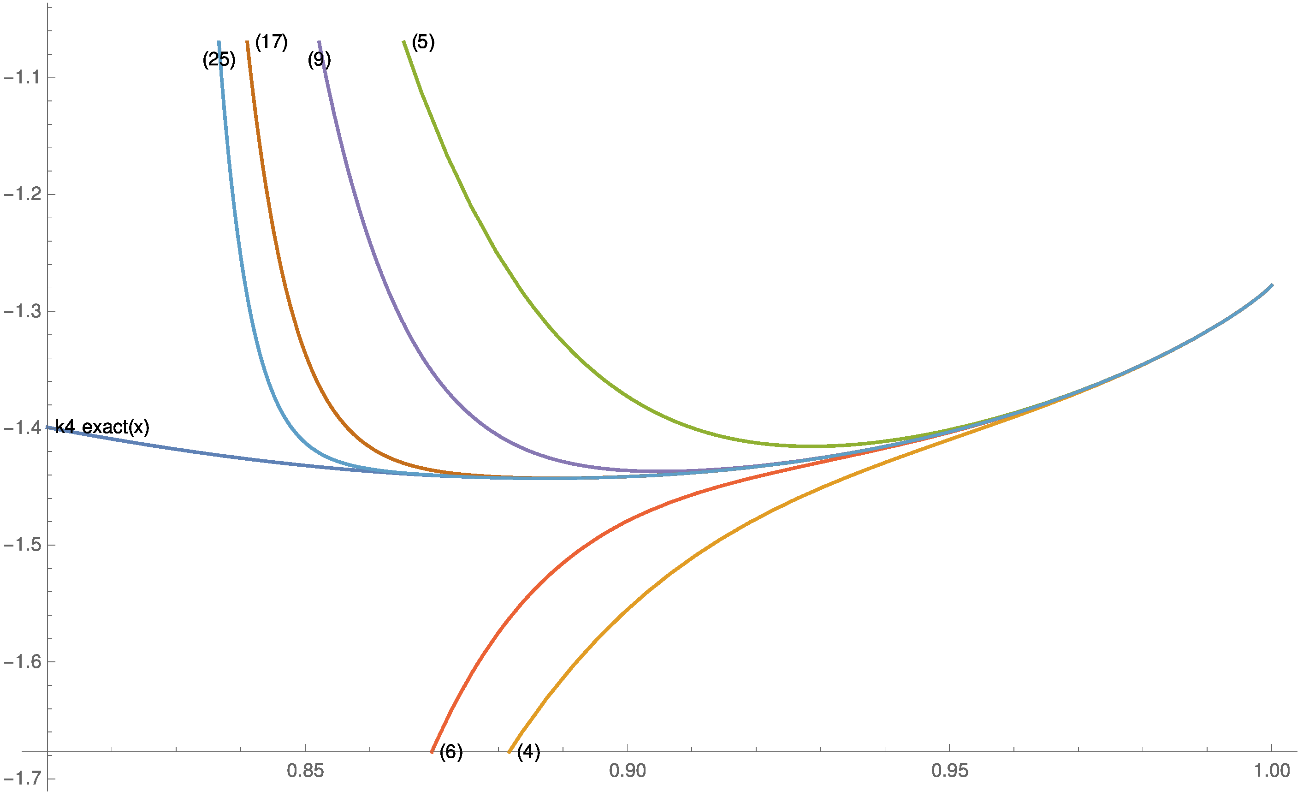
$$a_\mu^{\mathrm{HVP}}(\mathrm{NLO};4a) = \left(\frac{\alpha}{\pi}\right)^3 \int_0^1 \mathrm{d}x\, \bar\kappa^{(4)}(x)\, \Delta\alpha_{\mathrm{h}}(t(x)), \qquad t(x) = \frac{m^2x^2}{x-1}$$

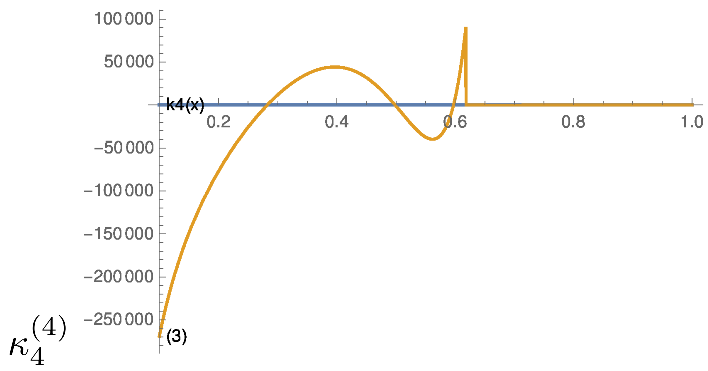
$$\bar\kappa^{(4)}(x) = \begin{cases} \frac{2-x}{x(1-x)}\, P\!\left(\frac{x^2}{1-x}\right), & 0 < x < x_\mu = (\sqrt{5}-1)/2 = 0.618\ldots \\ \frac{2-x}{x^3}\, L\!\left(\frac{1-x}{x^2}\right), & x_\mu < x < 1 \end{cases}$$

- *Original* ansatz had $\ln^2 r$ terms not fitted ($i.e.$ $H=0$) $\to$ Error of 6% on $a_\mu^{\mathrm{HVP}}(\mathrm{NLO};\textit{total})$,
- Error $\boxed{\text{eliminated}}$ by the *exact* NLO kernel $\kappa^{(4)}(x)$!



