Reconstruction Approximants to Hadronic Vacuum Polarization and the MUonE proposal

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The Evaluation of the Leading Hadronic Contribution to the Muon g-2:
Towards the MUonE Experiment – 17th Nov. 2022

In collaboration with Eduardo de Rafael

Based on D.G. and E. de Rafael, arXiv [hep-ph] 2202.10810, JHEP **05** (2022) 084

Anatomy of the muon anomaly

T. Aoyama *et al.* (Muon g-2 Theory Initiative), Phys. Rep. 887 1 (2020) ≐ White Paper (2020).

From the new experimental average value FNAL (2021),

$$a_\mu^{\text{exp.}} = 116\,592\,061(41) \cdot 10^{-11} (0.35\,\text{ppm})$$

there is a persistent discrepancy of 4.2σ with the SM evaluation WP (2020)

$$\Delta a_\mu = a_\mu^{\text{exp.}} - a_\mu^{\text{SM}} = (251 \pm 59) \cdot 10^{-11}$$

HVP-LO	6933 ± 25	A. Keshavarzi *et al.* (2018)
	6931 ± 40	White Paper (2020)
HVP-NLO	-99.3 ± 7	F. Jegerlehner (2017)
	-98.7 ± 7	White Paper (2020)
HVP-NNLO	$+12.2 \pm 1$	F. Jegerlehner (2017)
	$+12.4 \pm 1$	White Paper (2020)
HLbL	$+105 \pm 26$	J. Prades *et al.* (2009)
	$+92 \pm 18$	White Paper (2020)
EW	153.6 ± 1	White Paper (2020)

All in 10^{-11} units

The evaluation of the HVP-LO is the most important to consider

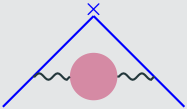
HVP contribution to the anomaly

HVP and a_μ

The two-point VV correlator Π obeys to a one sub. disp. rep. $Q^2 = -(q^2 = t) > 0$

$$= \Pi(-Q^2) = \int_{4m_\pi^2}^{\infty} \frac{\mathrm{d}t}{t} \left[\frac{-Q^2}{t + Q^2} \right] \frac{1}{\pi} \mathrm{Im}\,\Pi(t)$$

which gives a contribution



$$a_\mu^{\mathsf{HVP}} = \frac{\alpha}{\pi} \int_{4m_\pi^2}^{\infty} \frac{\mathrm{d}t}{t} \underbrace{\int_0^1 \mathrm{d}x \frac{x^2(1-x)}{x^2 + \frac{t}{m_\mu^2}(1-x)}}_{\doteq K\left(\frac{t}{m_\mu^2}\right)} \frac{1}{\pi} \mathrm{Im}\,\Pi(t)$$

Determination of a_μ^{HVP}

$$a_\mu^{\mathsf{HVP}} = \int_0^1 \mathrm{d}x(1-x) \int_{4m_\pi^2}^{\infty} \frac{\mathrm{d}t}{t} \left[\frac{\frac{x^2}{1-x} m_\mu^2}{t + \frac{x^2}{1-x} m_\mu^2} \right] \frac{1}{\pi} \mathrm{Im}\,\Pi(t) \qquad \longleftarrow \sigma[e\bar{e} \xrightarrow[\gamma *]{} \mathsf{Hadr.}](t) = \frac{4\pi^2 \alpha}{t} \mathrm{Im}\,\Pi(t)$$

$$= -\int_0^1 \mathrm{d}x\,(1-x)\,\Pi\left(-\frac{x^2}{1-x} m_\mu^2\right) \qquad \longleftarrow \text{Euclidean representation}$$

C. Bouchiat and L. Michel (1961); B. Lautrup and E. de Rafael (1969); E. de Rafael (1994)

The new Lattice QCD result on a_μ^{HVP} – BMW (2021)

Sz. Borsanyi *et al.*, Nature **593** 51 (2021)

LQCD reconstructs the Euclidean part $\Pi(-Q^2)$ and gives

$$a_\mu^{\text{HVP}}(\text{BMW}) = (7075 \pm 55) \cdot 10^{-11}$$

which differs by 2.1σ of the WP evaluation $a_\mu^{\text{HVP}}(\text{WP}) = (6845 \pm 40) \cdot 10^{-11}$ and

$$\Delta a_\mu = a_\mu^{\text{exp.}} - a_\mu^{\text{SM}}(\text{BMW}) = (21 \pm 30) \cdot 10^{-11}$$

which is a difference of 1.6σ (for the WP is 4.2σ).

