work in progress

The role of Padé approximants as fitting functions in the context of the MuonE experiment

Diogo Boito[1], **Cristiane Yumi London**[1,2,3], Pere Masjuan[2,3]

[1]Instituto de Física de São Carlos, Universidade de São Paulo, Brazil
[2]Departament de Física, Universitat Autònoma de Barcelona, Spain
[3]Institut de Física d'Altes Energies, The Barcelona Institute of Science and Technology, Spain

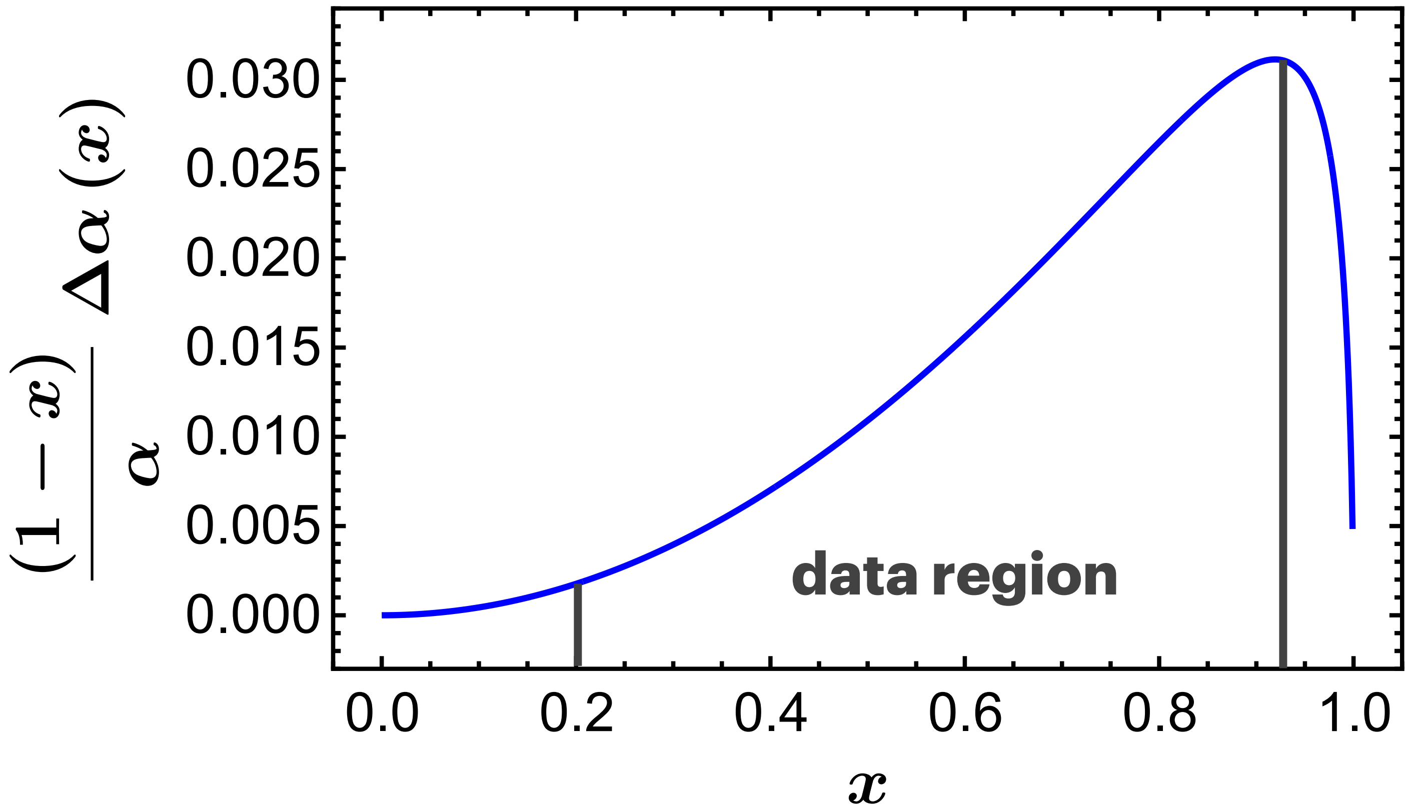
IFSC UNIVERSIDADE DE SÃO PAULO
Instituto de Física de São Carlos

IFAE
Institut de Física d'Altes Energies

UAB
Universitat Autònoma de Barcelona

Introduction

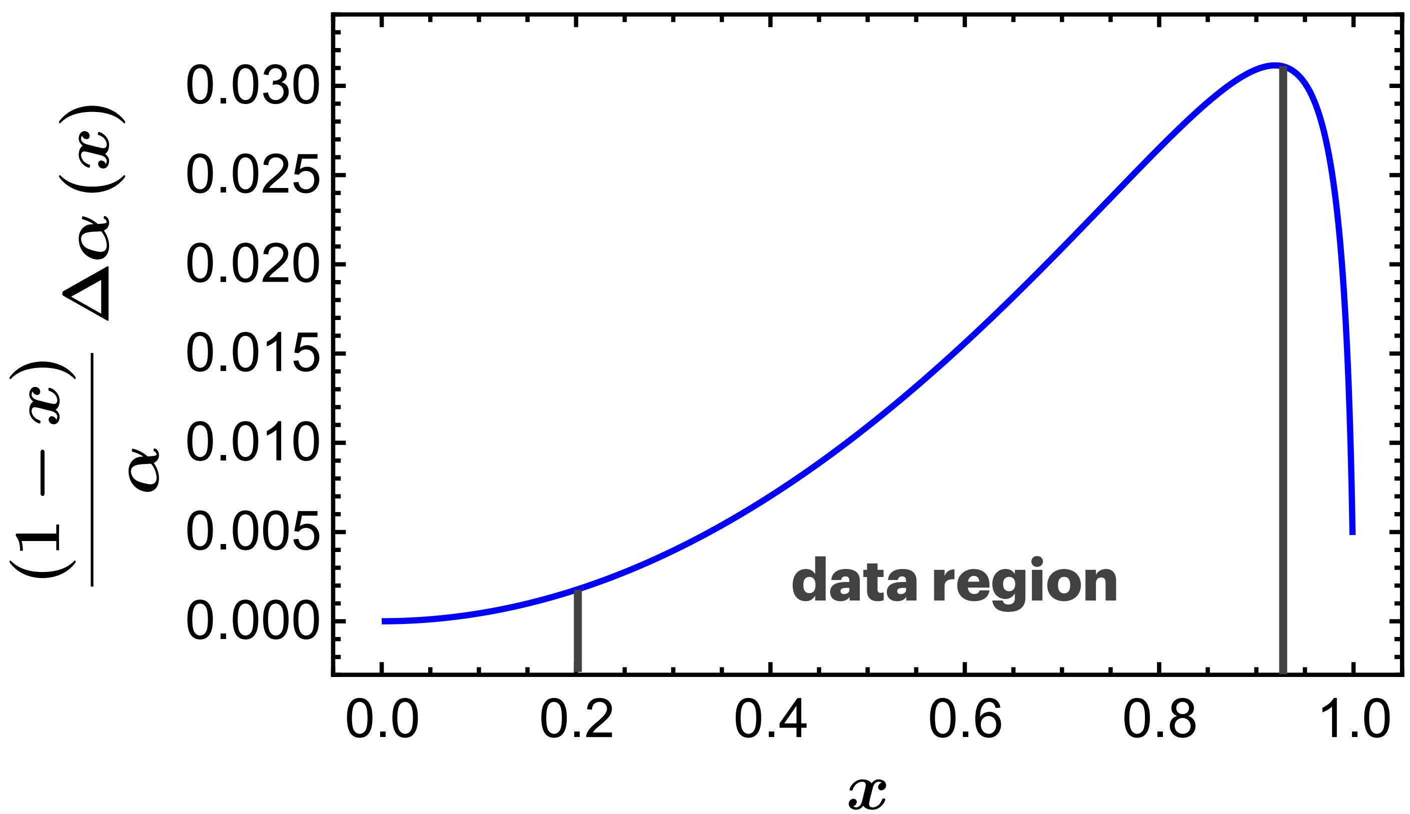
Motivation



Problem

Finding a reliable method to fit the data + good ***extrapolation*** outside data region

Motivation



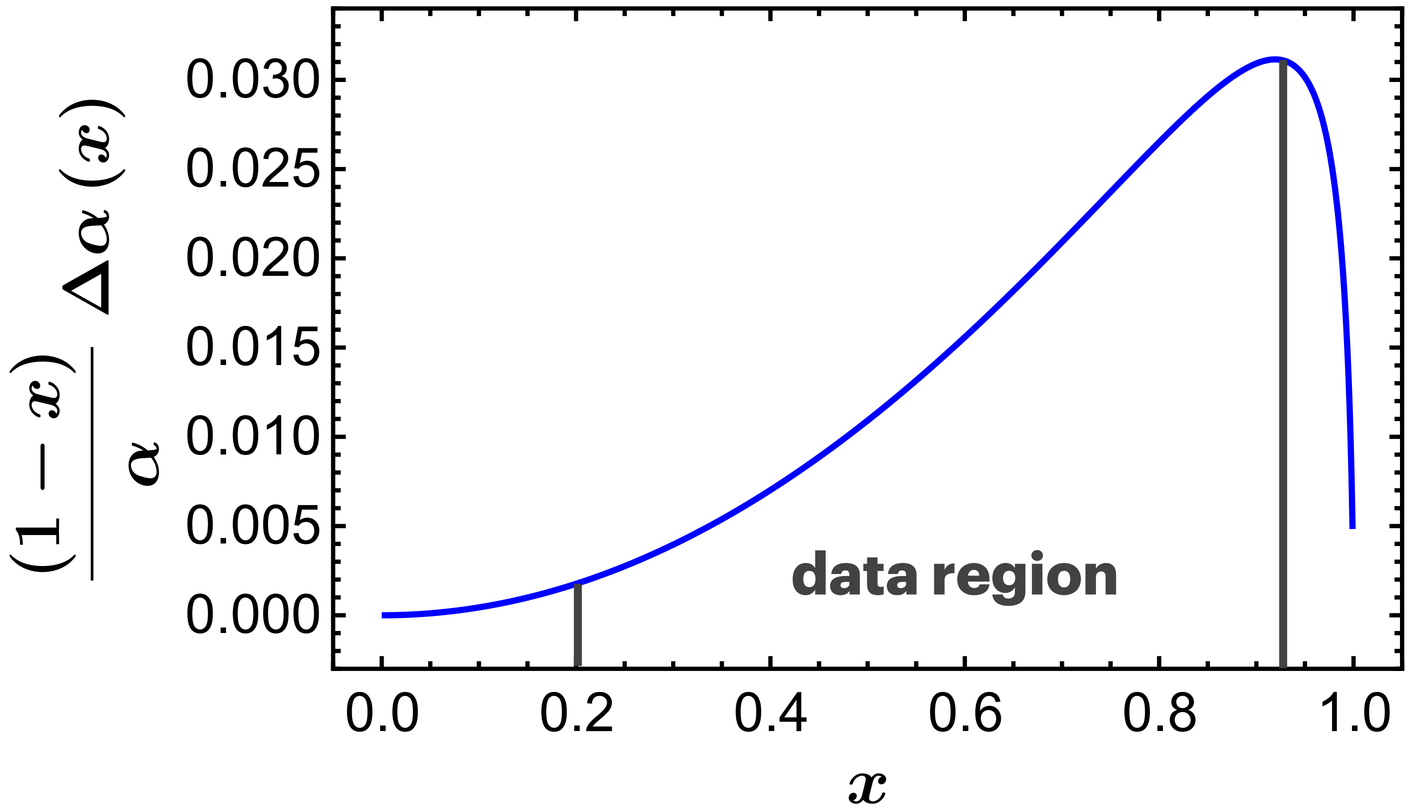
Problem

Finding a reliable method to fit the data + good ***extrapolation*** outside data region

Padé Approximants

$$P_N^M(z) = \frac{Q_M(z)}{R_N(z)} = \frac{a_0 + a_1 z + \ldots + a_M z^M}{1 + b_1 z + \ldots + b_N z^N}$$