Plot of the approximated kernels $\bar{\kappa}_n^{(4)}(x)$ for $n = 4$, 5, 6, 9, 17, 25 compared with exact $\kappa^{(4)}(x)$





- $\bar{\kappa}_n^{(4)}(x)$: good approximation for x near 1 ($t \to -\infty$);
- Discontinuity for $x = (\sqrt{5}-1)/2 \approx 0.618$ ($t = -m^2$)
- Wild oscillations for small x, worse for large n.
- For $n = 25$ up to $\sim \pm 10^{30}$! But the integral reproduces the exact result with error 10^{-20} $\to$ deep numerical cancellations!.
- Large n not necessary! $\boxed{n = 4}$ reproduces $a_\mu^{\mathrm{HVP}}(\mathrm{NLO}; 4a)$ with error $\lesssim 0.1\%$
- Let's use this method of approximation at NNLO!

$K^{(6a)}(s/m^2)$: Only the first 4 terms of the expansion in power series of $r = m^2/s$ are known $\rightarrow$n=4

Kurz, Liu, Marquard, Steinhauser, PLB734 (2014) 144

They contain terms with $r^n \ln r$, $r^n \ln^2 r$ and $r^n \ln^3 r$. As in NLO, we use an integral ansatz:

$$K^{(6a)}(s/m^2) = r\int_0^1 \mathrm{d}\xi \left[\frac{L^{(6a)}(\xi)}{\xi + r} + \frac{P^{(6a)}(\xi)}{1 + r\xi}\right] \qquad L^{(6a)}(\xi) = G^{(6a)}(\xi) + H^{(6a)}(\xi)\ln\xi + J^{(6a)}(\xi)\ln^2\xi \quad \text{new@NNLO}$$

$G^{(6a)}$, $H^{(6a)}$,$J^{(6a)}$, $P^{(6a)}$ polynomials

$$G^{(6a)}(\xi) = \sum_{i=0}^{3} g_i^{(6a)}\xi^i, \quad H^{(6a)}(\xi) = \sum_{i=0}^{3} h_i^{(6a)}\xi^i, \quad J^{(6a)}(\xi) = \sum_{i=0}^{3} j_i^{(6a)}\xi^i, \quad P^{(6a)}(\xi) = \sum_{i=0}^{3} p_i^{(6a)}\xi^i$$

We integrate in ξ, expand in r, and we find $g_i^{(6a)}$, $h_i^{(6a)}$, $j_i^{(6a)}$ and $p_i^{(6a)}$, $i = 0, 1, 2, 3$, in order to fit the known coefficients of the asymptotic expansion in r of $K^{(6a)}(s/m^2)$. Then approximated kernel $\bar{\kappa}^{(6a)}(x)$ is

$$a_\mu^{HVP}(\mathrm{NNLO}; 6a) = \left(\frac{\alpha}{\pi}\right)^3 \int_0^1 \mathrm{d}x\, \bar{\kappa}^{(6a)}(x)\, \Delta\alpha_{\mathrm{h}}(t(x)),$$

$$\bar{\kappa}^{(6a)}(x) = \begin{cases} \frac{2-x}{x(1-x)}\, P^{(6a)}\!\left(\frac{x^2}{1-x}\right), & 0 < x < x_\mu = (\sqrt{5}-1)/2 = 0.618\ldots \\ \frac{2-x}{x^3}\, L^{(6a)}\!\left(\frac{1-x}{x^2}\right), & x_\mu < x < 1 \end{cases}$$

