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NLO HVP contributions to the muon $g-2$ in the space-like region

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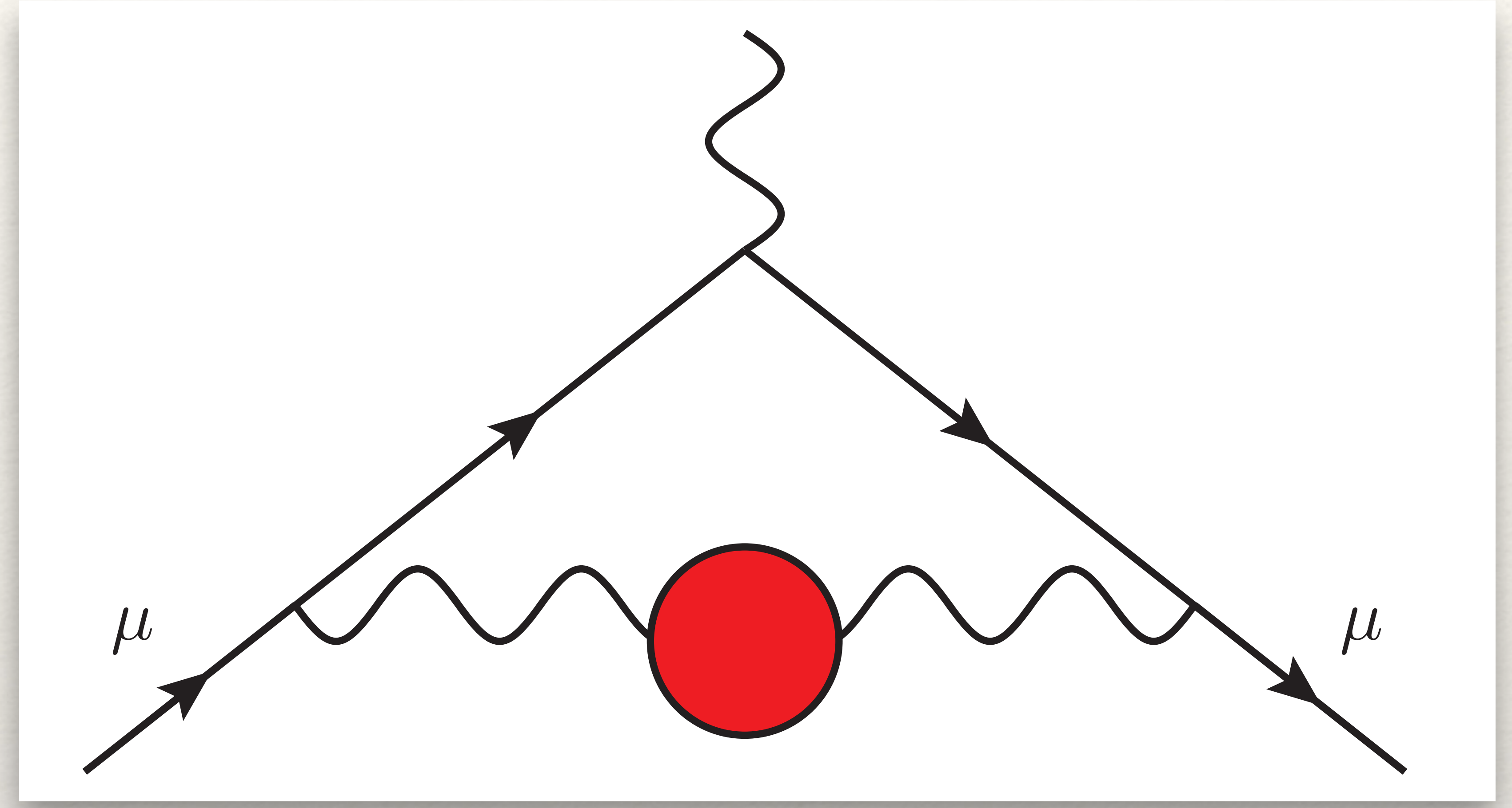
Work in Collaboration with S. Laporta and M. Passera, *Phys.Lett.B* 834 (2022) 137462

The hadronic LO contribution: time-like method



$$a_\mu^{HVP}(LO) = \frac{\alpha}{\pi^2}\int_{s_0^2}^{\infty}\frac{ds}{s}K^{(2)}(s/m^2)\mathrm{Im}\Pi_{\mathrm{h}}(\mathrm{s})$$