Confirmed by:

M. Cé et al., arXiv:2206.06582 [hep-lat]

C. Alexandrou et al., arXiv:2206.15084 [hep-lat]

Conclusion

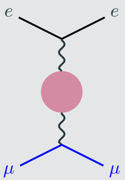
It is clearly important to understand this discrepancy both theoretically and experimentally.

The MUonE proposal at CERN

The MUonE proposal at the SPSC at CERN

G. Abbiendi *et al.* Letter of Intent: The MUonE Project, CERN-SPSC-2019-026 / SPSC-I-252 (2019)

Measuring the hadronic contribution to the Bhabha scattering $e\mu \to e\mu$

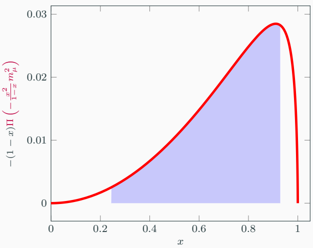


$$\frac{d\sigma^{\text{LO}}_{\text{Had.}}}{dQ^2} = 2\,\Pi(-Q^2)\,\frac{d\sigma_0}{dQ^2}$$

where the Born term is

$$\frac{d\sigma_0}{dQ^2} = 4\pi\alpha^2\,\frac{2(m_\mu^2+m_e^2)^2 - 2su - Q^4}{2Q^4\lambda(s,m_\mu^2,m_e^2)}$$

and $s - Q^2 + u = 2m_e^2 + 2m_\mu^2$, $s = 2m_e E_\mu + m_\mu^2 + m_e^2$, λ is the Källen function



$$a_\mu^{\text{HVP}} = -\int_0^1 \mathrm{d}x\,(1-x)\,\Pi\left(-\frac{x^2}{1-x}m_\mu^2\right)$$

The expected range of measurements

$$0.23 \leqslant x \leqslant 0.93$$

This area represents at the best 87% of the total value.

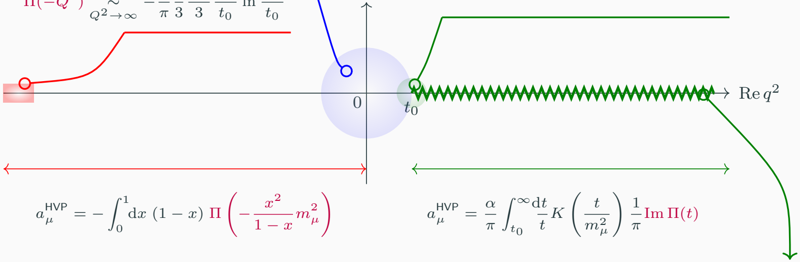
We need an efficient extrapolation of Π !

HVP structure and expansions

$$\Pi(q^2) \underset{q^2\to 0}{\sim} \sum_{n=1}^{\infty} \mathcal{M}(1-n) \left(\frac{q^2}{t_0}\right)^n$$

$$\Pi(-Q^2) \underset{Q^2\to\infty}{\sim} -\frac{\alpha}{\pi}\frac{5}{3}\frac{N_c}{3}\frac{Q^2}{t_0}\ln\frac{Q^2}{t_0}$$

$$\frac{1}{\pi}\mathrm{Im}\,\Pi(q^2) \underset{q^2\to t_0}{\sim} \frac{\alpha}{\pi}\frac{1}{12}\left(\frac{q^2}{t_0}-1\right)^{\frac{3}{2}}$$