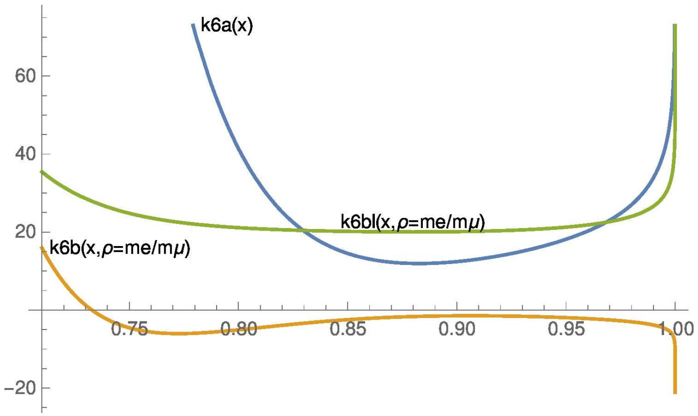
- The contributions of classes ($6b$) and ($6bll$) can be calculated similarly to class ($6a$).
- $a_\mu^{\mathrm{HVP}}(\mathrm{NNLO}; 6a) = +8.0 \times 10^{-11}$ $\quad a_\mu^{\mathrm{HVP}}(\mathrm{NNLO}; 6b) = -4.1 \times 10^{-11}$ $\quad a_\mu^{\mathrm{HVP}}(\mathrm{NNLO}; 6bll) = +9.1 \times 10^{-11}$
- The uncertainty due to the series approximations of $K^{(6a)}$, $K^{(6b)}$, $K^{(6bll)}$ are estimated to be less than $O(10^{-12})$

(6a)	
$j_0 = 0;$	$h_0 = -\frac{359}{36};$
$j_1 = -\frac{3793}{864};$	$h_1 = \frac{122293}{5184};$
$j_2 = \frac{35087}{21600};$	$h_2 = -\frac{43879427}{648000};$
$j_3 = \frac{1592093}{43200};$	$h_3 = \frac{14388407}{48000};$

$$g_0 = \frac{1301}{144} - \frac{19\pi^2}{9};$$
$$g_1 = \frac{441277}{10368} + \pi^2\left(-\frac{355}{648} + \ln 4\right) + \frac{25\,\zeta(3)}{2};$$
$$g_2 = -\frac{5051645167}{38880000} + \pi^2\left(\frac{221411}{32400} - 18\ln 2\right) - \frac{3919\,\zeta(3)}{60};$$
$$g_3 = \frac{14588342017}{38880000} + \pi^2\left(-\frac{2479681}{64800} + 112\ln 2\right) + \frac{3113\,\zeta(3)}{10};$$

$$p_0 = -\frac{1808080780513}{14580000} + \frac{41851\pi^4}{15} + \frac{8432\ln^4 2}{3} + 67456\, a_4 + \frac{2085448\,\zeta(3)}{15} + $$
$$+\pi^2\left(-\frac{11944163099}{194400} + \frac{272}{3}(180 - 31\ln 2)\ln 2 + \frac{115072\,\zeta(3)}{3}\right) - \frac{575360\,\zeta(5)}{3};$$
$$p_1 = \frac{134017456919}{96000} - \frac{4481182\pi^4}{135} - \frac{98420\ln^4 2}{3} - 787360\, a_4 + 2255200\,\zeta(5) + $$
$$+\pi^2\left(\frac{23549054249}{32400} - 201122\ln 2 + \frac{98420\ln^2 2}{3} - 451040\,\zeta(3)\right) - \frac{57189259\,\zeta(3)}{36};$$
$$p_2 = -\frac{13069081405453}{3888000} + \frac{330073\pi^4}{4} + 80790\ln^4 2 + 1938960\, a_4 + \frac{77371609\,\zeta(3)}{20} + $$
$$+\pi^2\left(-\frac{729995599}{405} + 6(85313 - 13465\ln 2)\ln 2 + 1114360\,\zeta(3)\right) - 5571800\,\zeta(5);$$
$$p_3 = \frac{1274611832039}{583200} - \frac{986377\pi^4}{18} - 53340\ln^4 2 - 1280160\, a_4 + \frac{11057200\,\zeta(5)}{3} + $$
$$+\pi^2\left(\frac{5809659289}{4860} + 420\ln 2(-823 + 127\ln 2) - \frac{2211440\,\zeta(3)}{3}\right) - \frac{22833188\,\zeta(3)}{9};$$

Table 1: The coefficients $g_i^{(6a)}$, $h_i^{(6a)}$, $j_i^{(6a)}$, $p_i^{(6a)}$ $(i = 0,1,2,3)$. The superscript $(6a)$ has been dropped for simplicity. In the above coefficients, the Riemann zeta function $\zeta(k) = \sum_{n=1}^{\infty} 1/n^k$ and $a_4 = \sum_{n=1}^{\infty} 1/(2^n n^4) = \mathrm{Li}_4(1/2)$.