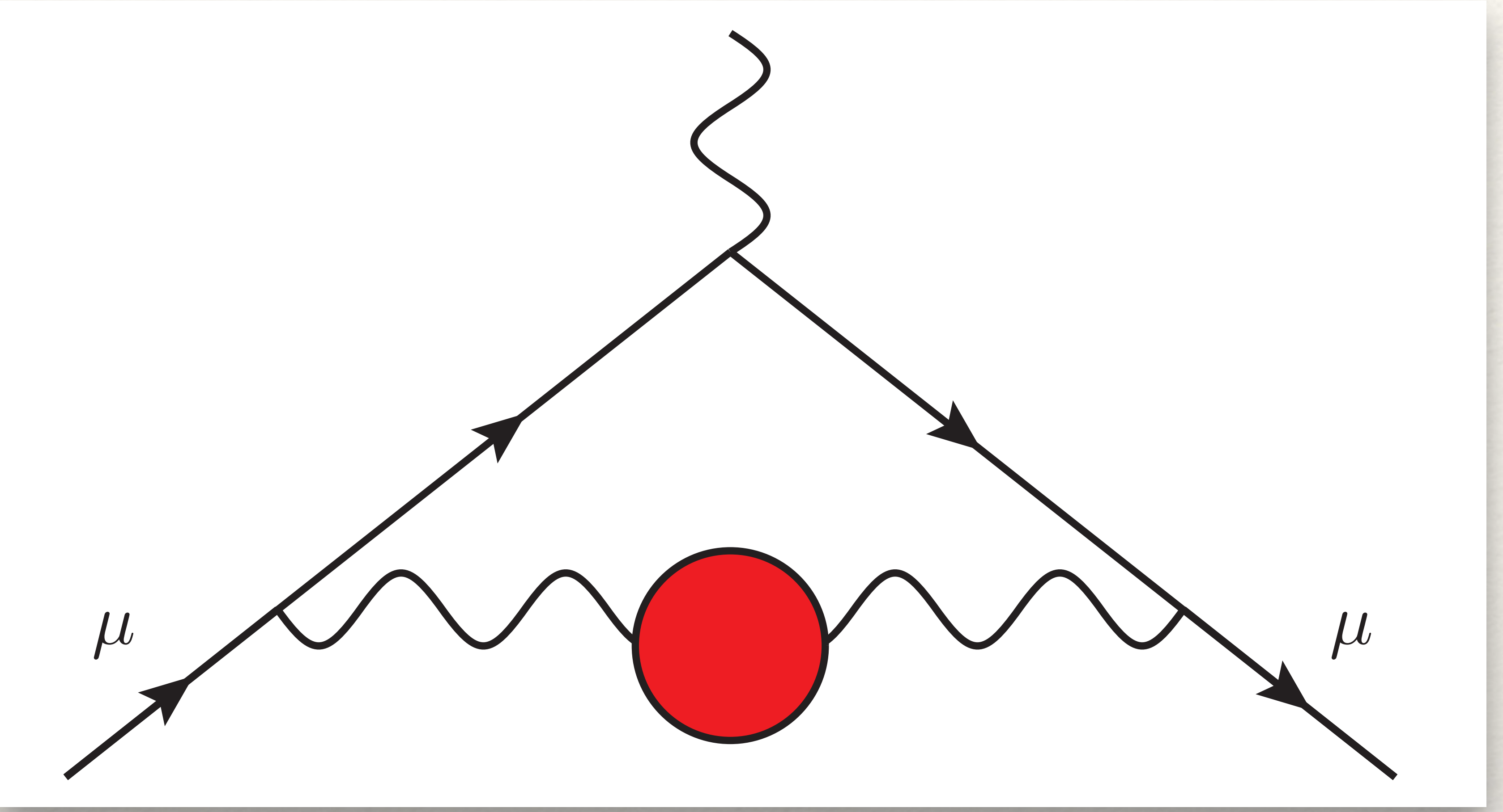
$$\mathrm{Im}\Pi_{\mathrm{h}}(\mathrm{s}) = \frac{\alpha}{3}\mathrm{R(s)} \qquad \mathrm{R(s)} = \frac{\sigma(\mathrm{e^+e^-} \to \mathrm{hadrons})}{4\pi\alpha^2/(3\mathrm{s})}$$

$$K^{(2)}(s/m^2) = \int_0^1 dx\frac{x^2(1-x)}{x^2+(1-x)(s/m^2)}$$

Defining $z = \dfrac{q^2}{m^2}$ & $y(z) = \dfrac{z-\sqrt{z(z-4)}}{z+\sqrt{z(z-4)}}$:

$$K^{(2)}(z) = \frac{1}{2} - z + \left(\frac{z^2}{2} - z\right)\ln z + \frac{\ln y(z)}{\sqrt{z(z-4)}}\left(z - 2z^2 + \frac{z^3}{2}\right)$$

The hadronic LO contribution: time-like method



$$a_\mu^{HVP}(LO) = 6895\ (33) \times 10^{-11}$$ F. Jegerlehner, arXiv:1711.06089

$$= 6939\ (40) \times 10^{-11}$$ Davier, Hoecker, Malaescu, Zhang, arXiv:1908.00921