Motivation



Problem

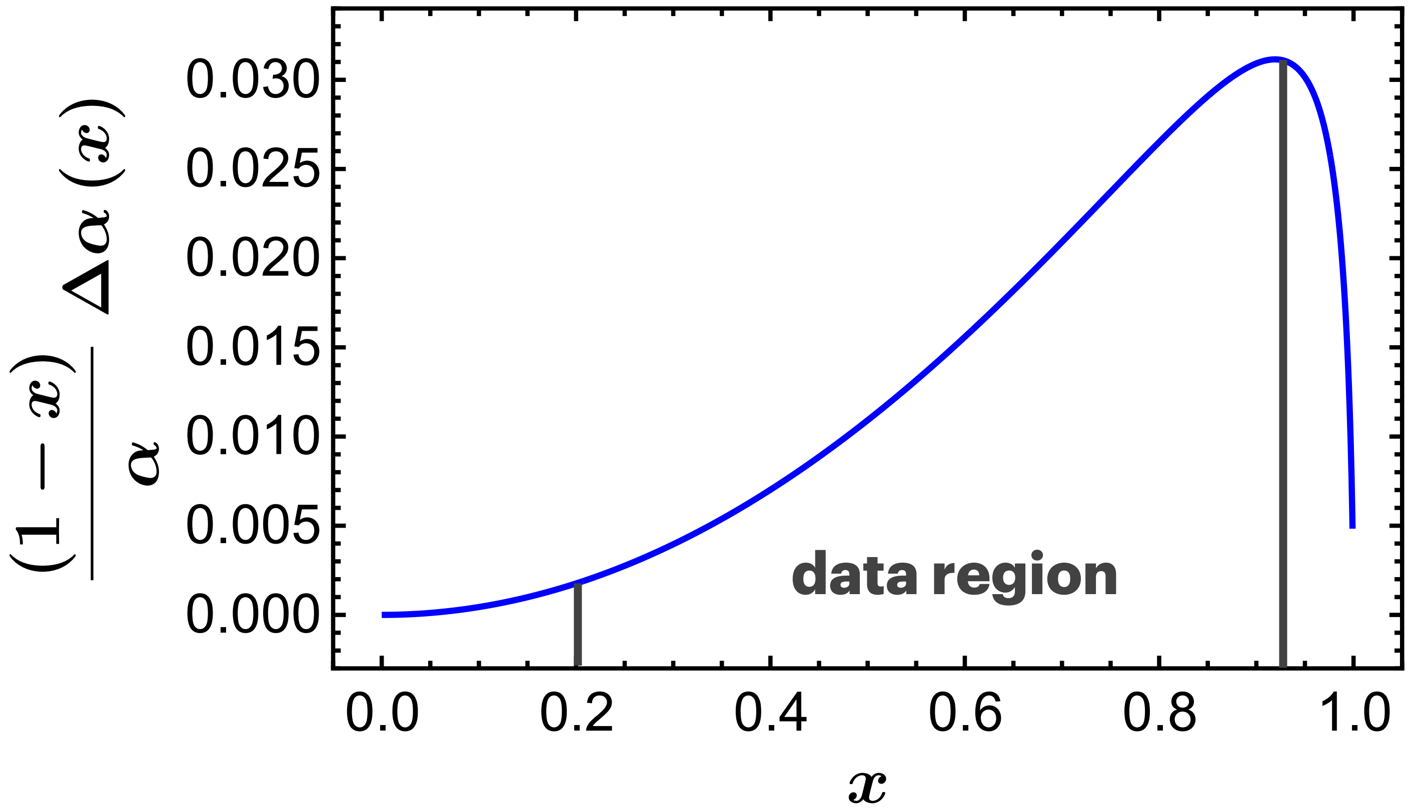
Finding a reliable method to fit the data + good ***extrapolation*** outside data region

Padé Approximants

$$P_N^M(z) = \frac{Q_M(z)}{R_N(z)} = \frac{a_0 + a_1 z + \ldots + a_M z^M}{1 + b_1 z + \ldots + b_N z^N}$$

- Already applied in similar contexts:
  - Masjuan, Peris (2009) arXiv:0903.0294 [hep-ph]
  - Golterman, Maltman, Peris (2014) arXiv:1405.2389 [hep-lat]
  - Chakraborty, Davies, de Oliveira, Koponen, Lepage, van de Water (2016) arXiv:1512.07555 [hep-lat]
  - Aubin, Blum, Chau, Golterman, Peris, Tu (2017) arXiv:1601.03071 [hep-lat]
  - Masjuan, Sanchez-Puertas (2017) arXiv:1701.05829 [hep-ph]

Motivation



Problem

Finding a reliable method to fit the data + good ***extrapolation*** outside data region

Padé Approximants

$$P_N^M(z) = \frac{Q_M(z)}{R_N(z)} = \frac{a_0 + a_1\, z + \ldots + a_M\, z^M}{1 + b_1\, z + \ldots + b_N\, z^N}$$

Advantages of PAs

- Systematic and model-independent method
- Partial reconstruction of analytical (physical) properties
- Efficient approximation
- Possible to provide a convergence error

Padé's Theory

Stieltjes function

$$f(z) = \int_0^\infty \frac{d\phi(u)}{1+zu} \qquad \phi(u) \text{ is a measure on } [0,\infty)$$

- There are convergence theorems for PAs to Stieltjes functions

- Some convergence properties:

  * poles of $P_N^{N+J}, \;\; J \geq -1$, are real and simple;

  * Padé sequences $P_N^{N+J}, \;\; J \geq -1$, uniformity converge to the original function;

  * Padés act as bounds of the function

$$\lim_{N\to\infty} P_N^N(t) \geq \bar{\Pi}(t) \geq \lim_{N\to\infty} P_N^{N+1}(t)$$

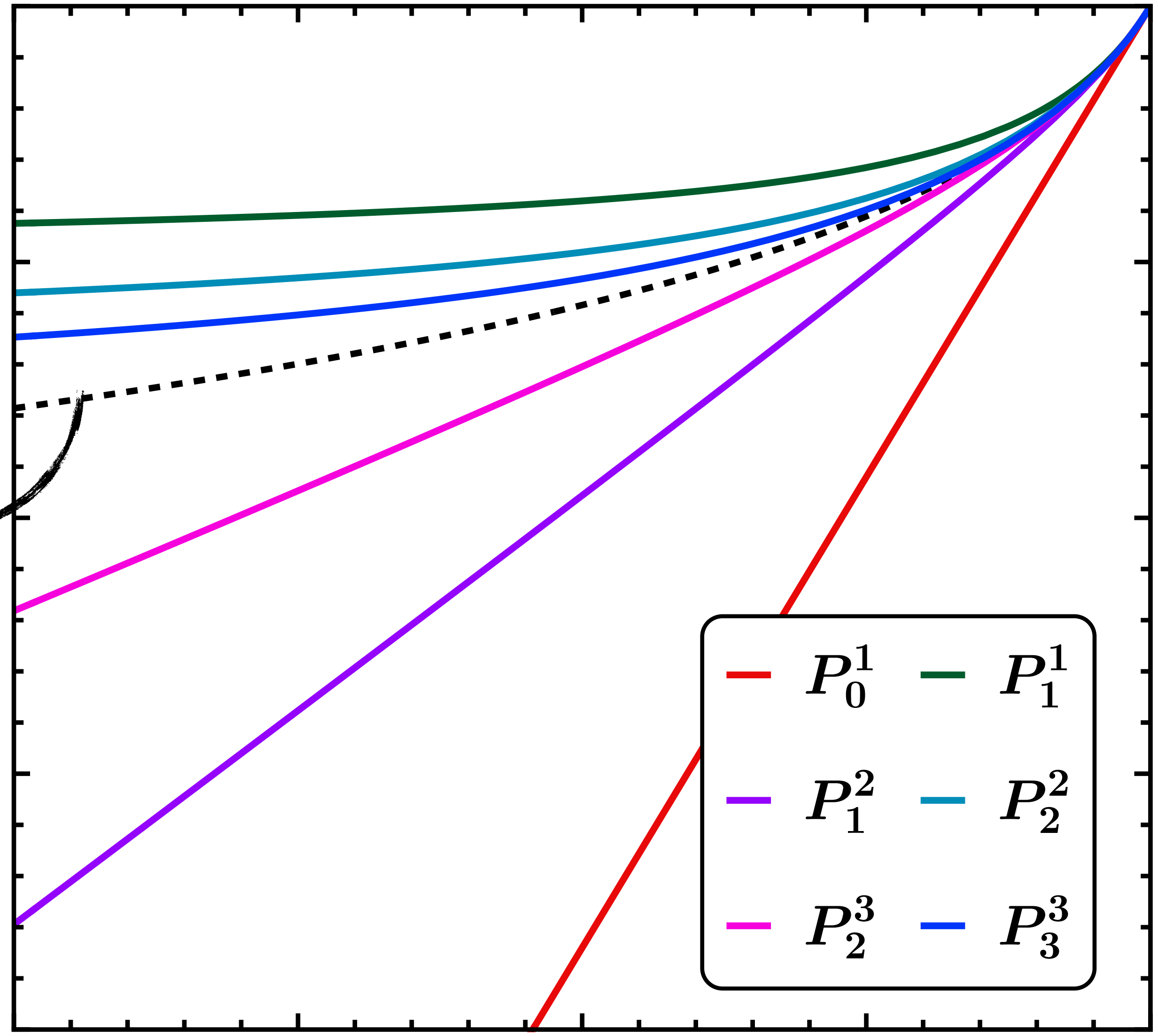
Padé's Theory

$f(z)$ … $,\infty)$

…verge…

…genc…

function
approximated
by PAs



* poles of P_N^{N+J}, …

* Padé sequenc… …ne original function;

* Padés act as bounds of the function

$$\lim_{N\to\infty} P_N^N(t) \geq \bar{\Pi}(t) \geq \lim_{N\to\infty} P_N^{N+1}(t)$$

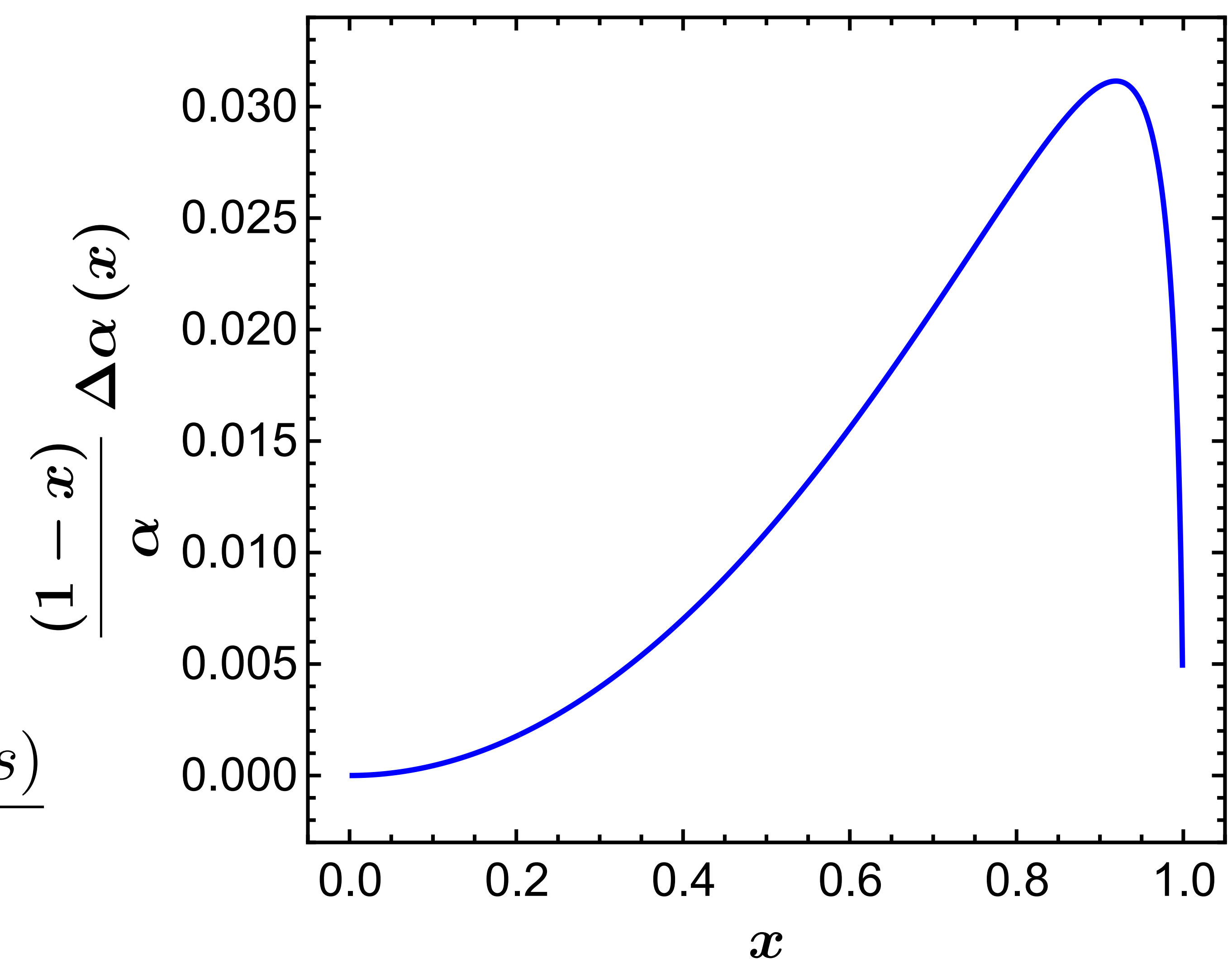
Method

Data Generation

$$a_\mu^{\mathrm{HVP}} = \frac{\alpha}{\pi}\int_0^1 \mathrm{d}x\ (1-x)\Delta\alpha_{\mathrm{had}}[t(x)]$$