(6b)	
$j_0 = 0;$	$h_0 = \frac{65}{54};$
$j_1 = \frac{11}{27};$	$h_1 = -\frac{3559}{1296} + \rho^2 + \frac{5}{18}\ln\rho;$
$j_2 = \frac{41}{120};$	$h_2 = \frac{3917}{432} - \frac{82\rho^2}{3} + \frac{61}{10}\ln\rho;$
$j_3 = -\frac{507}{40};$	$h_3 = -\frac{4109}{80} + \frac{2211\rho^2}{10} - \frac{1763}{30}\ln\rho;$

$$g_0 = \frac{1}{108}\left(259 - 72\rho^2 + 276\ln\rho\right);$$
$$g_1 = -\frac{9215}{1296} + \frac{65\pi^2}{162} - \frac{3\pi^2\rho}{4} + \frac{49\rho^2}{36} + \left(-\frac{301}{54} + 8\rho^2\right)\ln\rho + \frac{4}{3}\ln^2\rho + 2\,\zeta(3);$$
$$g_2 = \frac{501971}{40500} - \frac{113\pi^2}{36} + \frac{270\pi^2\rho}{36} - \frac{8417\rho^2}{180} + \left(\frac{3479}{900} - 44\rho^2\right)\ln\rho - 8\ln^2\rho - 12\,\zeta(3);$$
$$g_3 = -\frac{2523823}{324000} + \frac{625\pi^2}{36} - 49\pi^2\rho + \frac{84946\rho^2}{225} + \left(\frac{987}{50} + 200\rho^2\right)\ln\rho + \frac{112}{3}\ln^2\rho + 56\,\zeta(3);$$

$$p_0 = -\frac{95519053063}{486000} - 7275\pi^2\rho + \left(-\frac{587150693}{5400} + \frac{75272\rho^2}{3} + \frac{120800\pi^2}{9}\right)\ln\rho + \left(\frac{1135508}{9} + 96\rho^2\right)\zeta(3) + $$
$$+4720\ln^2\rho + \frac{1067115409\rho^2}{5400} + \pi^2\left(\frac{24382331}{810} - \frac{285184}{9}\ln 2\right) - 32\pi^2\rho^2(687 + \ln 4);$$
$$p_1 = \frac{279489728279}{121500} + \frac{179283\pi^2\rho}{2} + \left(\frac{2280933773}{1800} - 309540\rho^2 - \frac{1419328\pi^2}{9}\right)\ln\rho - \frac{10}{3}\left(446023 + 216\rho^2\right)\zeta(3) + $$
$$-\frac{174712}{3}\ln^2\rho - \frac{174350167\rho^2}{75} + \pi^2\left(-\frac{143574463}{405} + \frac{3352256\ln 2}{9}\right) + \frac{16}{3}\pi^2\rho^2(48481 + 90\ln 2);$$
$$p_2 = -\frac{229560199193}{40500} - \frac{912495\pi^2\rho}{4} + \left(-\frac{1867939691}{600} + 788488\rho^2 + \frac{1168336\pi^2}{3}\right)\ln\rho + \left(\frac{11034553}{3} + 1440\rho^2\right)\zeta(3) + $$
$$+148348\ln^2\rho + \frac{258653648\rho^2}{45} + \frac{4}{135}\pi^2(29597029 - 31048560\ln 2) - \frac{320}{3}\pi^2\rho^2(5989 + \ln 512);$$
$$p_3 = \frac{72762177677}{19440} + 154035\pi^2\rho - \frac{7}{108}\left(-31650719 + 3973440\pi^2 + 8220240\rho^2\right)\ln\rho - \frac{280}{9}\left(78283 + 27\rho^2\right)\zeta(3) + $$
$$-100240\ln^2\rho - \frac{513692207\rho^2}{135} + \frac{35}{162}\pi^2(-2687659 + 2816064\ln 2) + \frac{140}{3}\pi^2\rho^2(9055 + \ln 4096);$$

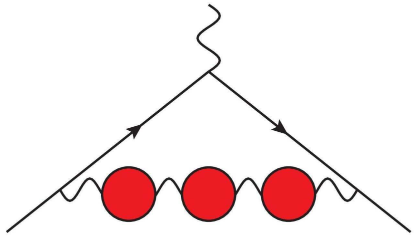
Table 2: The coefficients $g_i^{(6b)}$, $h_i^{(6b)}$, $j_i^{(6b)}$, $p_i^{(6b)}$ $(i = 0,1,2,3)$. The superscript $(6b)$ has been dropped for simplicity. In the above coefficients, $\rho = m_e/m$, the Riemann zeta function $\zeta(k) = \sum_{n=1}^{\infty} 1/n^k$, and $a_4 = \sum_{n=1}^{\infty} 1/(2^n n^4) = \mathrm{Li}_4(1/2)$.

(6bll)	
$j_0 = 0;$	$h_0 = -\frac{9}{2};$
$j_1 = \frac{4}{27} - \frac{9\rho^2}{2};$	$h_1 = \frac{59}{9} - \frac{275\rho^2}{36} - 18\rho^2\ln\rho;$
$j_2 = -\frac{41}{48} + \frac{2201\rho^2}{216};$	$h_2 = -\frac{485}{32} + \frac{1351\rho^2}{48} + \frac{659\rho^2}{18}\ln\rho;$
$j_3 = \frac{3037}{900} - \frac{5909\rho^2}{216};$	$h_3 = \frac{282617}{6750} - \frac{10481\rho^2}{108} - \frac{851\rho^2}{9}\ln\rho;$