$$= 6928\ (24) \times 10^{-11}$$ Keshavarzi, Nomura, Teubner, arXiv: 1911.00367

$$= 6931\ (40) \times 10^{-11} (0.6\%)$$ WP20 value

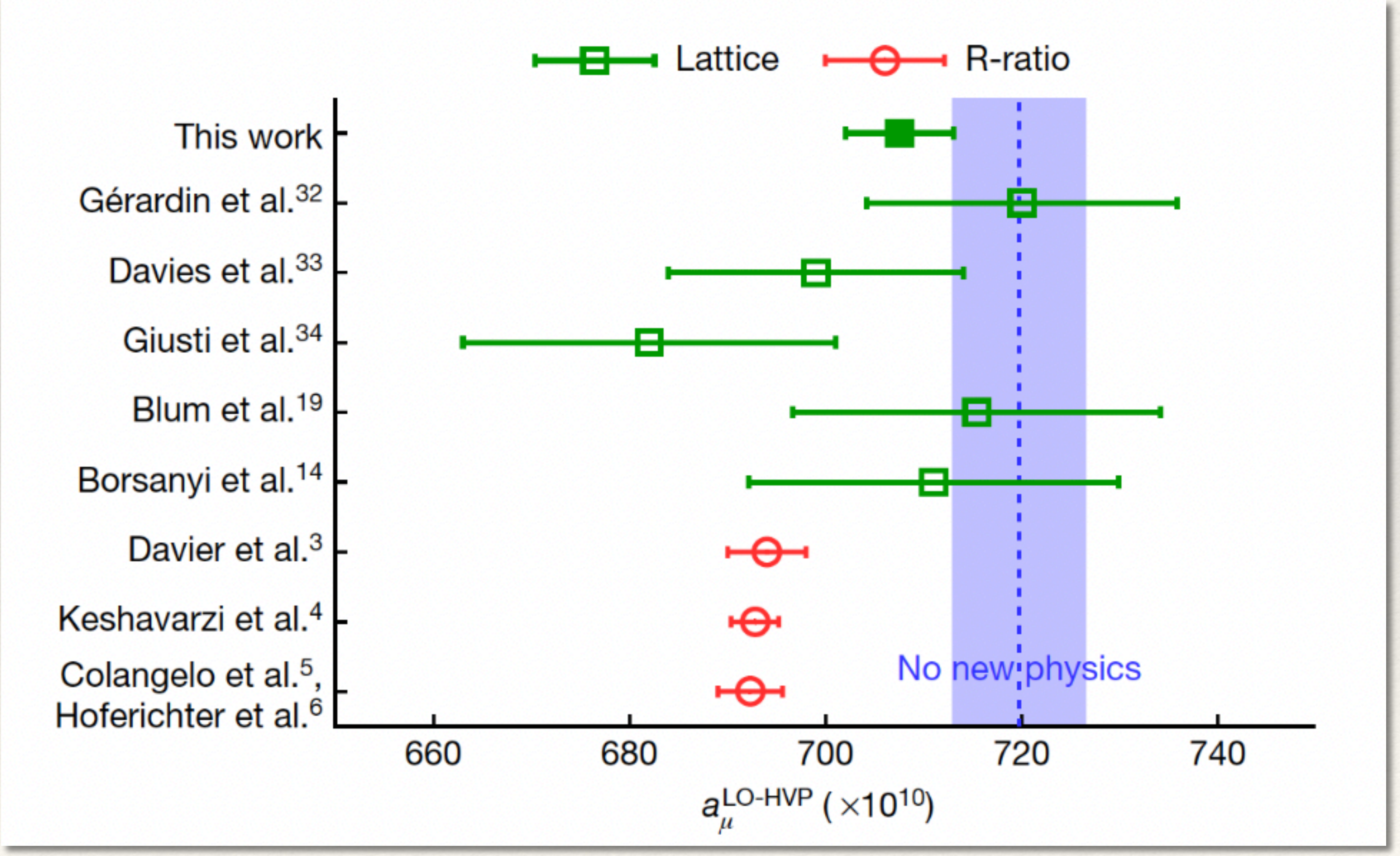
The hadronic LO contribution from lattice QCD

The BMW collaboration presented the first lattice QCD determination of a_μ^{HLO} with a 0.8 % precision

$$a_\mu^{HLO} = 7075(23)_{stat}(50)_{syst}[55]_{tot} \times 10^{-11}$$

2.2σ tension with the time-like data driven determination.