$$\Delta\alpha_{\mathrm{had}}(t) = -\,\mathrm{Re}[\bar{\Pi}_{\mathrm{had}}(t)]$$

$$\bar{\Pi}_{\mathrm{had}}[t(x)] \equiv \Pi_{\mathrm{had}}(t+i\epsilon) - \Pi_{\mathrm{had}}(0+i\epsilon)$$
$$= \frac{t}{\pi}\int_0^\infty \mathrm{d}s\ \frac{\mathrm{Im}\,\Pi_{\mathrm{had}}(s)}{s(s-t)}$$



instead of working with $\bar{\Pi}_{\mathrm{had}}$ we work with $R^{\mathrm{LO}}_{\mathrm{had}}$

physical quantity

$$R^{\mathrm{LO}}_{\mathrm{had}}(t) = \frac{\mathrm{d}\sigma^{\mathrm{LO}}{}_{(\Delta\alpha_{\mathrm{had}}(t)\neq 0)}}{\mathrm{d}\sigma^{\mathrm{LO}}{}_{(\Delta\alpha_{\mathrm{had}}(t)=0)}} = [1-\Delta\alpha_{\mathrm{had}}(t)]^{-2} \approx 1 + 2\,\Delta\alpha_{\mathrm{had}}(t)$$

Model of Greynat and de Rafael

$$\mathrm{Im}\,\Pi_{\mathrm{had}}(s) = \alpha\left(1-\frac{4m_\pi^2}{s}\right)^{3/2}\left[\frac{1}{12}\,|F(s)|^2 + \Theta(s)\sum_{\mathrm{quarks}} q_f^2\right]\theta(s-4m_\pi^2)$$

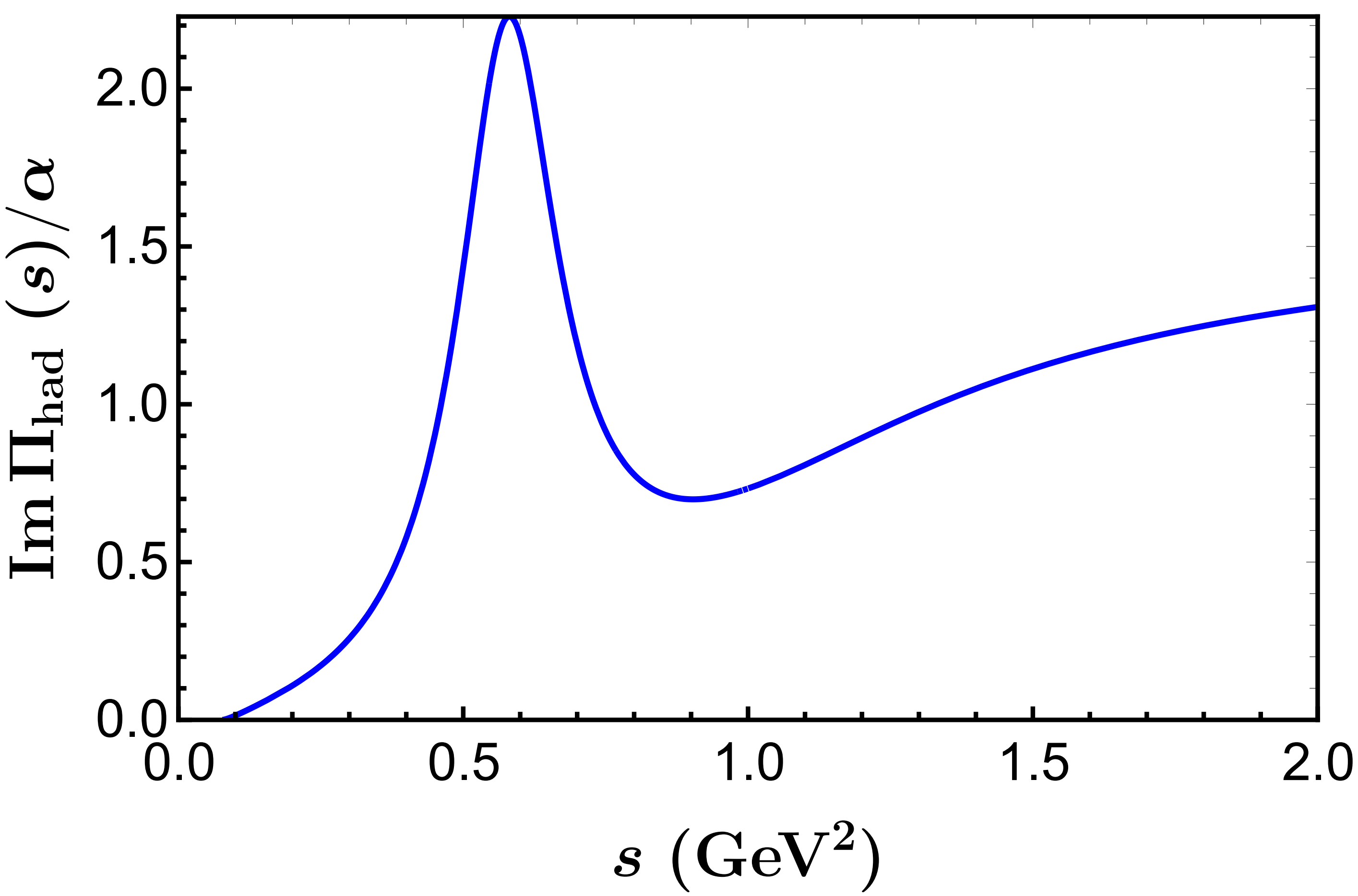
model used to generate toy data

Greynat, de Rafael (2022)

$$\Gamma(s) = \frac{m_\rho\, s}{96\pi f_\pi^2}\left[\left(1-\frac{4m_\pi^2}{s}\right)^{3/2}\theta(s-4m_\pi^2) + \frac{1}{2}\left(1-\frac{4m_\pi^2}{s}\right)^{3/2}\theta(s-4m_K^2)\right]$$

$$|F(s)|^2 = \frac{m_\rho^4}{(m_\rho^2-s)^2+m_\rho^2\,\Gamma^2(s)}$$

$$\Theta(s) = \frac{2}{\pi}\left[\frac{\arctan(\frac{s-s_c}{\Delta}) - \arctan(\frac{4m_\pi^2-s_c}{\Delta})}{\arctan(\frac{4m_\pi^2-s_c}{\Delta})}\right]$$



Fitting Method

- Padés as fitting functions

$$(\theta_\mu, R_{\rm had})$$

$$0.16 \leq \theta_{\mu\,(\rm mrad)} \leq 4.8$$

$$(x, R_{\rm had}^{\rm mod})$$

$$0.214 \leq x \leq 0.932$$

30 equally spaced data points

$$R_{\rm had}^{\rm mod} = (R_{\rm had} - 1) \times 10^5 \approx 10^5 \times 2\,\Delta\alpha_{\rm had}$$

$$\Delta\alpha_{\rm had}(x) = \sum_{n=0}^{\infty} c_n\, x^n = c_2\, x^2 + c_3\, x^3 + \ldots$$

- PA parameters: χ^2 minimization

- Error propagation: Monte Carlo

Example

Toy data set

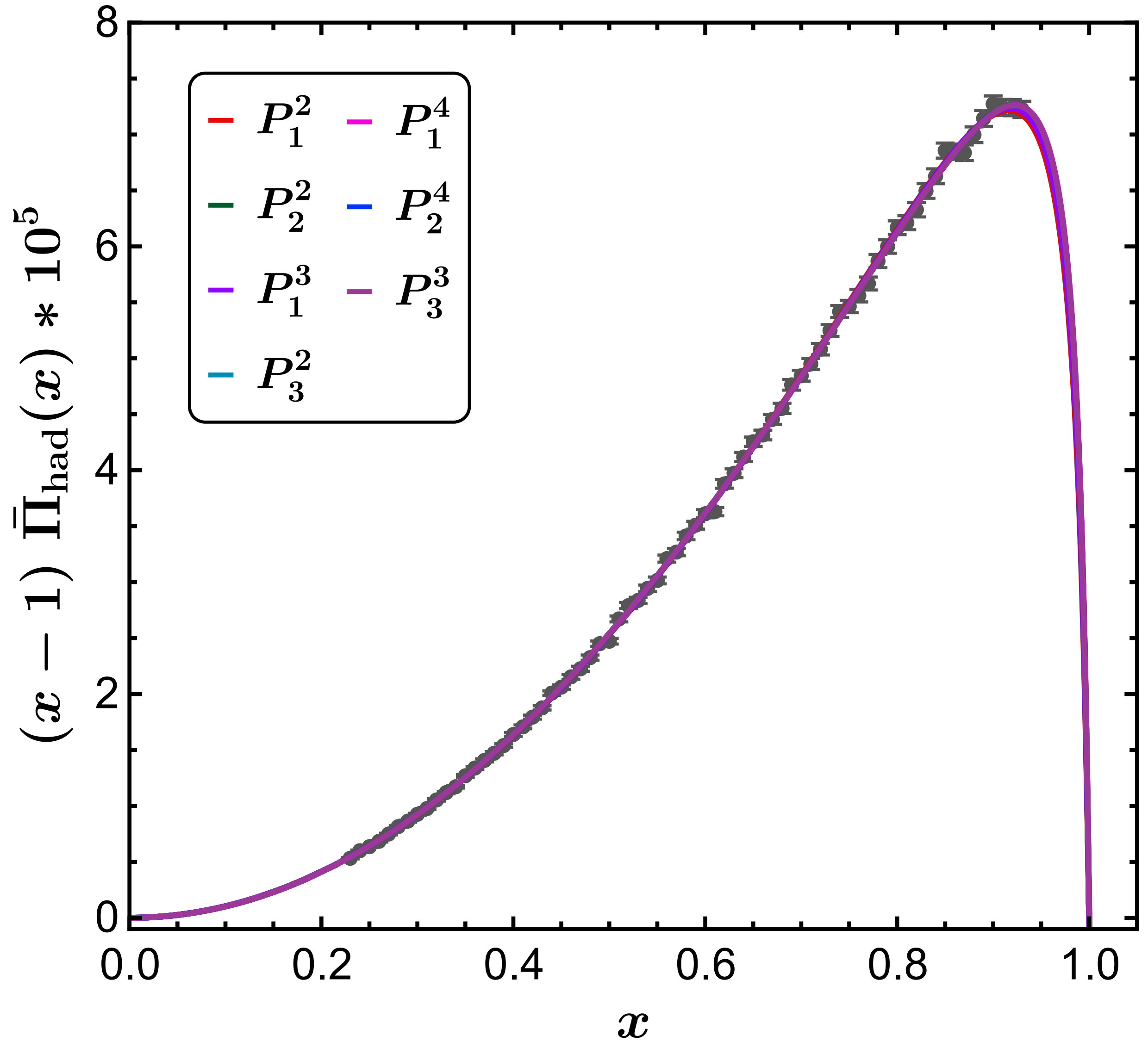
- Data points $(x, \bar{\Pi}_{\rm had})$ generated in the interval $0.23 \leq x \leq 0.93$ in steps of 0.01

- Central value randomly chosen from a gaussian distribution with $1\,\%$ error

- Padés as fitting and extrapolation functions to $\bar{\Pi}_{\rm had}(x) \times 10^5$