$$g_0 = \frac{43}{8} - 4\pi^2\rho + 15\rho^2 + \pi^2\rho^2 - 18\rho^2\ln\rho + 6\rho^2\ln^2\rho;$$
$$g_1 = -\frac{73}{81} + \frac{8\pi^2}{81} + \frac{40\pi^2\rho}{9} + \frac{2437\rho^2}{108} + \frac{17\pi^2\rho^2}{9} + \frac{607\rho^2}{18}\ln\rho - \frac{20\rho^2}{3}\ln^2\rho + \frac{2}{3}\zeta(3) + 2\rho^2\zeta(3);$$
$$g_2 = -\frac{385}{162} - \frac{41\pi^2}{72} - \frac{28\pi^2\rho}{3} - \frac{89873\rho^2}{5184} - \frac{997\pi^2\rho^2}{324} - \frac{1961\rho^2}{72}\ln\rho + 14\rho^2\ln^2\rho - \frac{5}{2}\zeta(3) - \frac{16\rho^2}{3}\zeta(3);$$
$$g_3 = \frac{2691761}{202500} + \frac{3037\pi^2}{1350} + 24\pi^2\rho + \frac{655429\rho^2}{97200} + \frac{2359\pi^2\rho^2}{324} + \frac{6943\rho^2}{360}\ln\rho - 36\rho^2\ln^2\rho + \frac{42}{5}\zeta(3) + 15\rho^2\zeta(3);$$

$$p_0 = -\frac{343277101}{45000} - \frac{33156604927\rho^2}{583200} + \pi^2\left(-\frac{615427}{4050} + \frac{6776\rho}{3} + \frac{763121\rho^2}{972}\right) - \frac{4\pi^4}{135}\left(7817 + 3212\rho^2\right) + $$
$$+\left(-\frac{7290521}{3240} + \frac{49622\pi^2}{27} - \frac{128\pi^4}{9}\right)\rho^2\ln\rho + \left(-3388 - \frac{80\pi^2}{3}\right)\rho^2\ln^2\rho + $$
$$+\left(25642 + \frac{1515724\rho^2}{27} - 128\pi^2\rho^2 - 160\rho^2\ln\rho\right)\zeta(3) - \frac{1280}{3}\rho^2\zeta(5);$$
$$p_1 = \frac{89280434843}{972000} + \frac{248834878697\rho^2}{388800} - \frac{1}{324}\pi^2\left(-533001 + 9110736\rho + 3110417\rho^2\right) + \frac{2}{135}\pi^4\left(180247 + 73530\rho^2\right) + $$
$$+\left(\frac{11101973}{1080} - \frac{193400\pi^2}{9} + \frac{320\pi^4}{3}\right)\rho^2\ln\rho + \frac{2}{3}\left(63269 + 300\pi^2\right)\rho^2\ln^2\rho + $$
$$+\frac{1}{45}\left(-13410977 + 100\left(-292301 + 432\pi^2\right)\rho^2 + 54000\rho^2\ln\rho\right)\zeta(3) + 3200\rho^2\zeta(5);$$
$$p_2 = -\frac{6209532853}{27000} - \frac{29997466847\rho^2}{19440} + \pi^2\left(-\frac{114521}{30} + 71840\rho + \frac{1970140\rho^2}{81}\right) - \frac{4}{9}\pi^4\left(14685 + 6032\rho^2\right) + $$
$$-\frac{1}{54}\left(190613 - 2847360\pi^2 + 11520\pi^4\right)\rho^2\ln\rho - 80\left(1347 + 5\pi^2\right)\rho^2\ln^2\rho + $$
$$-\frac{10}{9}\left(-658509 + \left(-1431463 + 1728\pi^2\right)\rho^2 + 2160\rho^2\ln\rho\right)\zeta(3) - 6400\rho^2\zeta(5);$$
$$p_3 = \frac{49726331179}{324000} + \frac{7324831423\rho^2}{7290} + \pi^2\left(\frac{3897971}{1620} - \frac{145880\rho}{3} - \frac{3977785\rho^2}{243}\right) + \frac{14}{27}\pi^4\left(8269 + 3419\rho^2\right) + $$
$$+\frac{7}{81}\left(-81551 - 401520\pi^2 + 1440\pi^4\right)\rho^2\ln\rho + \frac{140}{3}\left(1563 + 5\pi^2\right)\rho^2\ln^2\rho + $$
$$+\frac{35}{27}\left(-371889 + 16\left(-50437 + 54\pi^2\right)\rho^2 + 1080\rho^2\ln\rho\right)\zeta(3) + \frac{11200}{3}\rho^2\zeta(5);$$

Table 3: The coefficients $g_i^{(6bll)}$, $h_i^{(6bll)}$, $j_i^{(6bll)}$, $p_i^{(6bll)}$ $(i = 0,1,2,3)$. The superscript $(6bll)$ has been dropped for simplicity. In the above coefficients, $\rho = m_e/m$, the Riemann zeta function $\zeta(k) = \sum_{n=1}^{\infty} 1/n^k$, and $a_4 = \sum_{n=1}^{\infty} 1/(2^n n^4) = \mathrm{Li}_4(1/2)$.