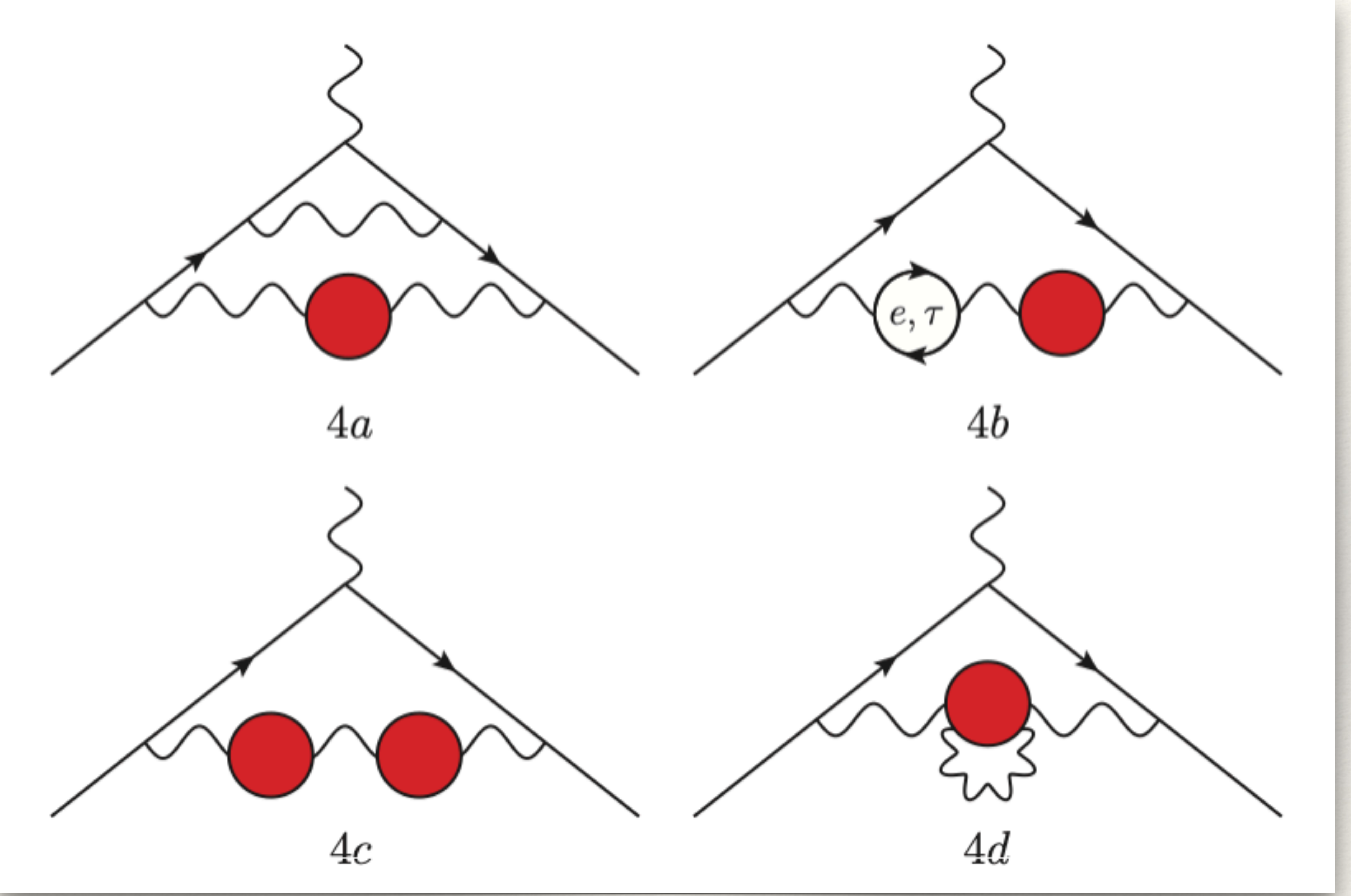
BMW collaboration 2021



Borsanyi et al (BMWc), Nature 2021

The hadronic NLO VP contribution

- $O(\alpha^3)$ contributions containing HVP insertion

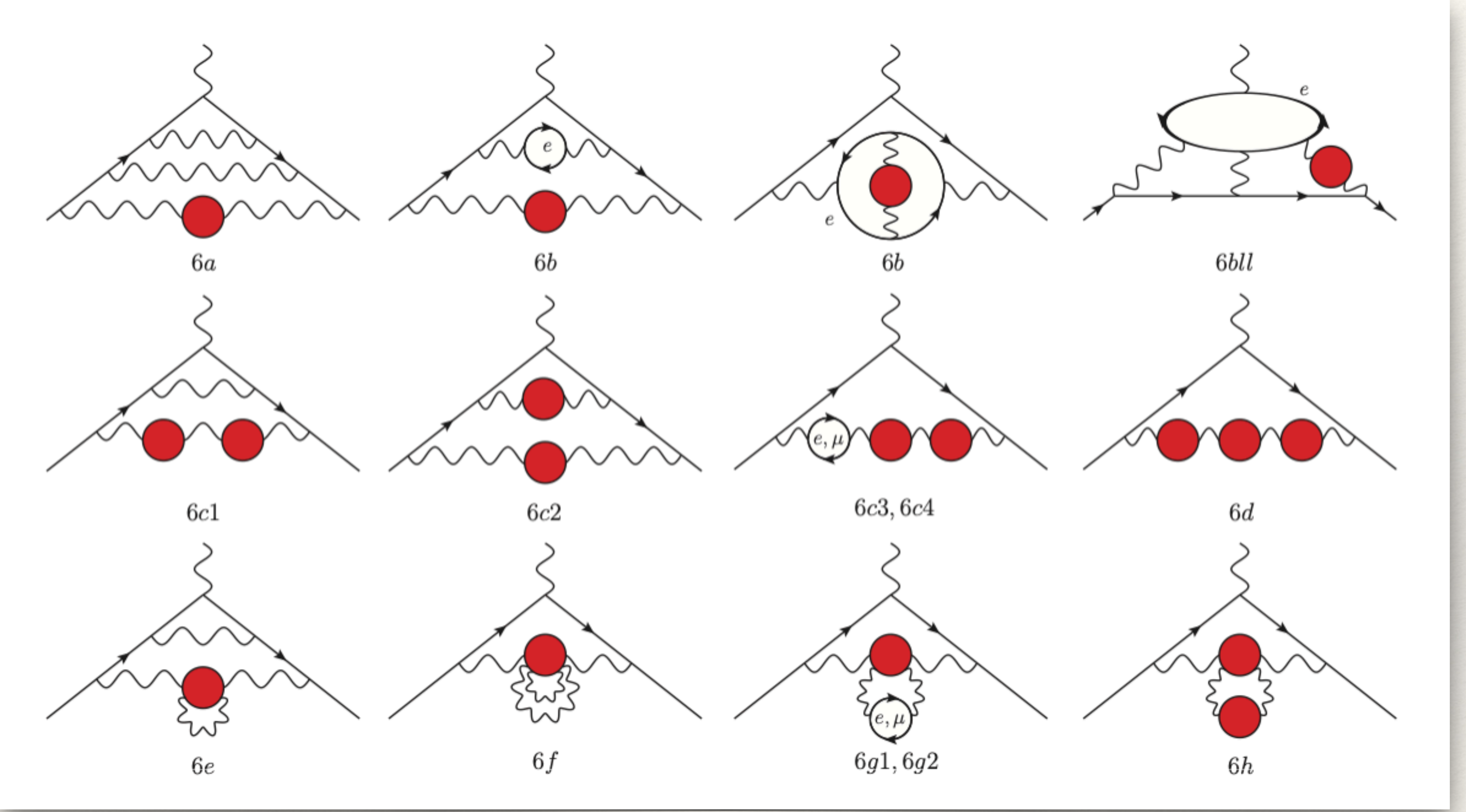


$$a_\mu^{HVP}(NLO) = -98.3(7) \times 10^{-11}$$

Krause '96; Keshevarzi, Nomura, Teubner 2019; WP20

The hadronic NNLO VP contribution

- $O(\alpha^4)$ contributions containing HVP insertion

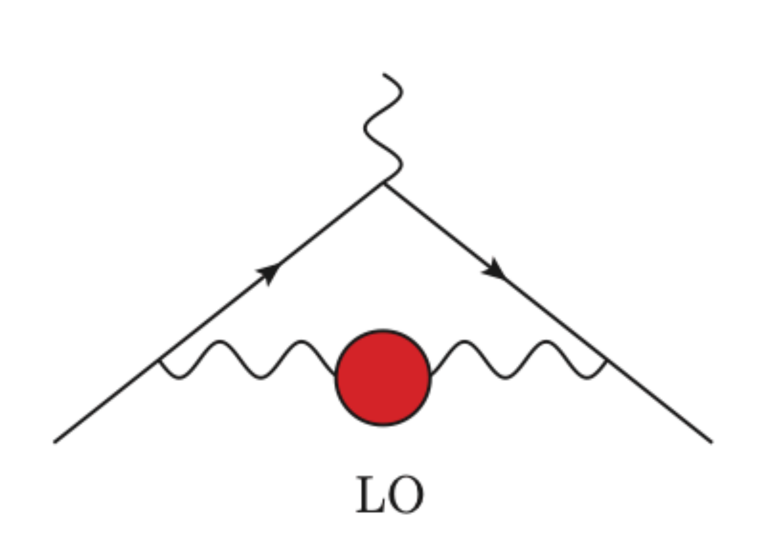


See Laporta's talk

$$a_\mu^{HVP}(NNLO) = 12.4(1) \times 10^{-11}$$

Kurz, Liu, Marquard, Steinhauser 2014

Space-like method: LO hadronic vacuum polarization contribution



$$K^{(2)}(z) = \frac{1}{\pi}\int_{-\infty}^{0} dz' \frac{\mathrm{Im}K^{(2)}(z')}{z'-z}, \qquad z > 0$$

$$\frac{\Pi(q^2)}{q^2} = \frac{1}{\pi}\int \frac{ds}{s}\frac{\mathrm{Im}\Pi(s)}{s-q^2}, \qquad q^2 < 0$$