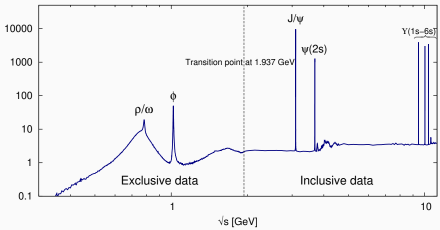
$$a_\mu^{\text{HVP}} = -\int_0^1 \mathrm{d}x\,(1-x)\,\Pi\left(-\frac{x^2}{1-x}m_\mu^2\right)$$

$$a_\mu^{\text{HVP}} = \frac{\alpha}{\pi}\int_{t_0}^{\infty}\frac{\mathrm{d}t}{t}K\left(\frac{t}{m_\mu^2}\right)\frac{1}{\pi}\mathrm{Im}\,\Pi(t)$$

Spectral Function Moments

The moments of $\frac{1}{\pi}\mathrm{Im}\,\Pi$ $(n \in \mathbf{N})$

$$\mathcal{M}(1-n) = \int_{t_0}^{\infty}\frac{\mathrm{d}t}{t}\left(\frac{t_0}{t}\right)^n\frac{1}{\pi}\mathrm{Im}\,\Pi(t) = \frac{t_0^n}{n!}\frac{\partial^n\Pi(t)}{\partial t^n}\bigg|_{t=0}$$



From A. Keshavarzi *et al.* Phys. Rev. D **97** (2018)

What should be a good HVP extrapolation?

- The anomaly requires to know:
  - $\Pi(-Q^2)$ for $0 < Q^2 < \infty$
  - Or $\mathrm{Im}\,\Pi(q^2)$ for $t_0 < q^2 < \infty$.

- This is equivalent to know all the moments $\mathcal{M}(1-n)$.

- We are looking for an extrapolation method which satisfies the analytic properties of QCD

A mathematical interlude

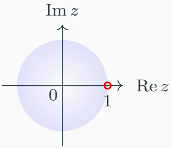
Flajolet and Odlyzko's transfer theorem

Ph. Flajolet and A.M. Odlyzko, SIAM Journal Discrete Math.**3** 2 216 (1990)

Ph. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, (2009)

Transfer Theorem

"The asymptotic expansion of a function near its dominant singularities provides the asymptotic expansion of the function's coefficients."



Assume that $f\colon \mathbf{C} \to \mathbf{C}$ is analytic inside the unit disc,

$$f(z) \underset{|z|<1}{=} \sum_{n=1} f_n z^n \ ,$$

and at $z = 1$, f has the singular behaviour – $(\alpha, \beta) \in \mathbf{C}^2$

$$f(z) \underset{z\to 1}{\sim} (1-z)^{\alpha} \ln^{\beta}(1-z)$$

then the *transfer theorem states*

$$f_n \underset{n\to\infty}{\sim} f_n^{\mathrm{AS}} \doteq \frac{\ln^{\beta-1} n}{n^{\alpha+1}} \sum_{k=-|\lfloor \alpha+1 \rfloor|}^{\infty} \frac{P_k(\ln n)}{n^k}$$

where P_k are polynomials.

Transfer Theorem: some examples

Ph. Flajolet and A.M. Odlyzko, SIAM Journal Discrete Math.**3** 2 216 (1990)

Ph. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, (2009)

Some examples how the *transfer* works

$$f(z) \underset{z\to 1}{\sim} \frac{1}{1-z} \qquad \longmapsto \qquad f_n \underset{n\to\infty}{\sim} f_n^{\mathrm{AS}} = 1$$

$$f(z) \underset{z\to 1}{\sim} \frac{1}{(1-z)^2} \qquad \longmapsto \qquad f_n \underset{n\to\infty}{\sim} f_n^{\mathrm{AS}} = n+1$$

$$f(z) \underset{z\to 1}{\sim} \sqrt{1-z} \qquad \longmapsto \qquad f_n \underset{n\to\infty}{\sim} f_n^{\mathrm{AS}} = -\frac{1}{2\sqrt{\pi}}\frac{1}{\sqrt{n}}\left[\frac{1}{n^2}+\frac{3}{8}\frac{1}{n^3}+\cdots\right]$$

$$f(z) \underset{z\to 1}{\sim} = \frac{1}{1-z}\ln\left(\frac{1}{1-z}\right) \qquad \longmapsto \qquad f_n \underset{n\to\infty}{\sim} f_n^{\mathrm{AS}} = \ln n + \gamma_E + \frac{1}{2n} + \cdots$$