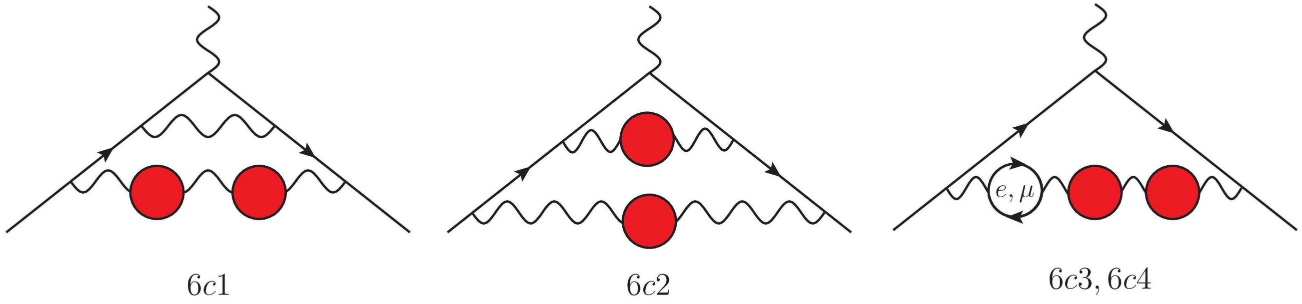


$6d$

$$a_\mu^{HVP}(\mathrm{NNLO}; 6d) \;=\; \frac{\alpha}{\pi} \int_0^1 \mathrm{d}x \, \kappa^{(2)}(x) \; [\Delta\alpha_{\mathrm{h}}(t(x))]^3 \, .$$

$$a_\mu^{\mathrm{HVP}}(\mathrm{NNLO}; 6d) = +0.005 \times 10^{-11}$$

very small contribution



$$a_\mu^{HVP}(\mathrm{NNLO};6c) = a_\mu^{HVP}(\mathrm{NNLO};6c1) + a_\mu^{HVP}(\mathrm{NNLO};6c2) + a_\mu^{HVP}(\mathrm{NNLO};6c3) + a_\mu^{HVP}(\mathrm{NNLO};6c4)$$

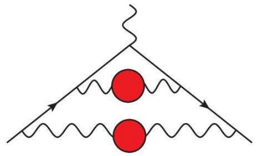
$$a_\mu^{HVP}(\mathrm{NNLO};6c1) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^1 \mathrm{d}x \left[\kappa^{(4)}(x) - \frac{2\pi}{\alpha}\kappa^{(2)}(x)\,\Delta\alpha_\mu^{(2)}(t(x))\right] \left[\Delta\alpha_{\mathrm{h}}(t(x))\right]^2$$

6c4 separated
multiplicity=3

$$a_\mu^{HVP}(\mathrm{NNLO};6c3) = \frac{3\alpha}{\pi}\int_0^1 \mathrm{d}x\,\kappa^{(2)}(x)\left[\Delta\alpha_{\mathrm{h}}(t(x))\right]^2 \Delta\alpha_e^{(2)}(t(x)).$$

$$a_\mu^{HVP}(\mathrm{NNLO};6c4) = \frac{3\alpha}{\pi}\int_0^1 \mathrm{d}x\,\kappa^{(2)}(x)\left[\Delta\alpha_{\mathrm{h}}(t(x))\right]^2 \Delta\alpha_\mu^{(2)}(t(x)).$$

$a_\mu^{HVP}(6c1) = -5\times 10^{-12}$, $a_\mu^{HVP}(6c3) = 0.9\times 10^{-12}$, $a_\mu^{HVP}(6c4) = 0.1\times 10^{-12}$ 6c2 ?

This class requires *double* integrals 

6c2

$$a_\mu^{HVP}(\mathrm{NNLO};6c2) = \frac{\alpha^2}{\pi^4}\int_{s_0}^{\infty}\frac{\mathrm{d}s}{s}\int_{s_0}^{\infty}\frac{\mathrm{d}s'}{s'}\,K^{(6c2)}(s/m^2,s'/m^2)\mathrm{Im}\Pi_\mathrm{h}(s)\,\mathrm{Im}\Pi_\mathrm{h}(s').$$

$$a_\mu^{HVP}(\mathrm{NNLO};6c2) = \left(\frac{\alpha}{\pi}\right)^2\int_{x_\mu}^{1}\mathrm{d}x\int_{x_\mu}^{1}\mathrm{d}x'\,\bar\kappa^{(6c2)}(x,x')\Delta\alpha_\mathrm{h}(t(x))\Delta\alpha_\mathrm{h}(t(x')),$$