$$a_\mu^{HVP}(LO) = -\frac{\alpha}{\pi^2}\int_{-\infty}^{0}\frac{dt}{t}\Pi_h(t)\mathrm{Im}K^{(2)}(t/m^2)$$

With the imaginary part:

$$\mathrm{Im}K^{(2)}(z+i\epsilon) = \pi\theta(-z)\left[\frac{z^2}{2} - z + \frac{z - 2z^2 + z^3/2}{\sqrt{z(z-4)}}\right] = \pi\theta(-z)F^{(2)}(1/y(z)), \qquad F^{(2)}(u) = \frac{u+1}{u-1}u^2$$

Changing the variable in the dispersive integral $t \to y \to x$: $t(x) = \dfrac{m^2x^2}{1-x^2}, \quad x = 1 + y$

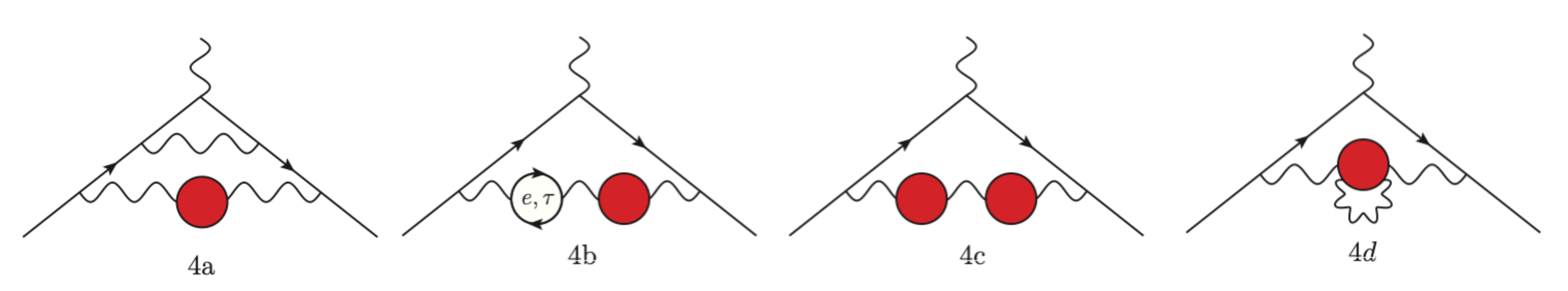
$$\Delta\alpha_h(t) = -\Pi_h(t)$$

$$a_\mu^{HVP}(LO) = \frac{\alpha}{\pi}\int_0^1 dx\, \kappa^{(2)}(x)\Delta\alpha_h(t(x))$$

$$\kappa^{(2)}(x) = 1 - x$$

Lautrup, Peterman, de Rafael 1972

Space-like method: NLO HVP contribution



$$a_\mu^{(4a)} = \frac{\alpha^2}{\pi^3}\int_{s_0}^{\infty}\frac{ds}{s}2K^{(4)}(s/m^2)\mathrm{Im}\Pi_{\mathrm{h}}(\mathrm{s})$$