$$f(z) \underset{z\to 1}{\sim} (1-z)^m \ln\left(\frac{1}{1-z}\right) \qquad \longmapsto \qquad f_n \underset{n\to\infty}{\sim} f_n^{\mathrm{AS}} = \frac{(-1)^m\Gamma(1+m)}{n^{m+1}}\sum_{k=0}^{\infty}\left\{ {m+k \atop m} \right\}\frac{1}{n^k}$$

for any m positive integers and $\{ {\cdot \atop \cdot} \}$ Stirling numbers of 2nd kind.

Reconstruction approximants

From the exact equality

$$f(z) = \sum_{n=1}^{\infty} f_n z^n = \sum_{n=1}^{\infty} \underbrace{\left(f_n - f_n^{\mathrm{AS}}\right)}_{=\mathcal{A}_n} z^n + \sum_{n=1}^{\infty} f_n^{\mathrm{AS}} z^n$$

where the f_n^{AS} are given by the *transfer theorem*, then f can be approximated by

$$f(z) \approx f_N(z) \doteq \sum_{n=1}^{N} \mathcal{A}_n z^n + \sum_{n=1}^{\infty} f_n^{\mathrm{AS}} z^n$$

where $\mathcal{A}_n$ are treated as free parameters. The *local systematic error* is then

$$\mathcal{E}_N(z) = \sum_{n=N+1}^{\infty} \mathcal{A}_n z^n$$

Moreover

$$\text{if} \quad f_n^{\mathrm{AS}} = \sum_{\ell=1}^{L} \frac{\mathcal{B}_\ell}{n^\ell} \quad \text{then} \quad \sum_{n=1}^{\infty} f_n^{\mathrm{AS}} z^n = \sum_{\ell=1}^{L} \mathcal{B}_\ell \sum_{n=1}^{\infty} \frac{z^n}{n^\ell} = \sum_{\ell=1}^{L} \mathcal{B}_\ell \, \mathrm{Li}_\ell(z)$$

where the Li_ℓ are the Polylogarithm functions. Therefore

$$f(z) \approx f_{N,L}(z) \doteq \sum_{n=1}^{N} \mathcal{A}_n z^n + \sum_{\ell=1}^{L} \mathcal{B}_\ell \, \mathrm{Li}_\ell(z)$$

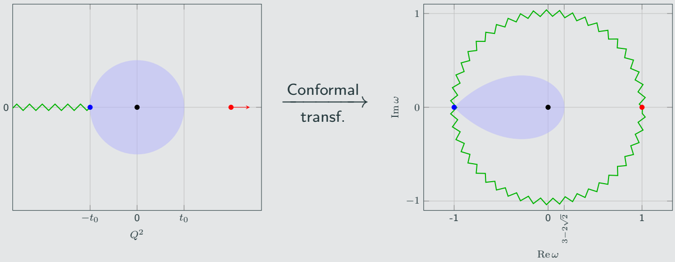
One can show that $f_{N,L}(z) \xrightarrow[N\to\infty]{} f(z)$ and $f_{N,L}(z) \xrightarrow[L\to\infty]{} f(z)$

How apply reconstruction approximants to HVP?

How to **transfert** asymptotic expansions of Π to its moments $\mathcal{M}(1-n)$?

By a *conformal change of variable*,

$$\frac{Q^2}{t_0} = \frac{4\omega}{(1-\omega)^2} \quad\longleftrightarrow\quad \omega = \begin{cases} \dfrac{\sqrt{1+\frac{Q^2}{t_0}}-1}{\sqrt{1+\frac{Q^2}{t_0}}+1} & \text{when } \dfrac{Q^2}{t_0} > -1 \\[2ex] \dfrac{i\sqrt{\tau-1}+1}{i\sqrt{\tau-1}-1} & \text{when } \dfrac{Q^2}{t_0} = -\tau < -1 \end{cases}$$



Π becomes

$$\Pi(-Q^2) \underset{|Q^2|<t_0}{=} \sum_{n=1}^{\infty} \mathcal{M}(1-n)\left(-\frac{Q^2}{t_0}\right)^n \quad\longleftrightarrow\quad \Pi\left(-\frac{4\omega}{(1-\omega)^2}\right) \underset{|\omega|<1}{=} \sum_{n=1}^{\infty} \Omega_n \omega^n$$

The *reconstruction approximants* can be applied to Π as a function of ω.