- Imposing that $c_0 = c_1 = 0$ feature of $\Delta\alpha_{\rm had}$

$$\Delta\alpha_{\rm had}(x) = \sum_{n=0}^{\infty} c_n\, x^n$$

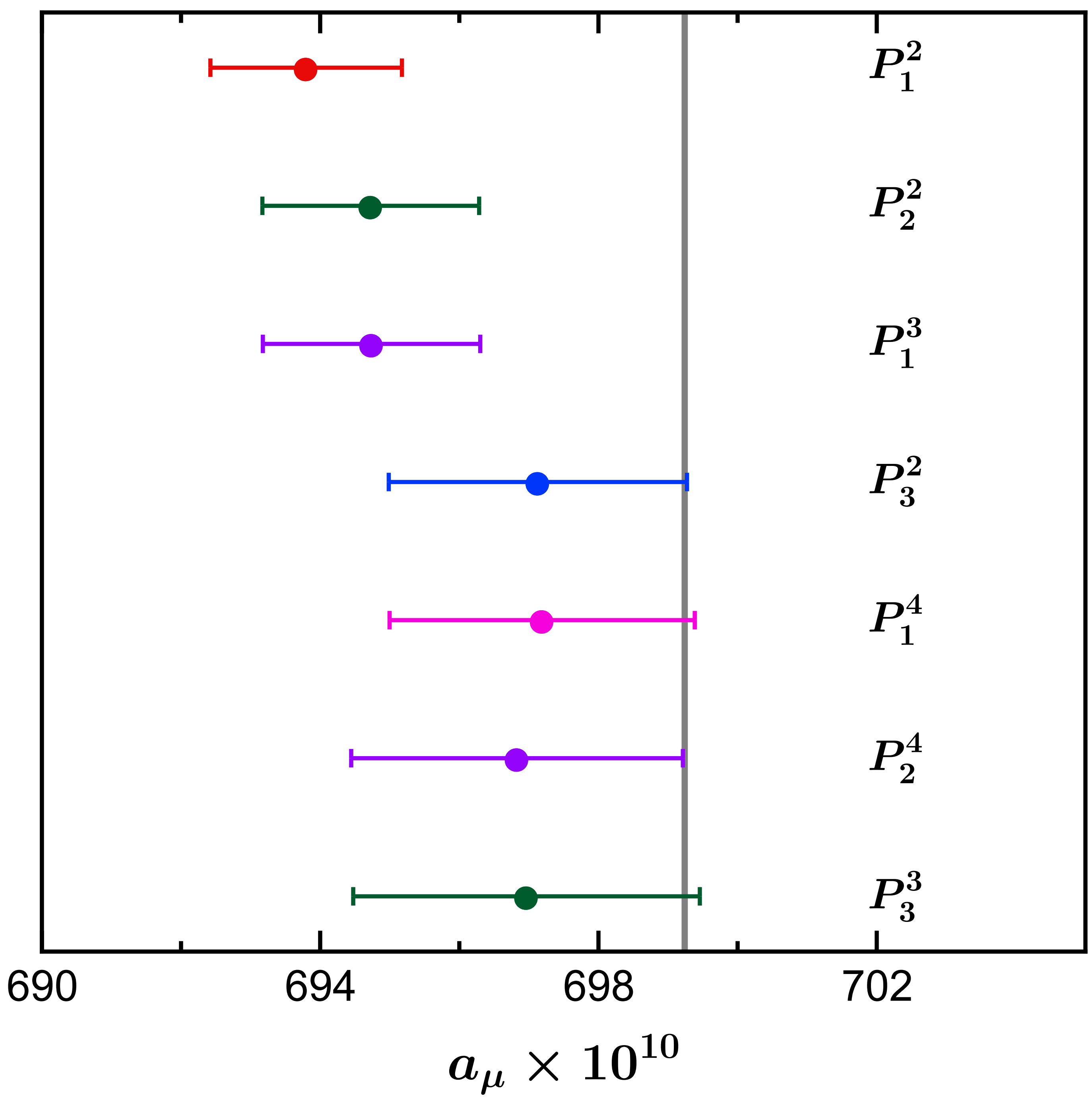


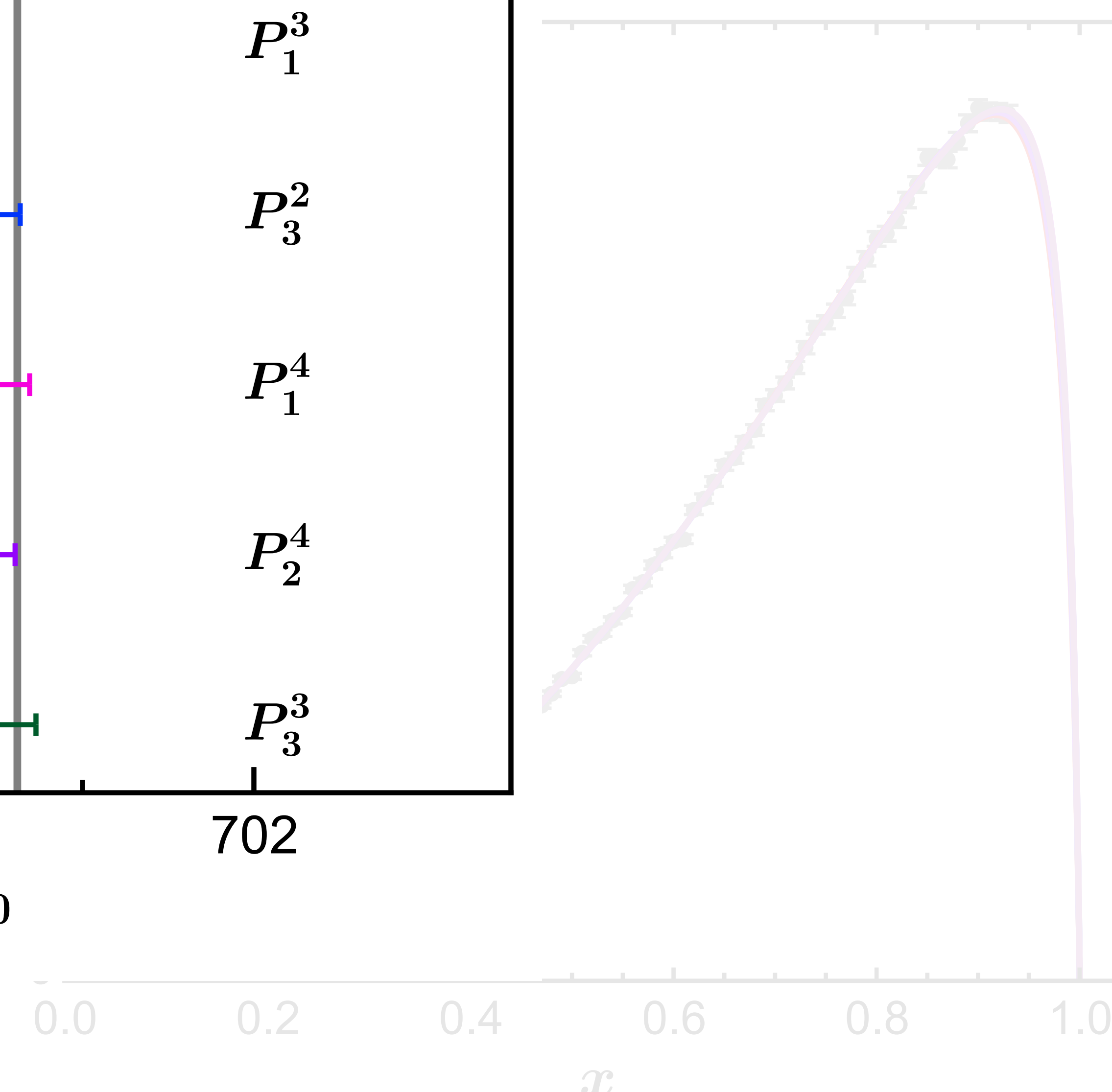
Padés reproduce well the data

Toy data set

- Data points $(x, \bar{\Pi}$ … in steps of 0.01
- Central value ra… …ith $1\,\%$ error
- Padés as fittir… functions to $\bar{\Pi}_{\rm ha}$…
- Imposing that $c_{($…

$\Delta\alpha_{\rm had}(x$…





Padés reprod…

Padés to the model

Canonical Padés

- Canonical Padé P_N^M:
  - determination of the PA coefficients through matching
  - PA reproduces the first $M+N+1$ Taylor series coefficients of the function