$K^{(4)}$ from Barbieri Remiddi 1975

$$a_\mu^{(4a)} = -\frac{\alpha^2}{\pi^3}\int_{-\infty}^{0}\frac{dt}{t}\Pi_h(t)\mathrm{ImK}^{(4)}(\mathrm{t/m^2})$$

$$\mathrm{ImK}^{(4)}(\mathrm{z}+\mathrm{i}\epsilon) = \pi\theta(-\mathrm{z})\mathrm{F}^{(4)}(1/\mathrm{y}(\mathrm{z}))$$

$$\begin{aligned} F^{(4)}(u) = {} & R_1(u) + R_2(u)\ln(-u) \\ & + R_3(u)\ln(1+u) + R_4(u)\ln(1-u) \\ & + R_5(u)\Big[4\,\mathrm{Li}_2(u) + 2\,\mathrm{Li}_2(-u) \\ & \quad + \ln(-u)\ln\left((1-u)^2(1+u)\right)\Big] \end{aligned}$$

$$R_1 = \frac{23u^6 - 37u^5 + 124u^4 - 86u^3 - 57u^2 + 99u + 78}{72(u-1)^2u(u+1)},$$

$$R_2 = \frac{12u^8 - 11u^7 - 78u^6 + 21u^5 + 4u^4 - 15u^3 + 13u + 6}{12(u-1)^3u(u+1)^2},$$

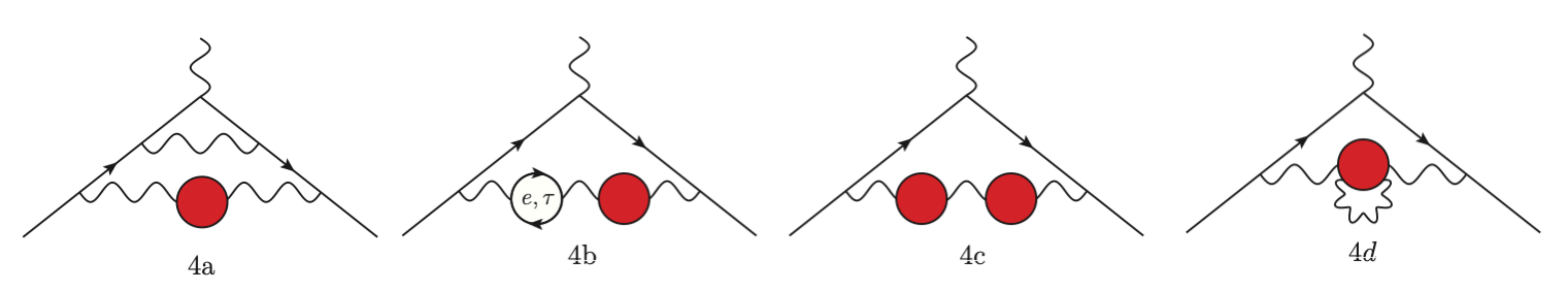
$$R_3 = \frac{(u+1)\left(-u^3 + 7u^2 + 8u + 6\right)}{12u^2},$$

$$R_4 = \frac{-7u^4 - 8u^3 + 8u + 7}{12u^2},$$

$$R_5 = -\frac{3u^4 + 5u^3 + 7u^2 + 5u + 3}{6u^2}.$$

Obtained independently by Nesterenko, arXiv: 2112.05009

Space-like method: NLO hadronic vacuum polarization contribution



$$a_\mu^{(4a)} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^1 dx\ \kappa^{(4)}(x)\Delta\alpha_h(t(x)),$$

$$\kappa^{(4)} = \frac{2(2-x)}{x(x-1)} F^{(4)}(x-1)$$

$$a_\mu^{(4b)} = \left(\frac{\alpha}{\pi}\right) \int_0^1 dx\ \kappa^{(2)}(x)\Delta\alpha_h(t(x)) \times 2\left[\Delta\alpha_e^{(2)}(t(x)) + \Delta\alpha_\tau^{(2)}(t(x))\right]$$

$$a_\mu^{(4c)} = \left(\frac{\alpha}{\pi}\right) \int_0^1 dx\ \kappa^{(2)}(x)[\Delta\alpha_h(t(x))]^2$$

Where $\Pi_\ell^{(2)}(t) = -\Delta\alpha_\ell^{(2)}(t)$ and $\Pi_\ell^{(2)}(t) = \frac{\alpha}{\pi}\left(\frac{8}{9} - \frac{\beta_\ell^2}{3} + \beta_\ell\left(\frac{1}{2} - \frac{\beta_\ell^2}{6}\right)\ln\frac{\beta_\ell - 1}{\beta_\ell + 1}\right)$, with $\beta_\ell = \sqrt{1 - 4m_\ell^2/t}$.