$\bar\kappa^{(6c2)}(x,x')$ space-like bidimensional kernel, $\quad x_\mu < \{x,x'\} < 1$

$$\bar\kappa^{(6c2)}(x,x') = \frac{2-x}{x^3}\frac{2-x'}{x'^3}\,G^{(6c2)}\!\left(\frac{1-x}{x^2},\frac{1-x'}{x'^2}\right)$$

From the <u>leading</u> terms of the known asymptotic expansion of $K^{(6c2)}(s/m^2,s'/m^2)$:
$s/s' \ll 1$ or $s/s' \approx 1$ or $s/s' \ggg 1$ and $s,s' \gg m^2$ we get the approximated space-like kernel

$$G^{(6c2)}(\xi,\xi') = \frac{1855-188\pi^2}{4\,(32\pi^2-315)}\frac{\min(\xi,\xi')}{\max(\xi,\xi')^2} + \frac{988\pi^2-9765}{4\,(32\pi^2-315)}\frac{\min(\xi,\xi')^2}{\max(\xi,\xi')^3} + \frac{6\left(435-44\pi^2\right)}{32\pi^2-315}\frac{\min(\xi,\xi')^3}{\max(\xi,\xi')^4}$$

Contribution of 6c2 class is $a_\mu^{HVP}(6c2) = -1.8\times 10^{-12}$

The uncertainty of this leading order approximation is estimated to be $\sim 10^{-13}$

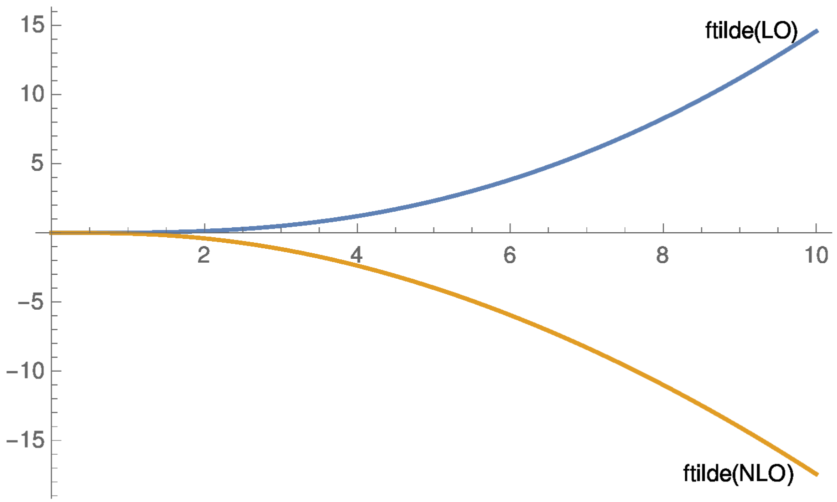
From lattice QCD (DellaMorte, Francis, Gulpers et al. JHEP10(2017)20)

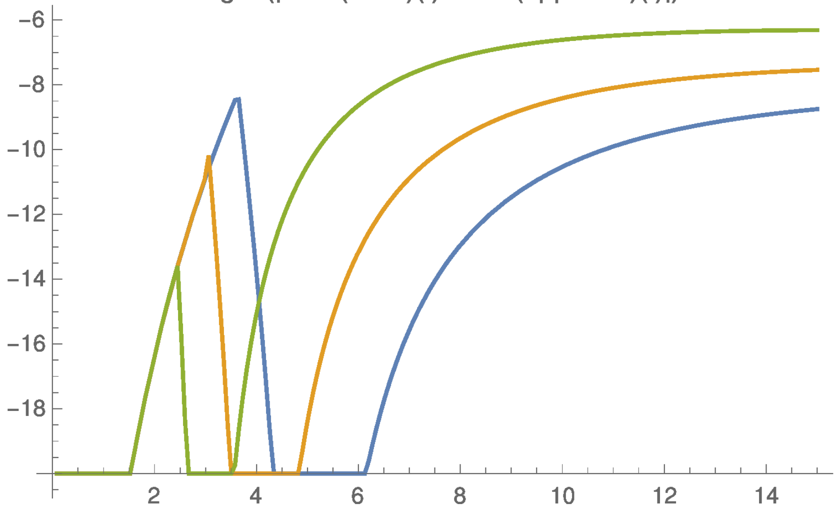
$$a_\mu^{HVP}(LO) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, G(t) \tilde{K}^{(2)}(t, m_\mu)$$

$$\tilde{K}^{(2)}(t, m_\mu = 1) = \tilde{f}^{(2)}(t) = 8\pi^2 \int_0^\infty \frac{d\omega}{\omega} \left(\omega^2 t^2 - 4\sin^2\left(\frac{\omega t}{2}\right)\right) f^{(2)}(\omega^2), \qquad f^{(2)}(\omega^2) = \frac{F^{(2)}(1/y(-\omega^2))}{\omega^3}\,, \qquad t = -\omega^2$$

$$\tilde{f}^{(2)}(t) = \frac{1}{4} G^{2,1}_{1,3}\left({\frac{3}{2} \atop 0,1,\frac{1}{2}} \middle| t^2 \right) + \frac{t^2}{4} + \frac{1}{t^2} + 2\ln(t) - \frac{2}{t} K_1(2t) + 2\gamma - \frac{1}{2}$$