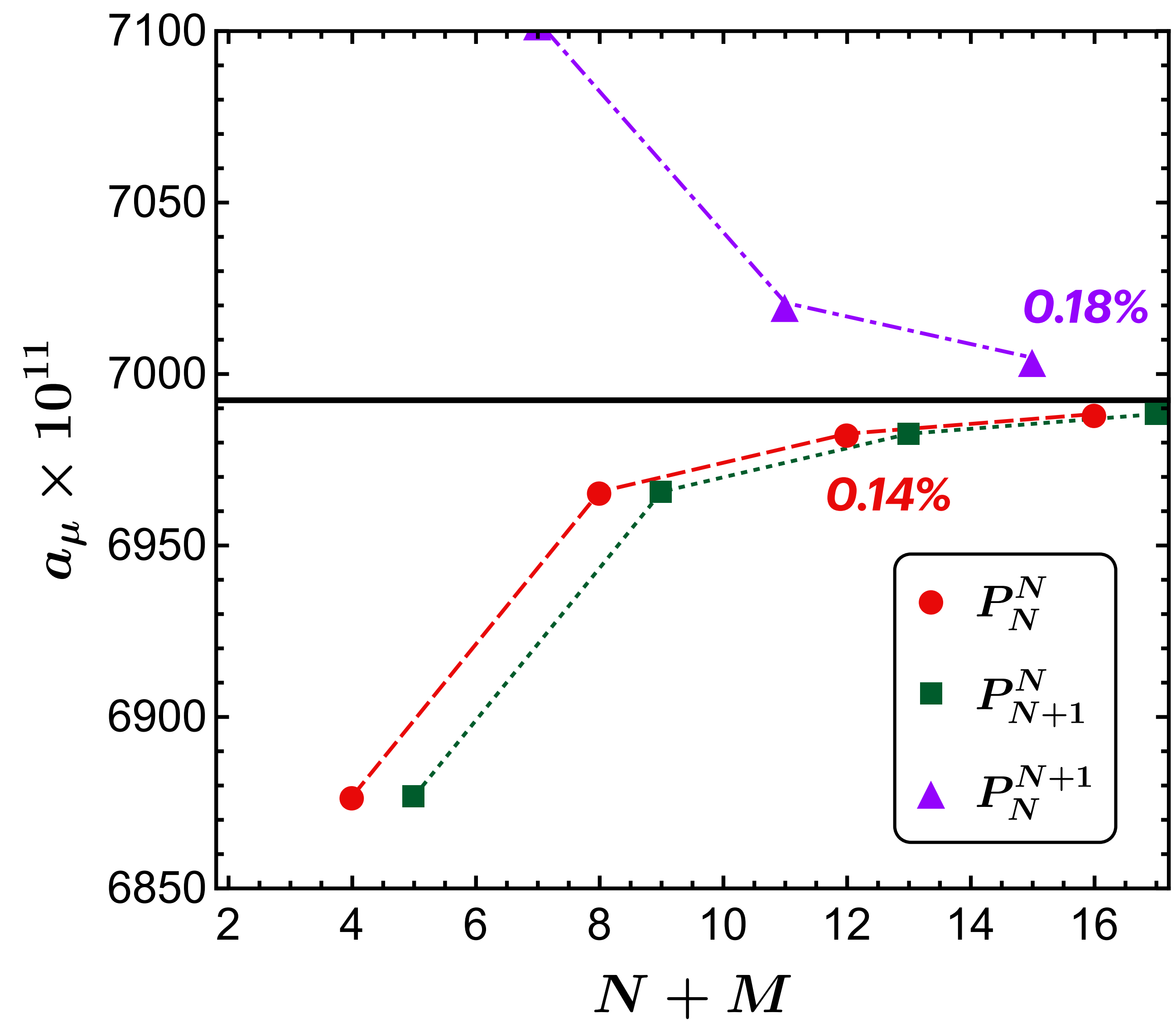
- Series approximated by the PAs

$$10^5 \times \Delta\alpha_{\text{had}}(x)$$

Stieltjes function

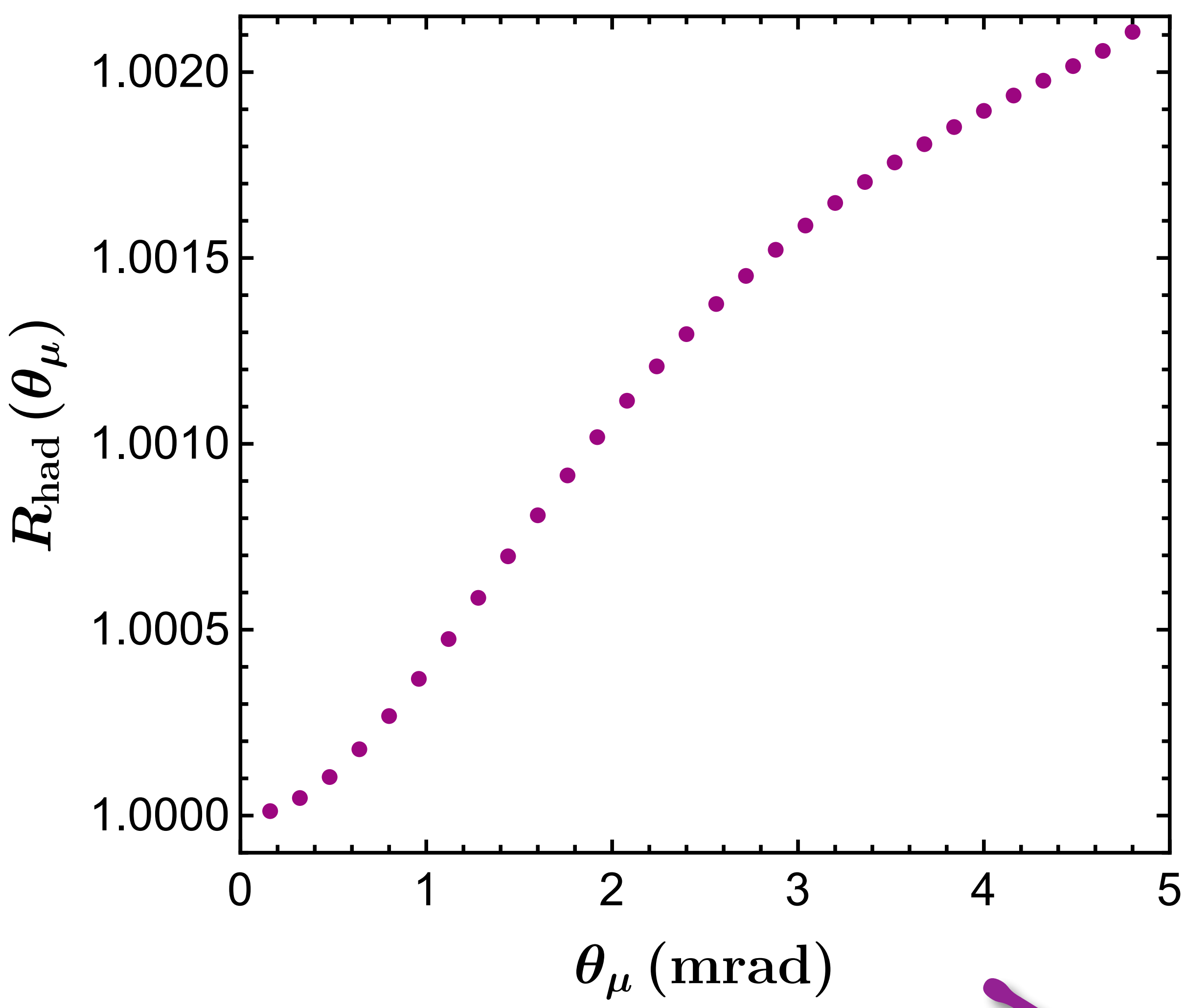
convergence theorems

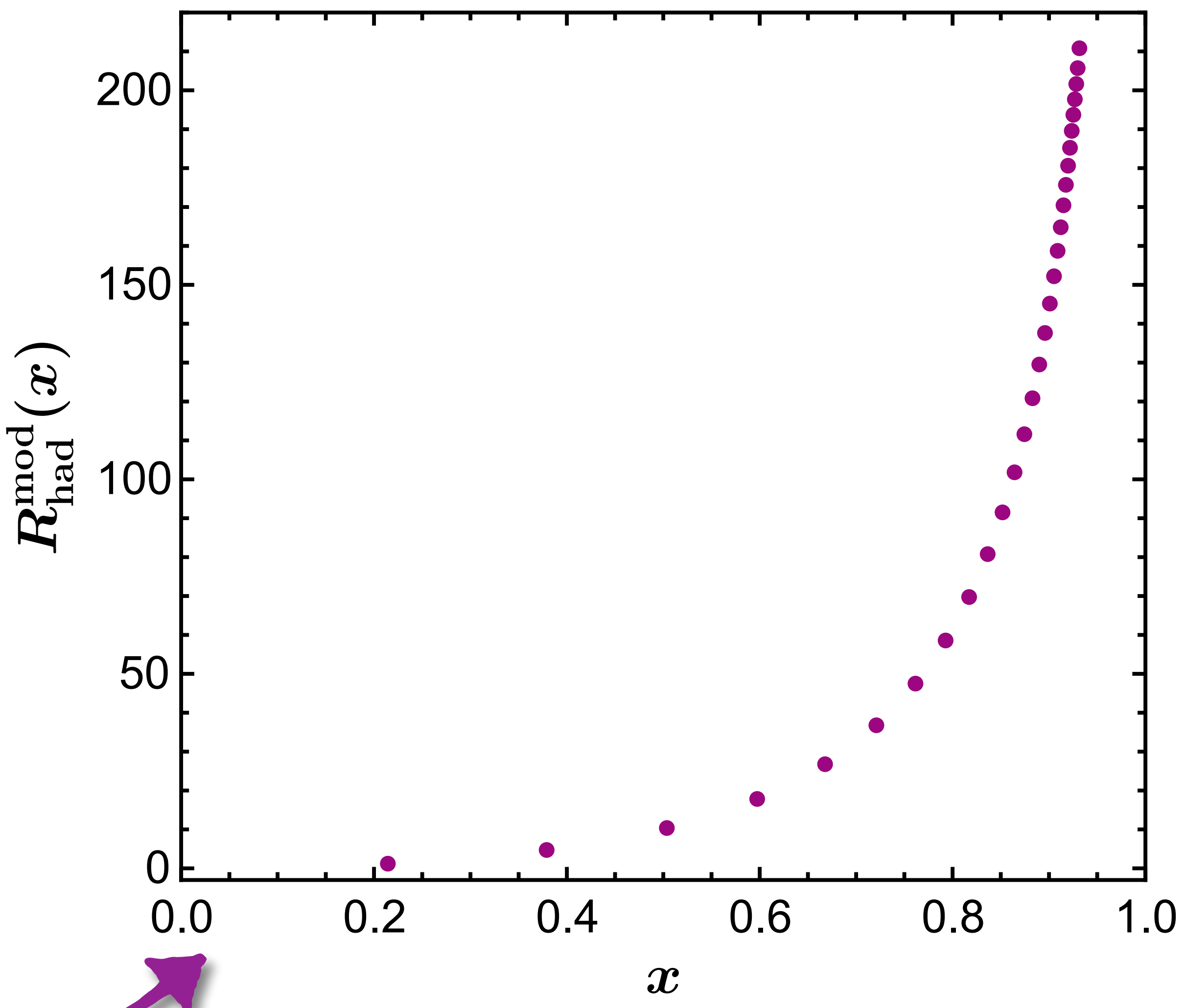
- PAs used to calculate a_μ in the whole region of $x \in [0,1]$



Ideal World

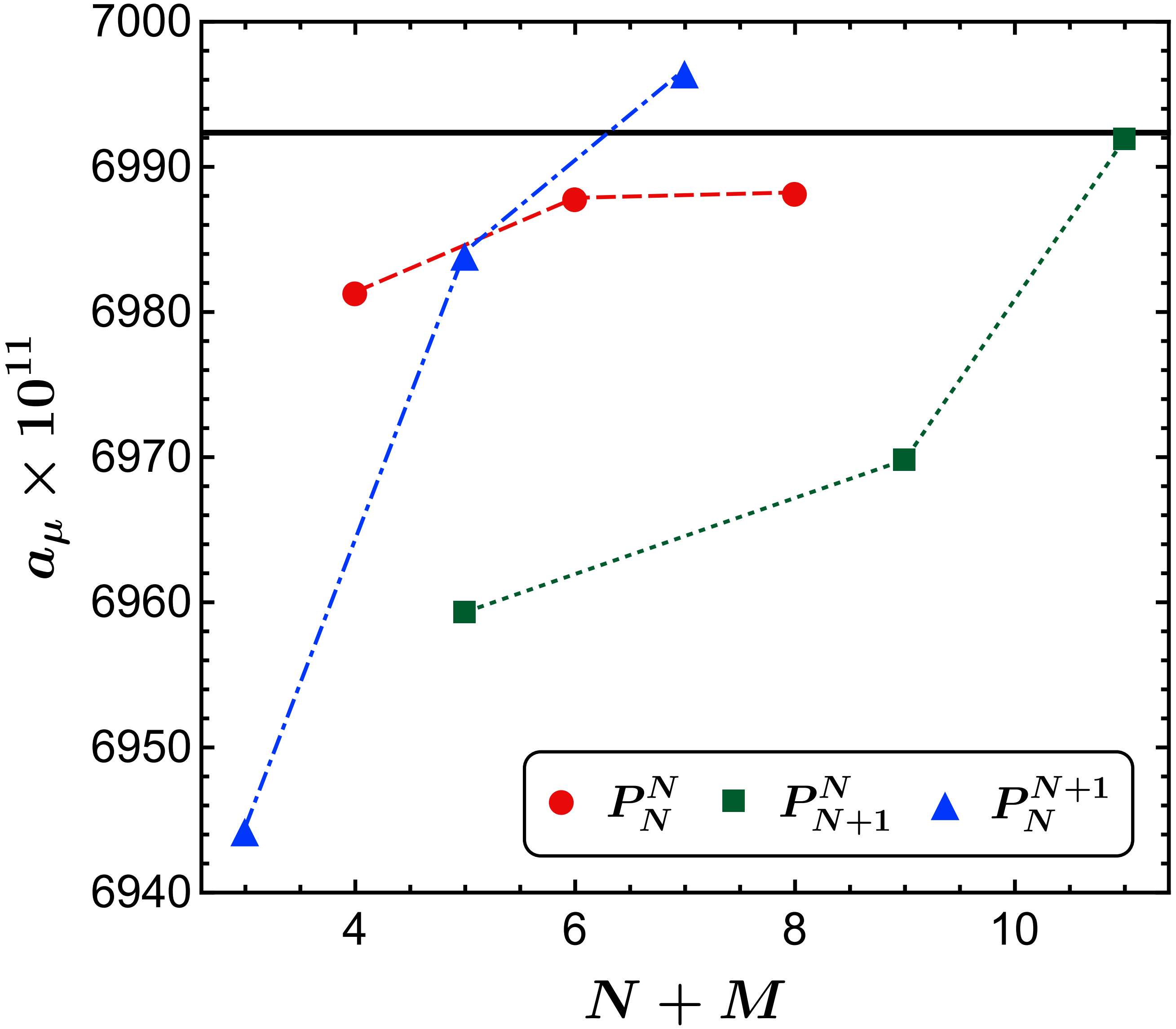
- Data without fluctuations or errors — estimate method uncertainty