Taylor expansion

The Taylor expansion at $\omega = 0$ is known since Ω_n are a linear combination of the moments $\mathcal{M}(1-n)$,

$$\mathcal{M}(1-n) = \frac{4^{-n}}{n} \sum_{k=1}^{n} \binom{2n}{n-k} k\, \Omega_k \longleftrightarrow \Omega_n = \sum_{p=1}^{n} \binom{n+p-1}{2p-1} \mathcal{M}(1-p)(-4)^p$$

The large Q^2 expansion

The generic large Q^2 expansion of Π in perturbative QCD is

$$\Pi(-Q^2) \underset{Q^2 \to \infty}{\sim} \sum_{\substack{p \geqslant 1 \\ k \geqslant 0}} \frac{(-1)^k}{k!} \mathcal{R}_{p,k} \left(\frac{Q^2}{t_0}\right)^{1-p} \ln^k \left(\frac{Q^2}{t_0}\right) ,$$

where the first coefficient is $\mathcal{R}_{1,1} = \frac{\alpha}{\pi} \frac{N_c}{3} \sum e^2_{\text{quarks}} = \frac{\alpha}{\pi} \frac{5}{3}$

In our case we take $k = 0, 1$ (compatible with all known phenomenological models in particular sum of Breit-Wigner's resonances)

$$\Pi\left(-\frac{4\omega}{(1-\omega)^2}\right) \underset{\omega \to 1}{\sim} \sum_{m \geqslant 0} \tilde{\mathcal{R}}_{m,1} (1-\omega)^m \ln\left(\frac{1}{1-\omega}\right) ,$$

The *transfer theorem* gives

$$\Omega_n \underset{n \to \infty}{\sim} \Omega_n^{\text{AS}} = \sum_{\ell \geqslant 0} \frac{\mathcal{B}_\ell}{n^\ell}$$

where $\mathcal{B}_1 = -2\mathcal{R}_{1,1}$ and $\mathcal{B}_\ell$ are a linear combination of the $\mathcal{R}_{p,1}$, with $\mathcal{B}_{2k} = 0$.

Spectral function

On the circle $|\omega| = 1$, we are moving on the cut $1 < \tau \doteq \frac{q^2}{t_0} < \infty$,

$$\mathrm{Im}\,\Pi\left(\tau = \frac{4\omega}{(1-\omega)^2}\right) = \sum_{n=1}^{\infty} \Omega_n \sin(n \arg \omega) = -\frac{2\sqrt{\tau - 1}}{\tau} \sum_{n=1}^{\infty} \Omega_n \, \mathrm{U}_{n-1}\left(1 - \frac{2}{\tau}\right)$$

where U_n are the Chebyshev polynomials of the 2nd kind. One gets the threshold expansion

$$\frac{1}{\pi}\mathrm{Im}\,\Pi\,(\tau) \underset{\tau \to 1}{\sim} \left[\frac{2}{\pi}\sum_{n=1}^{\infty} \Omega_n n(-1)^n\right]\sqrt{\tau - 1} + \left[-\frac{2}{3\pi}\sum_{n=1}^{\infty} \Omega_n n(1+2n^2)(-1)^n\right](\tau - 1)^{\frac{3}{2}}$$

According to Chiral Perturbation Theory,

$$\frac{1}{\pi}\mathrm{Im}\,\Pi(\tau) \underset{\tau \to 1}{\sim} \frac{\alpha}{\pi}\frac{1}{12}|F_\pi(t_0)|^2\,(\tau - 1)^{\frac{3}{2}}$$