- $\tilde{f}^{(2)}(t)$: analytical integration in ω is difficult but possible through Meijer G-functions
- $\tilde{f}^{(4)}(t)$, $F^{(2)} \to F^{(4)}$ [Li_2], analytical integrations of some of the integrals in ω have not been found so far.
- numerical approximations are unavoidable
- work in progress with E.Balzani and M.Passera: PRELIMINARY results



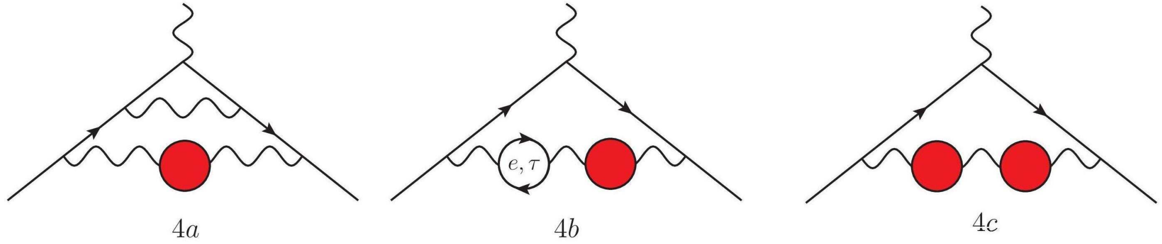


- Despite the lack of an analytical expression, we are able to get approximated space-like NNLO kernels from the first terms of the asymptotic expansions.
- For one set (6c2) containing two HVP insertions on *different* photon lines, we worked out a *bidimensional* space-like kernel.
- The precision of the contributions of these approximated space-like kernels obtained is at the level of 10^{-13}.
- Numerical approximations for the time-kernel at NLO were found.

The End

BACKUP SLIDES

NLO hadronic vacuum polarization contributions



- Class a: 1 HVP insertion in one photon line of 2-loop QED vertex diagrams
- Class b: 1 HVP insertion in the photon line of 2-loop QED vertex with one electron vacuum polarization
- Class c: 2 HVP insertion in the 1-loop QED vertex diagram

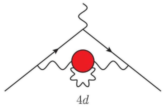
$$a_\mu^{\mathrm{HVP}}(\mathrm{NLO}; 4a) = -209.0 \times 10^{-11}$$

$$a_\mu^{\mathrm{HVP}}(\mathrm{NLO}; 4b) = +106.8 \times 10^{-11}$$

$$a_\mu^{\mathrm{HVP}}(\mathrm{NLO}; 4c) = +3.5 \times 10^{-11}$$

$$a_\mu^{\mathrm{HVP}}(\mathrm{NLO}; total) = -98.7(9) \times 10^{-11}$$

(Krause 1996, Hagiwara Liao Martin Nomura Toebner 2011, Kurz Liu Marquard Steinhauser 2014)



HVP insertion with internal corrections already incorporated in LO

$$y(z) = \frac{z-\sqrt{z(z-4)}}{z+\sqrt{z(z-4)}}$$

$$F^{(2)}(u) = \frac{u+1}{u-1}\, u^2.$$

$$\begin{aligned} F^{(4)}(u) = {} & R_1(u) + R_2(u)\ln(-u) \\ & + R_3(u)\ln(1+u) + R_4(u)\ln(1-u) \\ & + R_5(u)\Big[4\mathrm{Li}_2(u) + 2\mathrm{Li}_2(-u) \\ & \quad + \ln(-u)\ln\big((1-u)^2(1+u)\big)\Big], \end{aligned}$$

The rational functions $R_i(u)$ $(i = 1, \ldots, 5)$ are

$$\begin{aligned} R_1 &= \frac{23u^6-37u^5+124u^4-86u^3-57u^2+99u+78}{72(u-1)^2u(u+1)}, \\ R_2 &= \frac{12u^8-11u^7-78u^6+21u^5+4u^4-15u^3+13u+6}{12(u-1)^3u(u+1)^2}, \\ R_3 &= \frac{(u+1)\left(-u^3+7u^2+8u+6\right)}{12u^2}, \\ R_4 &= \frac{-7u^4-8u^3+8u+7}{12u^2}, \\ R_5 &= -\,\frac{3u^4+5u^3+7u^2+5u+3}{6u^2}. \end{aligned}$$

The dilogarithm is $\mathrm{Li}_2(u) = -\int_0^u (\mathrm{d}v/v)\ln(1-v)$.