Ideal World

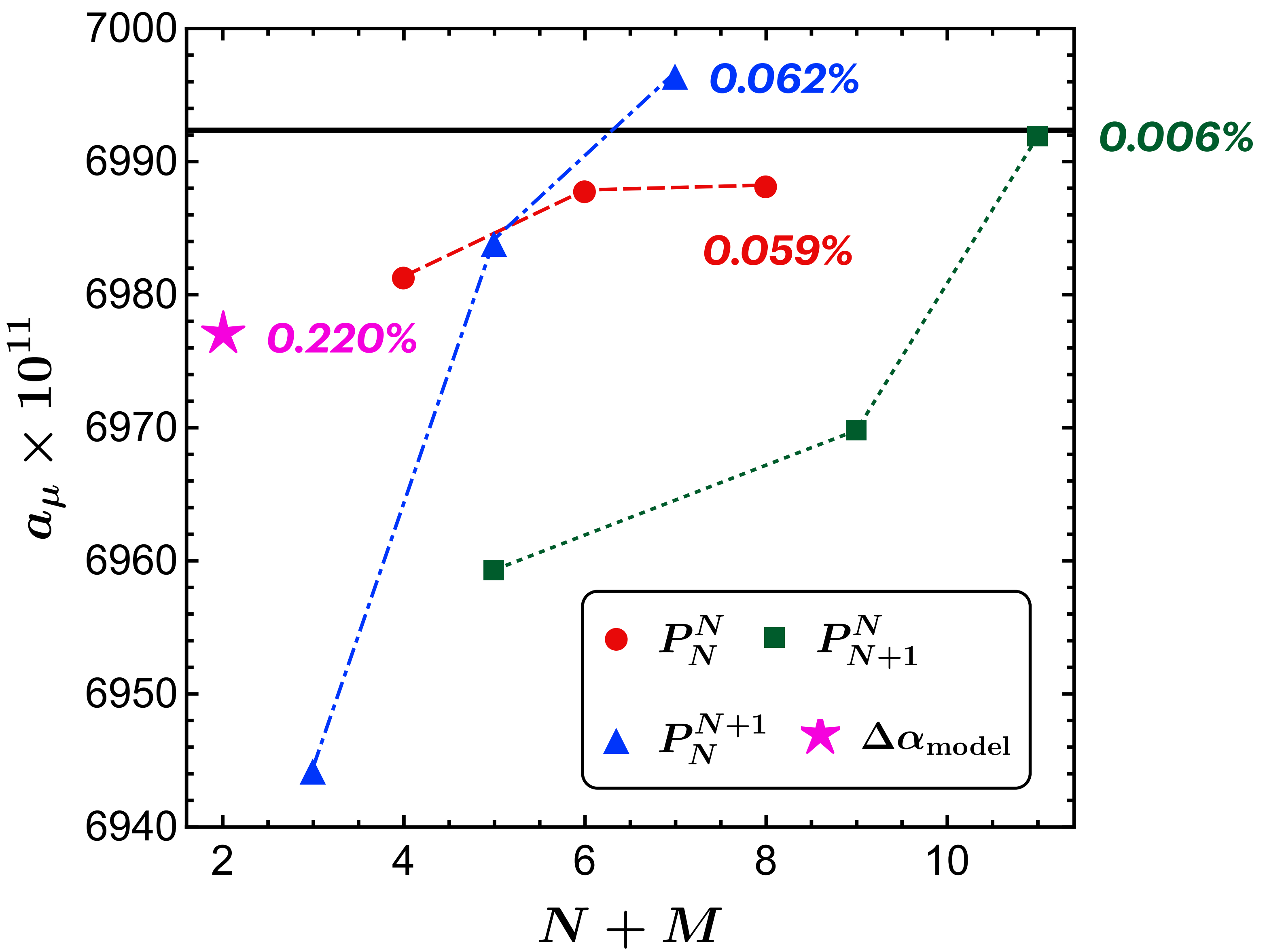
- Data without fluctuations or errors — estimate method uncertainty
- Padés as fitting functions imposing that $c_0 = c_1 = 0$
- PAs used to calculate a_μ in the whole region of $x \in [0,1]$
- Better results for higher-order PAs
- Uncertainty from the method is small



Ideal World

The MUonE Collaboration (2019)
Abbiendi (2022)

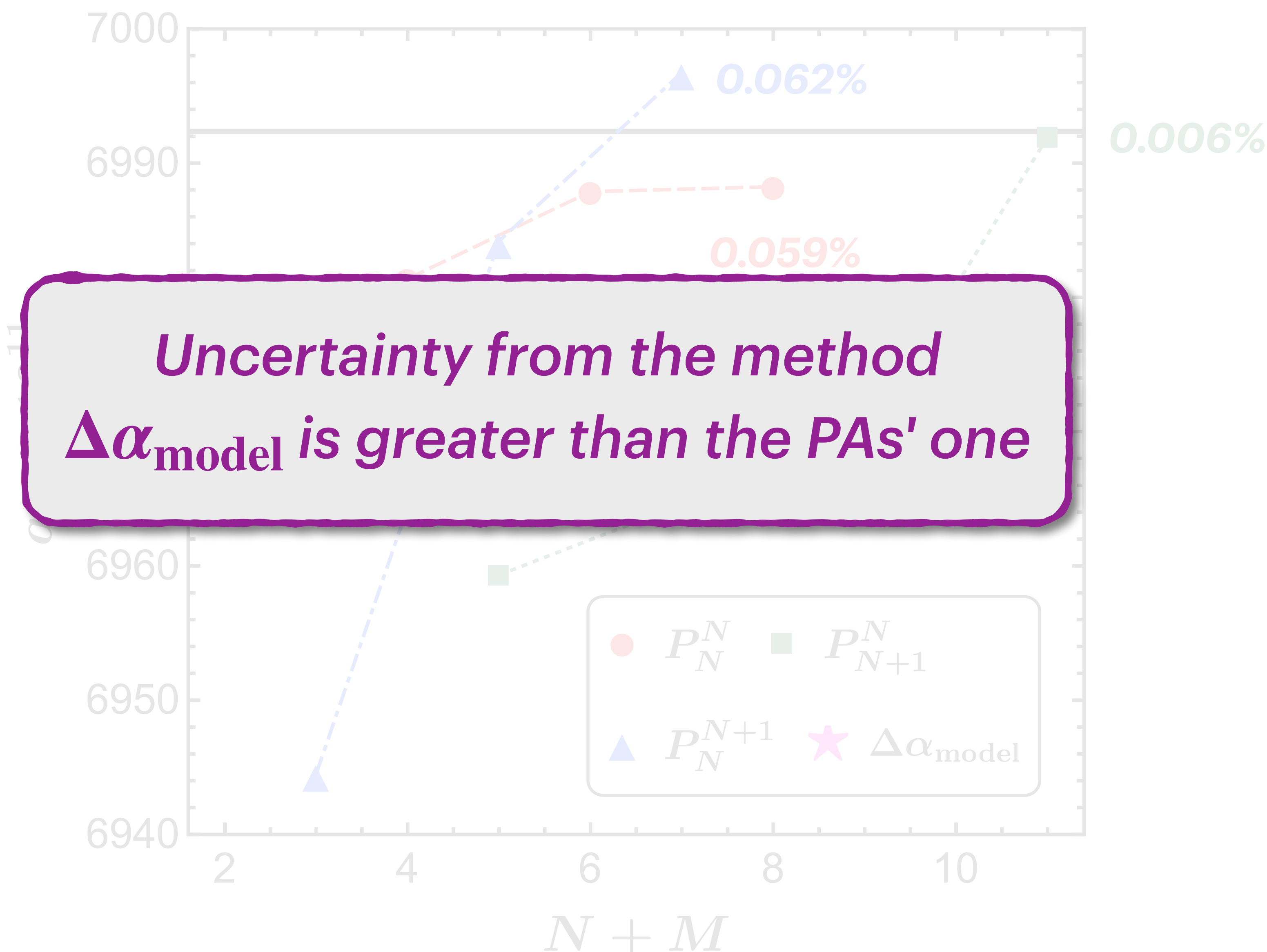
$$\Delta\alpha_{\mathrm{model}}(t) = k\left\{-\frac{5}{9}-\frac{4M}{3t}+\left(\frac{4M^2}{3t^2}+\frac{M}{3t}-\frac{1}{6}\right)\frac{2}{\sqrt{1-\frac{4M}{t}}}\log\left|\frac{1-\sqrt{1-\frac{4M}{t}}}{1+\sqrt{1-\frac{4M}{t}}}\right|\right\}$$



Ideal World

The MUonE Collaboration (2019)
Abbiendi (2022)

$$\Delta\alpha_{\mathrm{model}}(t) = k\left\{-\frac{5}{9} - \frac{4M}{3t} + \left(\frac{4M^2}{3t^2} + \frac{M}{3t} - \frac{1}{6}\right)\frac{2}{\sqrt{1-\frac{4M}{t}}}\log\left|\frac{1-\sqrt{1-\frac{4M}{t}}}{1+\sqrt{1-\frac{4M}{t}}}\right|\right\}$$



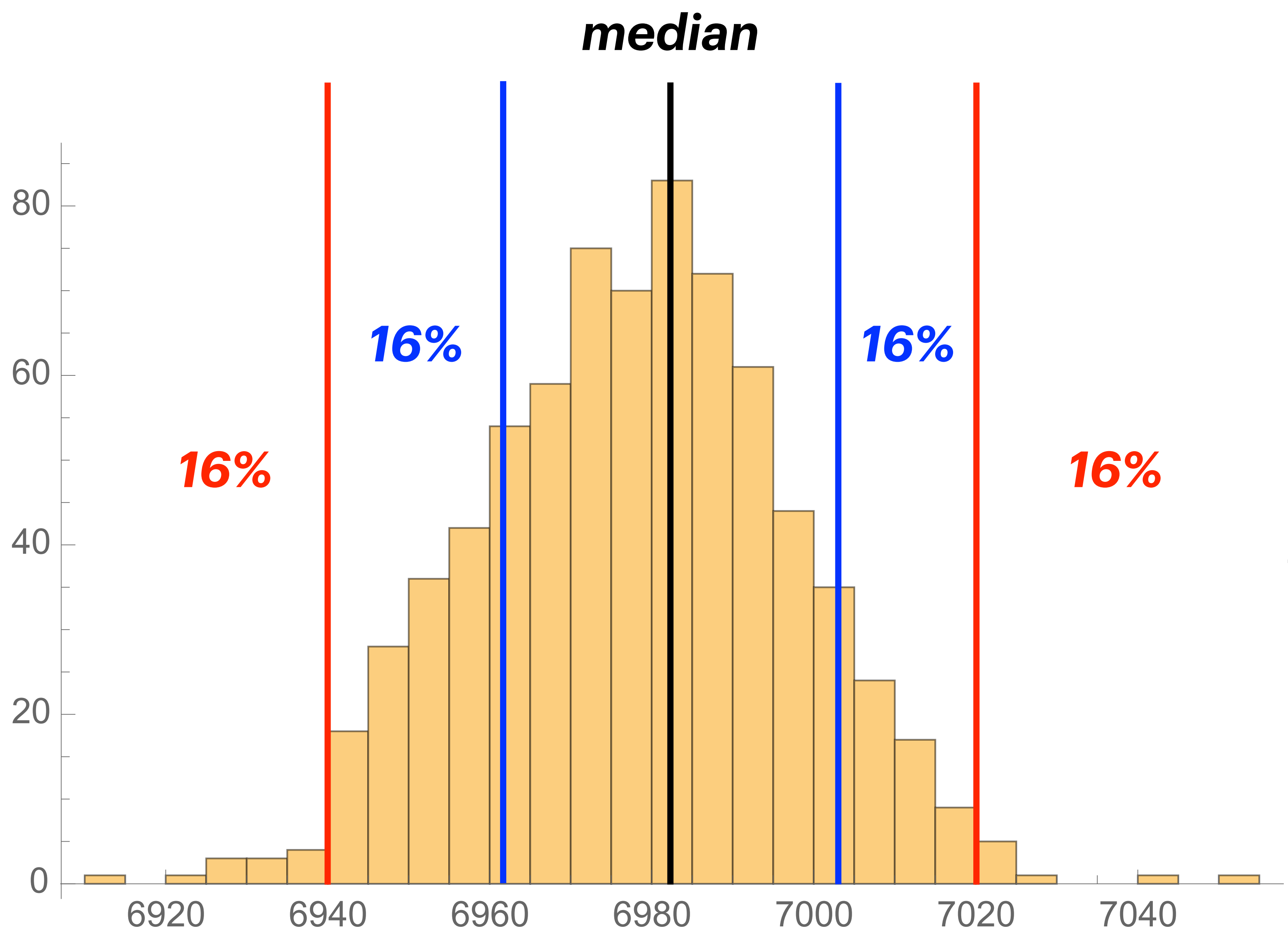
Uncertainty from the method $\Delta\alpha_{\mathrm{model}}$ is greater than the PAs' one

More Realistic Data

- 1000 toy data sets generated
- $(\theta_\mu, R_{\text{had}})$ — 30 data points equally spaced in the interval $0.16 \leq \theta_{\mu\,(\text{mrad})} \leq 4.8$
- Central value randomly chosen from a gaussian distribution with $0.001\,\%$ error — precision of $\mathcal{O}(10^{-5})$
- Selecting data sets whose relative error between its central value for a_μ^{partial} and the true one is lower than $1\,\%$
- Monte Carlo analysis of the fits for each Padé approximant
- PAs used to calculate a_μ in the whole x region

More Realistic Data

- 1000 toy data sets generated
- $(\theta_\mu, R_{\mathrm{had}})$ — … $(\mathrm{mrad}) \leq 4.8$
- Central val… $0.001\,\%$ error
  — precision…
- Selecting d… for a_μ^{partial} and
  the true one…
- Monte Carl…



- PAs used to calculate a_μ in the whole x region

More Realistic Data

- 1000 toy data sets generated
- $(\theta_\mu, R_{\mathrm{had}}) - 3$ … $_\mu(\mathrm{mrad}) \leq 4.8$
- Central valu … $0.001\,\%$ error — precision o
- Selecting da … s for a_μ^{partial} and the true one
- Monte Carlo
- PAs used to calculate a_μ in the whole x region