$F_\pi(t_0)$ is the e.m. form factor of the pion at the threshold. By identification, we have the *two constraints*

$$\sum_{n=1}^{\infty} \Omega_n n(-1)^n = 0 \quad \text{and} \quad -\frac{4}{3\pi}\sum_{n=1}^{\infty} \Omega_n n^3(-1)^n = \frac{\alpha}{\pi}\frac{1}{12}|F_\pi(t_0)|^2$$

Reconstruction Approximants of the HVP

The *reconstruction approximants* are

$$\Pi_N\left(-\frac{4\omega}{(1-\omega)^2}\right) = \sum_{n=1}^{N} \underbrace{\left(\Omega_n - \Omega_n^{\mathrm{AS}}\right)}_{=\mathcal{A}_n} \omega^n + \sum_{n=1}^{\infty} \Omega_n^{\mathrm{AS}} \omega^n$$

From the *transfer theorem* we know that

$$\Omega_n^{\mathrm{AS}} = \sum_{\ell=1}^{L} \frac{\mathcal{B}_\ell}{n^\ell} \implies \sum_{n=1}^{\infty} \Omega_n^{\mathrm{AS}} \omega^n = \sum_{\ell=1}^{L} \mathcal{B}_\ell \sum_{n=1}^{\infty} \frac{\omega^n}{n^\ell} = \sum_{\ell=1}^{L} \mathcal{B}_\ell \,\mathrm{Li}_\ell(\omega)$$

where the Li_ℓ are the Polylogarithm functions.

Reconstruction Approximants of the HVP

The Euclidean *reconstruction approximants* of the HVP are given by

$$\Pi\left(-Q^2\right) \approx \Pi_{N,L}\left(-Q^2\right) = \sum_{n=1}^{N} \mathcal{A}_n \left(\frac{\sqrt{1+\frac{Q^2}{t_0}}-1}{\sqrt{1+\frac{Q^2}{t_0}}+1}\right)^n + \sum_{\ell=1}^{L} \mathcal{B}_\ell \,\mathrm{Li}_\ell\left(\frac{\sqrt{1+\frac{Q^2}{t_0}}-1}{\sqrt{1+\frac{Q^2}{t_0}}+1}\right)$$

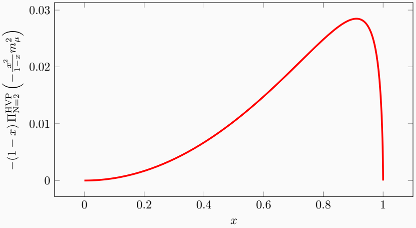
with the *constraints*

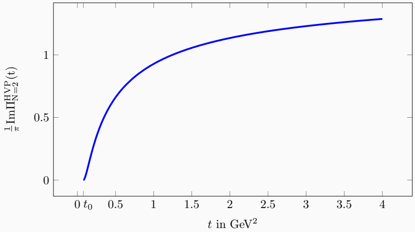
$$\sum_{n=1}^{N} \mathcal{A}_n n(-1)^n - \sum_{\ell=1}^{L} \mathcal{B}_\ell \,\mathrm{Li}_{\ell-1}(-1) = 0$$

$$\sum_{n=1}^{N} \mathcal{A}_n n^3(-1)^n - \sum_{\ell=1}^{L} \mathcal{B}_\ell \,\mathrm{Li}_{\ell-3}(-1) = -\frac{3\pi}{4} \cdot \frac{\alpha}{\pi} \frac{1}{12} |F_\pi(t_0)|^2$$

A first example

QCD asymptotic Freedom fixes: $\mathcal{B}_1 = (-2)\frac{\alpha}{\pi}\frac{5}{3}$ and *lowest order* χPT fixes $|F_\pi(t_0)|^2 = 1$. The two constraints give $\mathcal{A}_1 = 2.156$ and $\mathcal{A}_2 = 0.2450$





The anomaly

Using this first simple *reconstruction approximant*, we have a value for the anomaly

$$a_\mu^{\mathrm{HVP}}(N=2) = 6527.12 \cdot 10^{-11}$$

which is already at a level of 6% error from the WP prediction.