Space-like method: NLO hadronic vacuum polarization contribution

Chakraborty at al. (arXiv: 1806.08190) provided an approximated expression for the space-like kernel function. They started from :

$$K^{(4)}(s) = 2\frac{m^2}{s}\Bigg\{\left[\frac{223}{54} - 2\zeta(2) - \frac{23}{36}\ln\frac{s}{m^2}\right]$$
$$+\frac{m^2}{s}\left[\frac{8785}{1152} - \frac{37}{8}\zeta(2) - \frac{367}{216}\ln\frac{s}{m^2} + \frac{19}{144}\ln^2\frac{s}{m^2}\right]$$
$$+\frac{m^4}{s^2}\left[\frac{13072841}{432000} - \frac{883}{40}\zeta(2) - \frac{10079}{3600}\ln\frac{s}{m^2} + \frac{141}{80}\ln^2\frac{s}{m^2}\right]$$
$$+\frac{m^6}{s^3}\left[\frac{2034703}{16000} - \frac{3903}{40}\zeta(2) - \frac{6517}{1800}\ln\frac{s}{m^2} + \frac{961}{80}\ln^2\frac{s}{m^2}\right]\Bigg\}$$

With $r = \frac{m^2}{s}$

Krause '97

They exploited generating integral representation to fit the r^n and $r^n \ln r$, but not the $r^n \ln^2 r$ ones.

$$m^2\int_0^1 \frac{dxP(x)}{m^2x+s} = \frac{m^2}{s}\sum_n a_n\left(\frac{m^2}{s}\right)^n$$

for the power terms

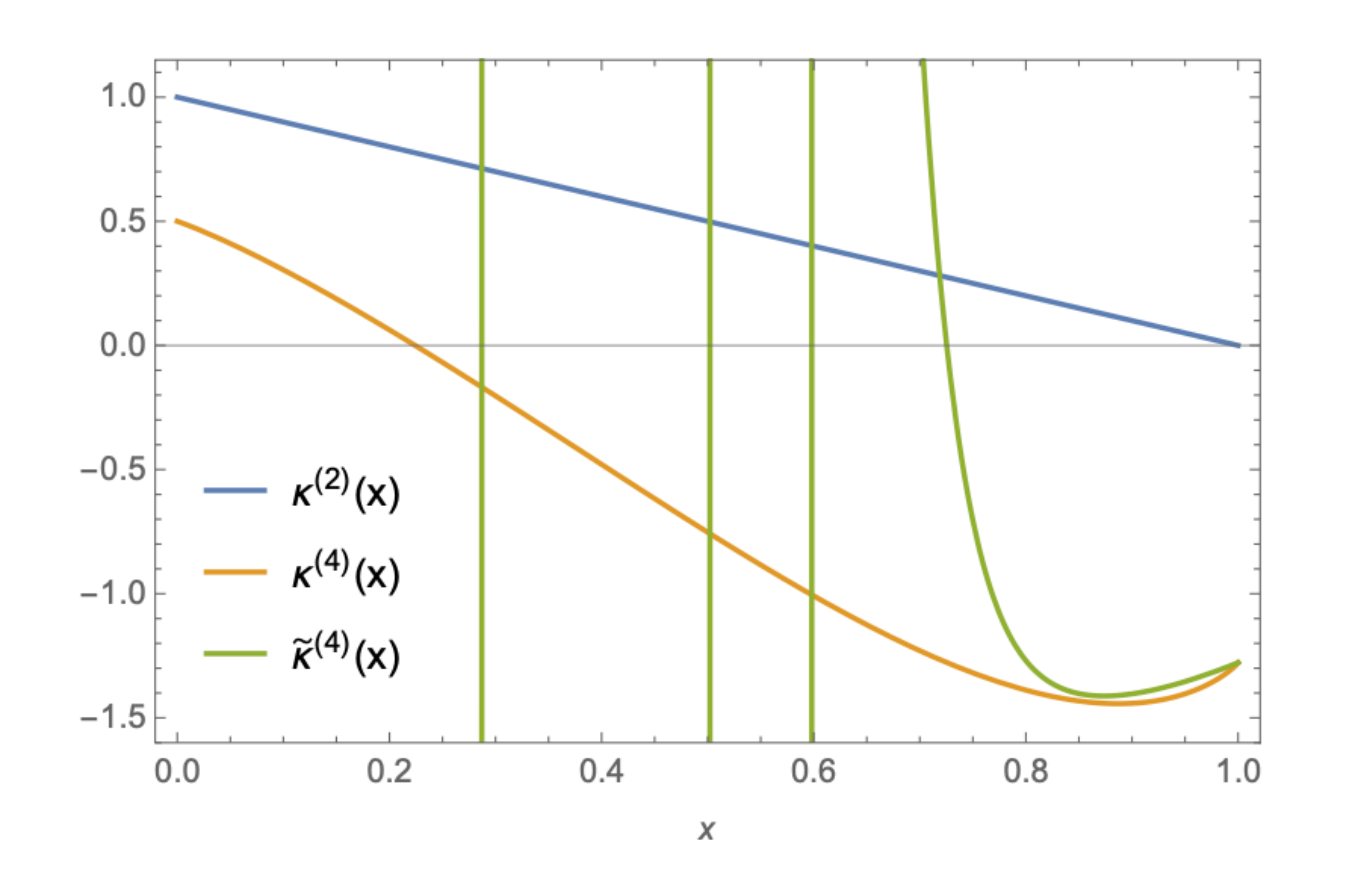
$$m^2\int_0^1 \frac{dxG(x)}{sx+m^2} = G_1\left(m^2/s\right) + G_2\left(m^2/s\right)\ln\left(\frac{s}{m^2}\right)$$