Constraints

- Quality of fit:
  - χ^2/dof
  - $p-\mathrm{value}$
- Quality of Padés:
  - only real poles;
  - poles outside the region $x \in [0,1]$;
  - no cancelation between pole and zero

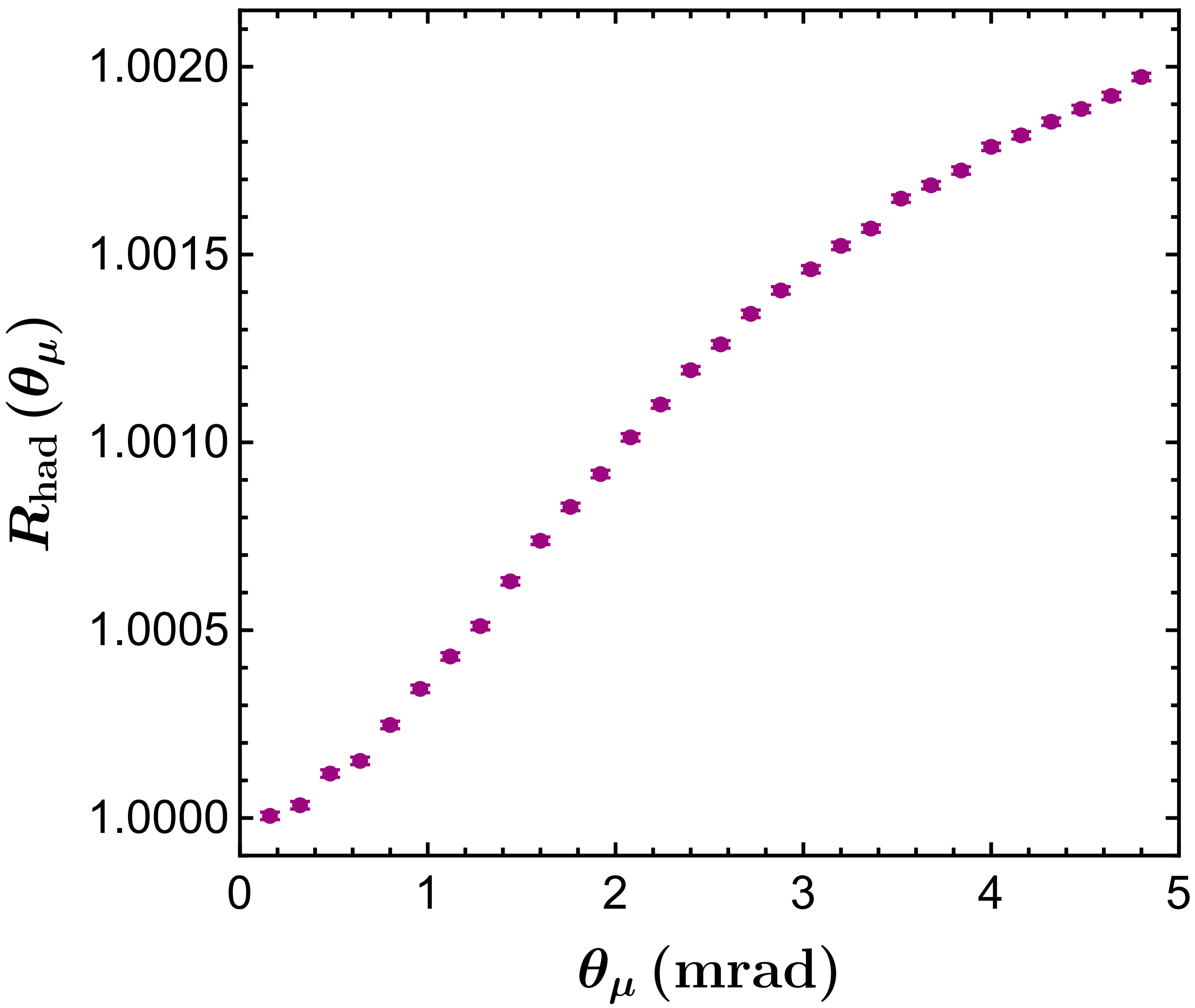
More Realistic Data

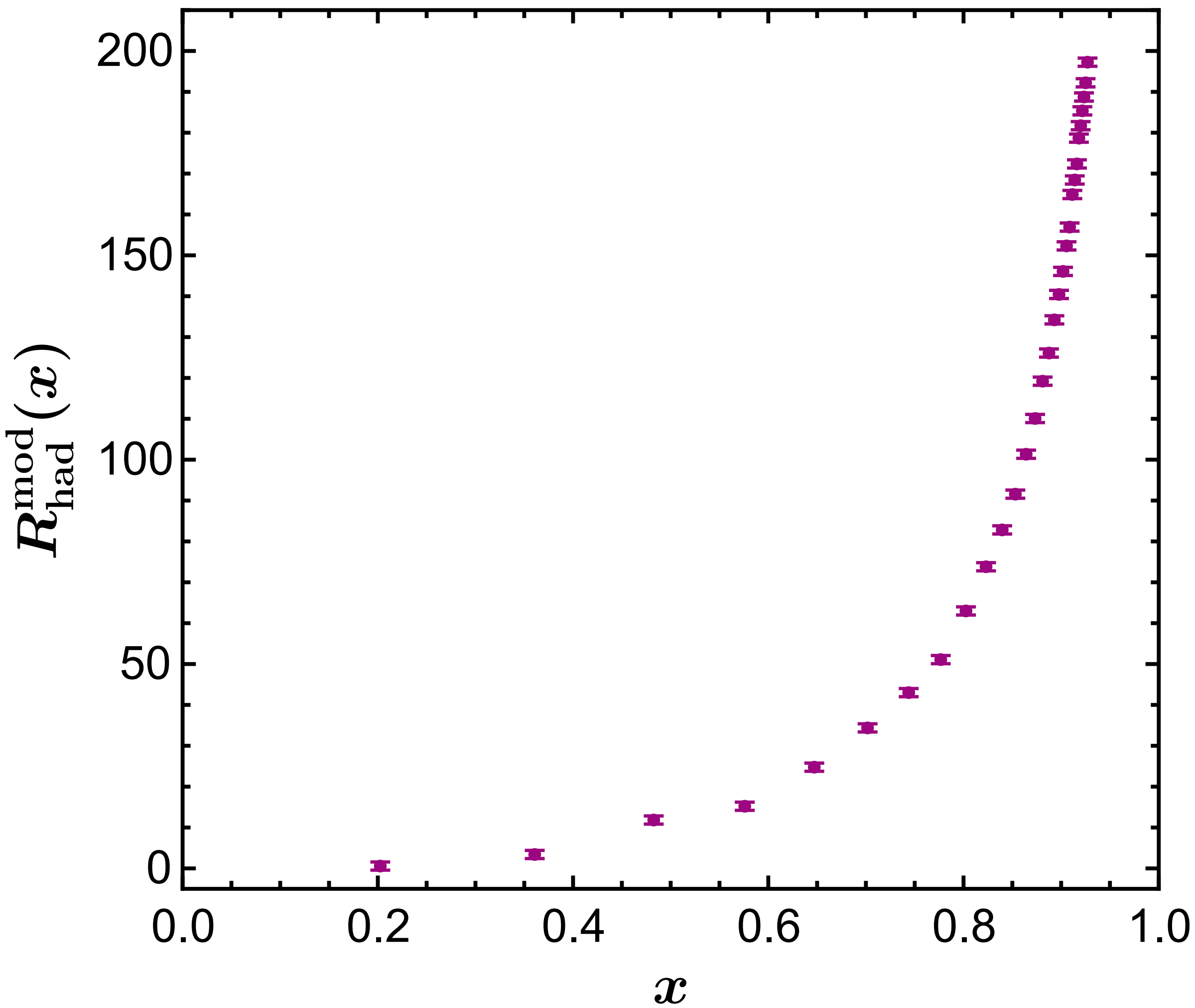
- 1000 toy data sets generated
- $(\theta_\mu, R_{\text{had}}) - 3$ … $_\mu(\text{mrad}) \leq 4.8$
- Central valu… $0.001\ \%$ error — precision c…
- Selecting da… for a_μ^{partial} and the true one
- Monte Carlo…
- PAs used to calculate a_μ in the whole x region

Constraints

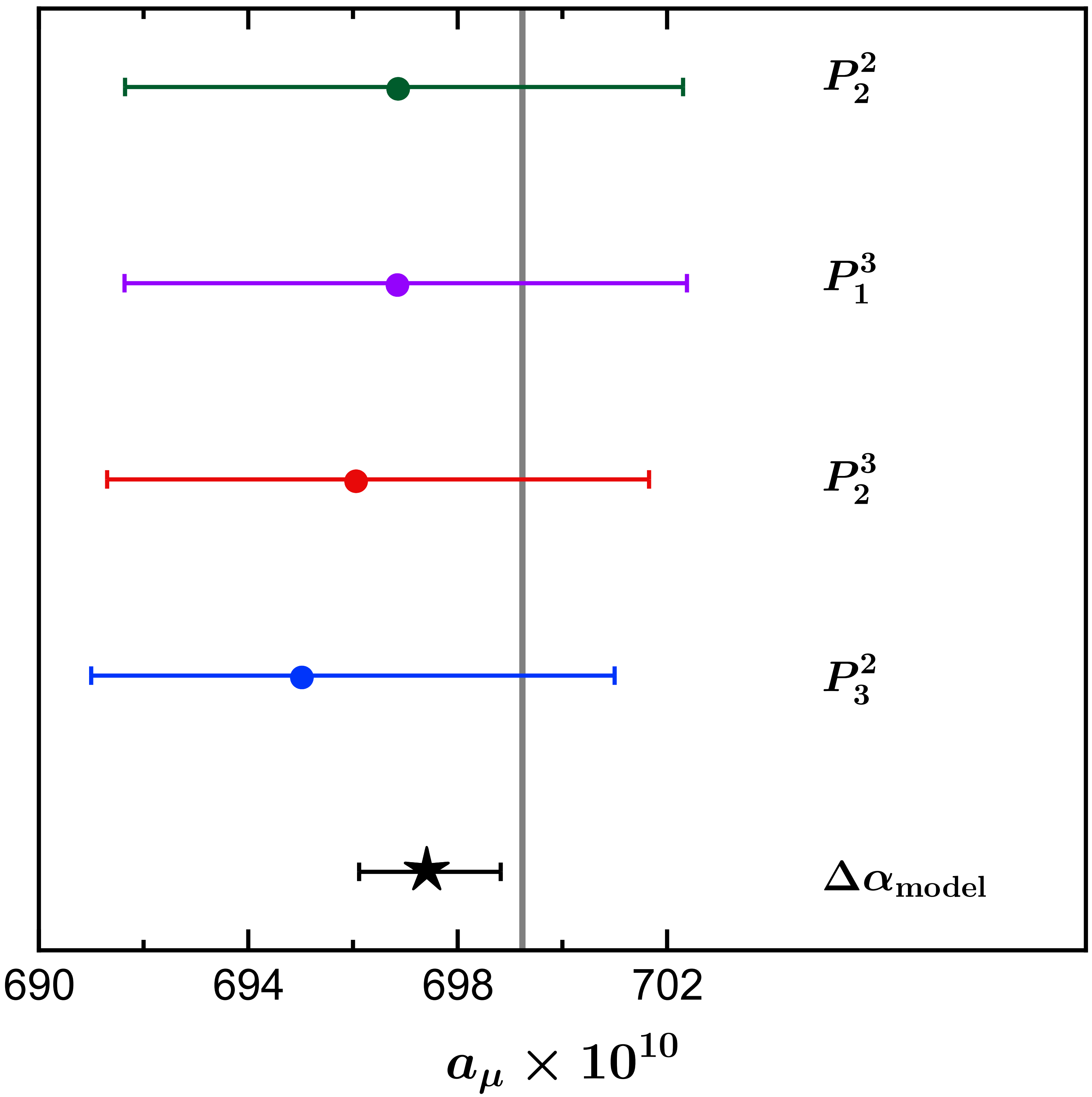
- Quality of fit:
  - χ^2/dof
  - $p - \text{value}$
- Quality of Padés:
  - only real poles;
  - poles outside the region $x \in [0,1]$;
  - no cancelation between pole and zero