Reconstruction approximants of HVP and the MUonE proposal

A toy-model for testing the efficiency

Phenomenological toy-model

We consider the spectral function

$$\frac{1}{\pi}\mathrm{Im}\,\Pi^{\mathrm{HVP}}_{\mathrm{model}}(t) = \frac{\alpha}{\pi}\left(1-\frac{4m_\pi^2}{t}\right)^{3/2}\left\{\frac{1}{12}|F(t)|^2 + \sum_{q\in\mathrm{quarks}} e_q^2\ \Theta(t,t_c,\Delta)\right\}\vartheta(t-4m_\pi^2)$$

It has a Breit-Wigner–like modulous squared form factor

$$|F(t)|^2 = \frac{M_\rho^4}{(M_\rho^2-t)^2+M_\rho^2\ \Gamma(t)^2}\,,$$

with an energy dependent width:

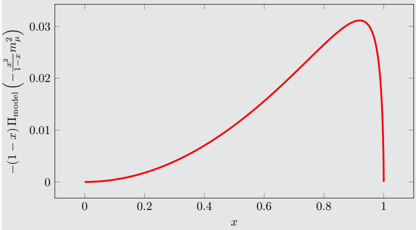
$$\Gamma(t) = \frac{M_\rho t}{96\pi f_\pi^2}\left[\left(1-\frac{4m_\pi^2}{t}\right)^{3/2}\vartheta(t-4m_\pi^2)+\frac{1}{2}\left(1-\frac{4M_K^2}{t}\right)^{3/2}\vartheta(t-4M_K^2)\right]\ ;$$

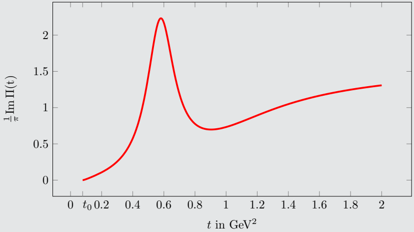
plus a function

$$\Theta(t,t_c,\Delta) = \frac{\frac{2}{\pi}\arctan\left(\frac{t-t_c}{\Delta}\right)-\frac{2}{\pi}\arctan\left(\frac{t_0-t_c}{\Delta}\right)}{1-\frac{2}{\pi}\arctan\left(\frac{t_0-t_c}{\Delta}\right)}\,,$$

with two arbitrary parameters t_c and Δ.

Phenomenological toy-model ($t_c = 1$ GeV^2, $\Delta = 0.5$ GeV^2)





This toy-model gives an anomaly:

$$a_\mu^{\rm HVP}(\text{Model}) = 6992.4 \cdot 10^{-11}$$

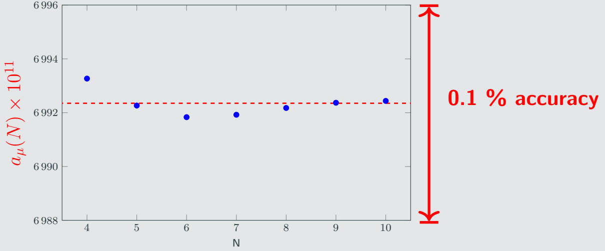
Procedure

1. To mimic the data points of MUonE, we consider 50 points equally spaced without "experimental errors" in the range $0.23 \leqslant x \leqslant 0.93$

2. We do linear fits of the parameters $\mathcal{A}_n$ in the reconstruction approximants formula for $\Pi_{N,L}\left(-Q^2\right)$.

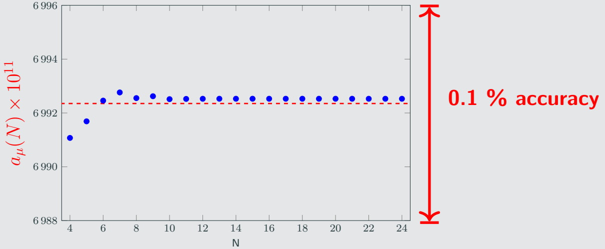
Fits # 1

We take $L = 1$ with the condition $\mathcal{B}_1 = (-2)\frac{\alpha}{\pi}\frac{5}{3}$



Fits # 2

We take $L = 5$ with the condition $\mathcal{B}_1 = (-2)\frac{\alpha}{\pi}\frac{5}{3}$, $\mathcal{B}_3$ and $\mathcal{B}_5$ fixed by constraints but $|F_\pi(t_0)|^2$ as fitted parameter.