for the log terms

Groote et al. 2002

Space-like method: NLO hadronic vacuum polarization contribution

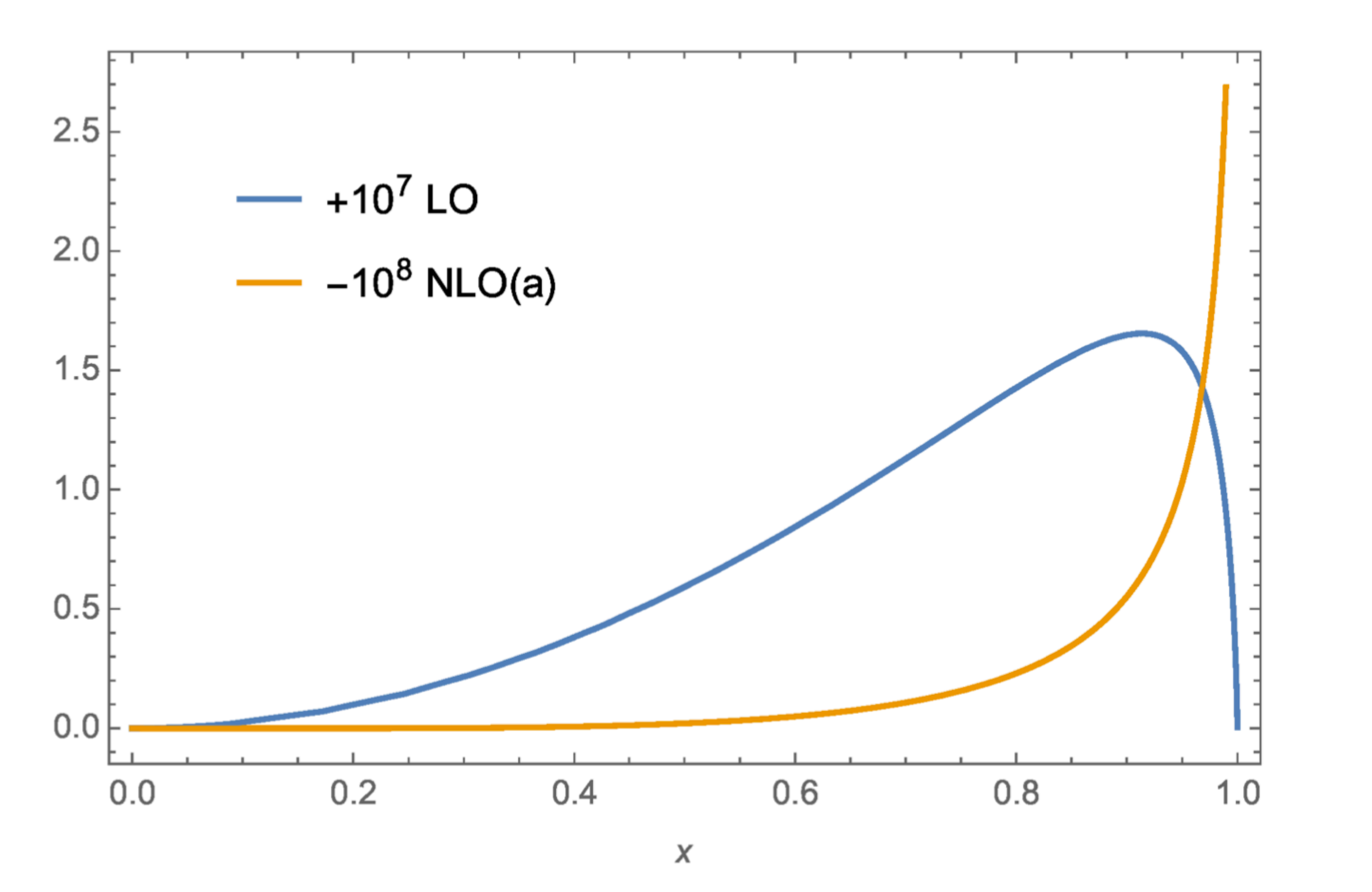
After simple changes of variables their approximation can be compared to our exact function $\kappa^{(4)}(x)$.



The authors added a $O(10\%)$ uncertainties to their final result.

This uncertainty can be eliminated using our exact formula $\kappa^{(4)}(x)$.

Space-like method: NLO hadronic vacuum polarization contribution



- The LO integrand has a peak at $x \sim 0.914$
- The NLO integrand has an integrable logarithmic singularity at $x \to 1$

Conclusions

- We provide simple analytic expressions to calculate the HVP contributions to the muon $g-2$ in the space-like region up to NLO.
- These results can be employed at MUonE, as well as in lattice QCD computations.
- An existing approximation for the NLO kernel induced large uncertainties. These can be eliminated using the exact expression $\kappa^{(4)}(x)$.
- Calculation of higher-order HVP corrections require a precise treatment of the QED radiative corrections to the HVP. MUonE will naturally include all of these corrections in the space-like approach.
- These results allow to compare time-like and space-like calculations of $a_\mu^{(HVP)}$ at NLO accuracy.

Thank you for your attention!