More Realistic Data



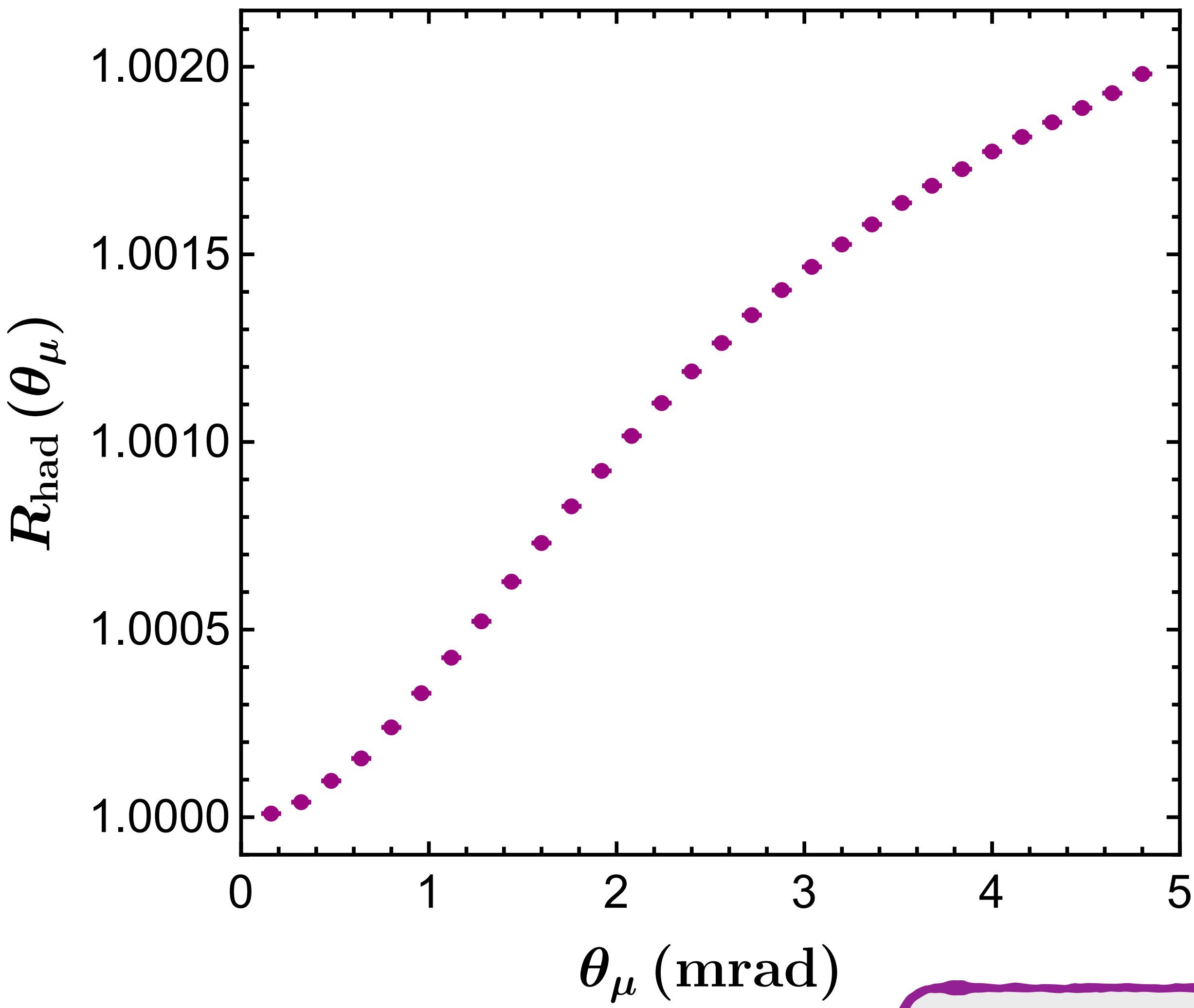


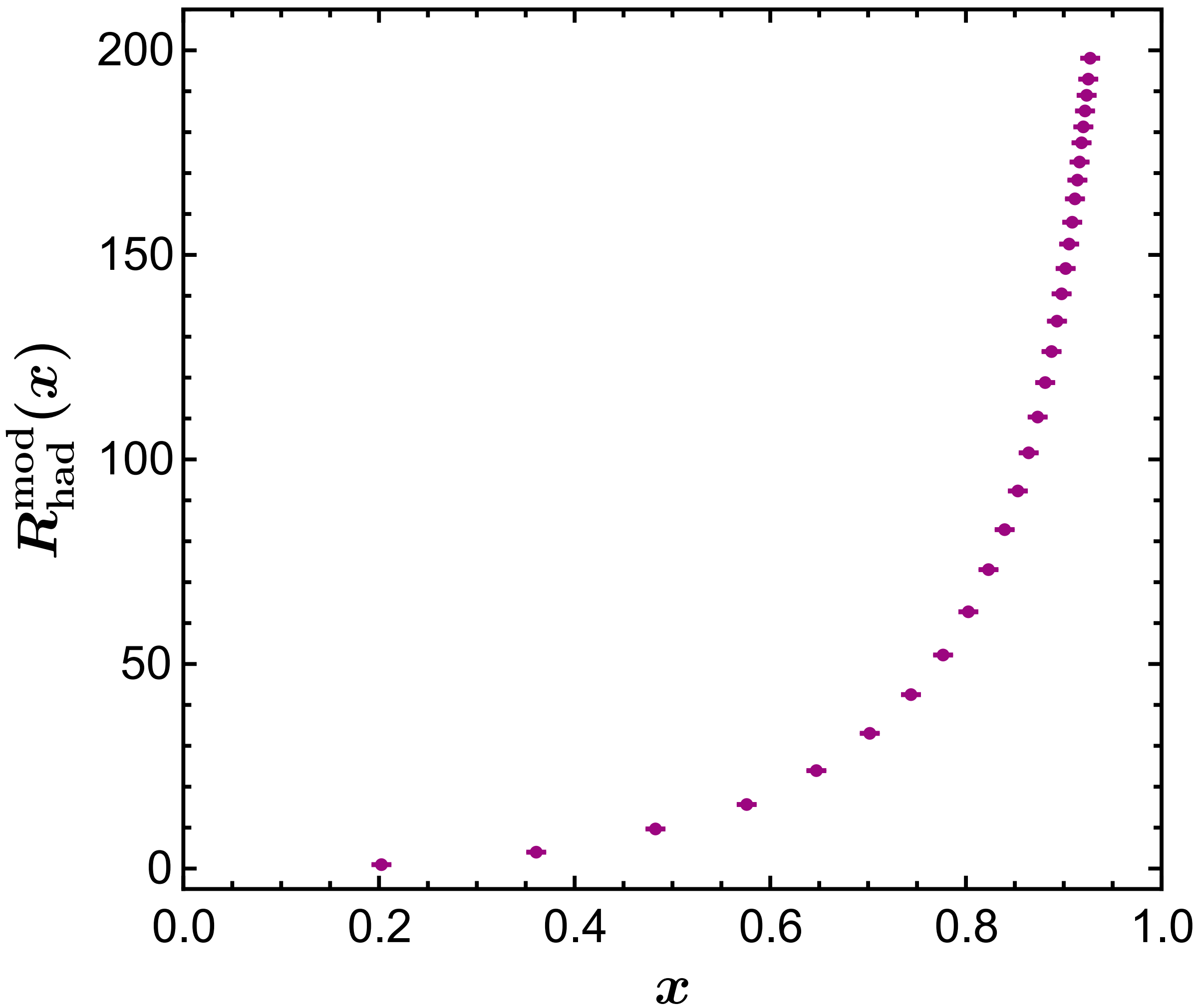
More Realistic Data



- Covariance between Padés predictions still needs to be computed
- Method uncertainty has to be added

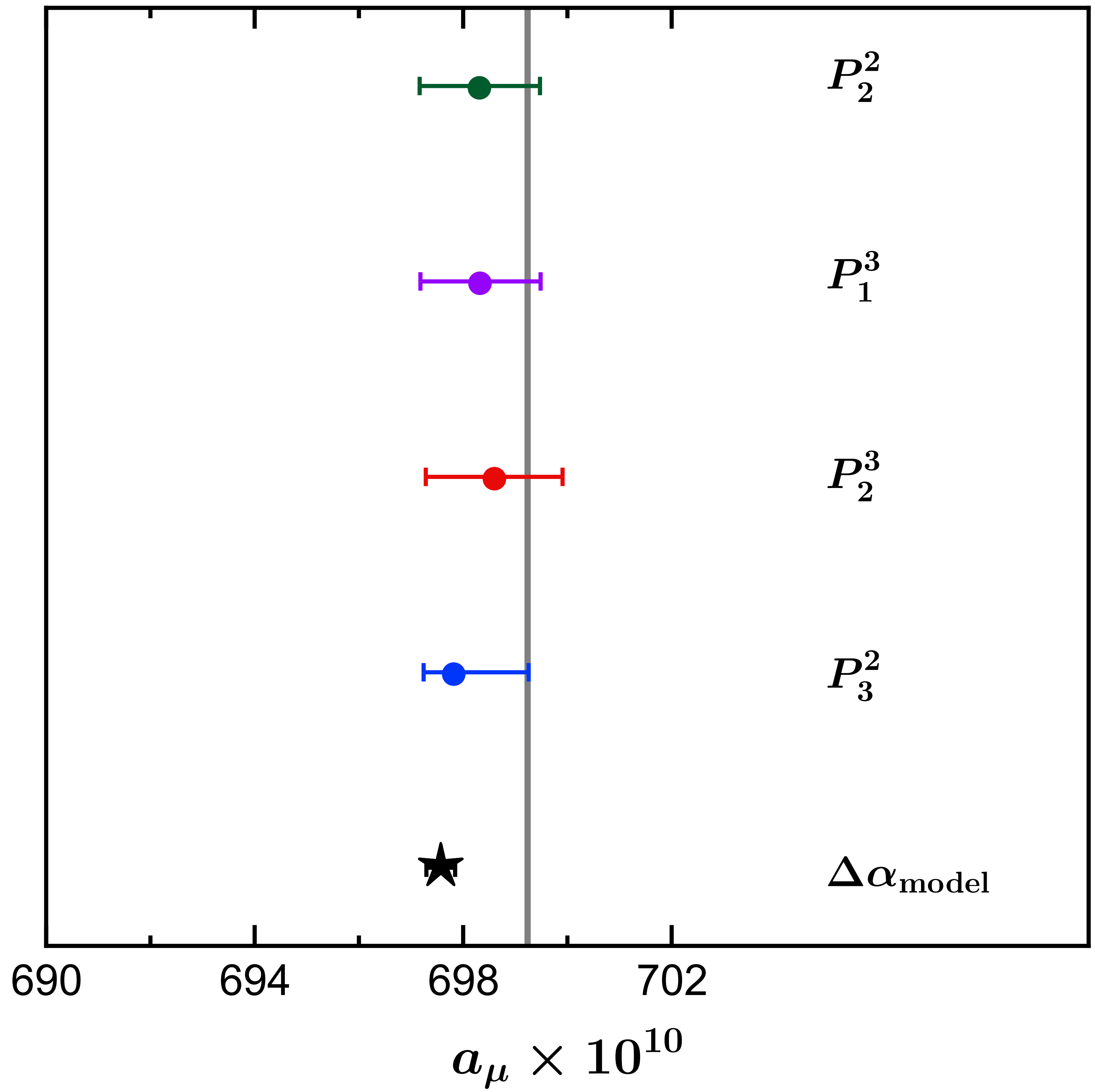
More Realistic Data





Dividing the error of the data points by 5

More Realistic Data



Dividing the error of the data points by 5

- No need for data points in larger values of x, but for lower errors
- Central value predicted by $\Delta\alpha_{\text{model}}$ is more distant from the real a_μ than the Padés

Conclusions and Outlooks

Conclusions

- Plots so far presented are based on a particular model
- Uncertainty due to the method can be estimated and is small
- Error of the model $\Delta\alpha_{\mathrm{model}}$ is larger than the one coming from Padés method — has to be computed in final value
- Larger value of E_μ is not needed but smaller errors of data points

Outlooks

- Results of Padés may be improved by employing:
    - partial Padés;
    - D-log Padés;
    - conformal mapping — Padés also converge;
    - include the moments of the correlator;
    - impose $c_2 = c_3$; **model independent!** $\Delta\alpha_{\rm had}(x) = \sum_{n=0}^{\infty} c_n\, x^n$
    - impose other constraints in PAs analysis

- Padés as fitting functions to data need more studies — no convergence theorems

Thank you for your attention!

FAPESP

IFAE
Institut de Física
d'Altes Energies

MITP
TOPICAL
WORKSHOP

The Evaluation of the Leading Hadronic Contribution to the Muon g-2: Toward the MUonE Experiment

14 – 18 November 2022

https://indico.mitp.uni-mainz.de/event/248

μONe

mitp
Mainz Institute for Theoretical Physics

© Luca Trentadue