Conclusion

Conclusion

- We have built reconstruction approximants to HVP in the Euclidean region using the Flajolet and Odlyzko's transfert theorem. This approximation method
  - is valid for any value of Q^2.
  - preserves the analytic structure (cuts,...) – contrary to Padé approximants.
- Well adapted to the MUonE experimental proposal.
- The method can be improved by adding more approximants.
- We are working on adapting the method to Lattice QCD evaluations of the Time Momentum Representation of HVP.

Backup slides

Relations on coefficients

From the Mellin singular expansion

$$\mathcal{M}(s) \asymp \sum_{\substack{p \geqslant 1 \\ k \geqslant 0}} \frac{(-1)^p \mathcal{R}_{p,k}}{(s-p)^{k+1}}$$

we get

$$\Pi(-Q^2) \underset{Q^2 \to \infty}{\sim} \sum_{\substack{p \geqslant 1 \\ k \geqslant 0}} \frac{(-1)^k}{k!} \mathcal{R}_{p,k} \left(\frac{Q^2}{t_0} \right)^{1-p} \ln^k \left(\frac{Q^2}{t_0} \right)$$

which becomes for $k = 0, 1$ in the conformal plane

$$\Pi \left(-\frac{4\omega}{(1-\omega)^2} \right) \underset{\omega \to 1}{\sim} \sum_{m \geqslant 0} \tilde{\mathcal{R}}_{m,1} \, (1-\omega)^m \ln \left(\frac{1}{1-\omega} \right)$$

where we have the explicit expressions

$$\widetilde{\mathcal{R}}_{0,1} = -2\mathcal{R}_{1,1} \text{ and } \widetilde{\mathcal{R}}_{m,1} = -2 \sum_{\mathsf{p}=2}^{\lfloor \frac{m+2}{2} \rfloor} \binom{m-\mathsf{p}}{\mathsf{p}-2} 4^{1-\mathsf{p}} \mathcal{R}_{\mathsf{p},1} \text{ for } m \geq 1$$

And the *transfert theorem* leads to

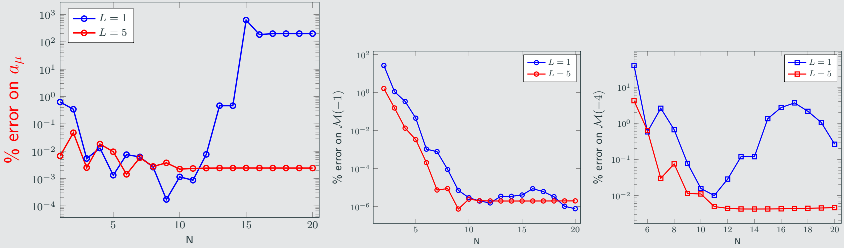
$$\Omega_n^{\mathrm{AS}} = \sum_{j=0}^{\infty} \sum_{m \geqslant 0} \widetilde{\mathcal{R}}_{m,1} \left\{ {m+j \atop m} \right\} \frac{(-1)^m \Gamma(m+1)}{n^{m+1+j}}$$

which can be rewritten as

$$\Omega_n^{\mathrm{AS}} = \sum_{l=0}^{\infty} \frac{\mathcal{B}_l}{n^l} \quad \text{where } \mathcal{B}_1 = -2\mathcal{R}_{1,1} \quad \text{and } \mathcal{B}_l = \sum_{m=1}^{l-1} \widetilde{\mathcal{R}}_{m,1} \left\{ {l-1 \atop m} \right\} (-1)^m \Gamma(m+1) \, .$$

Comments on errors and efficiency

Absolute error



This just an illustrative example to show the stability and the efficiency *a posteriori* of the reconstruction approximants.

Systematic error

There are several ways to estimate the systematic error

$$\mathcal{E}_N(\omega) = \sum_{n=N+1}^{\infty} \mathcal{A}_n \omega^n$$

but it must be discussed in the context of the MUonE experiment and adapted.

Imaginary part