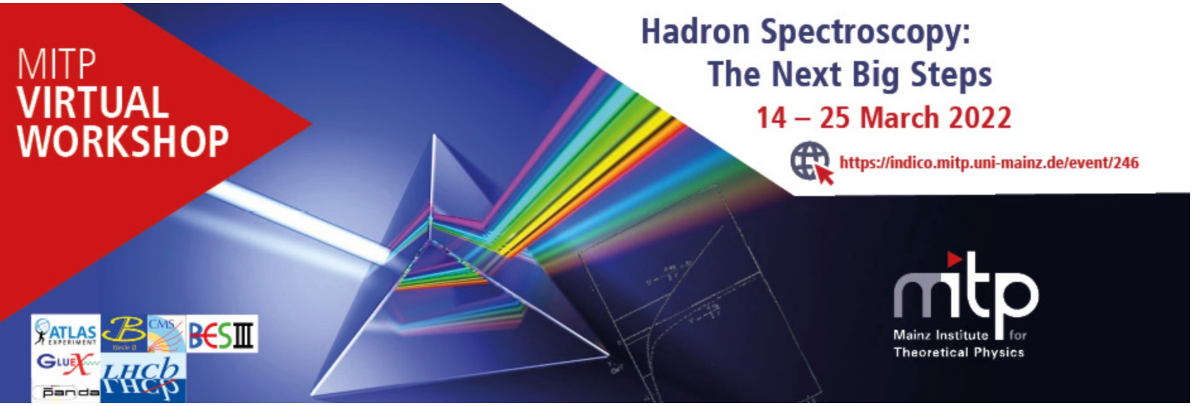



# Near-threshold structures in heavy hadron spectroscopy

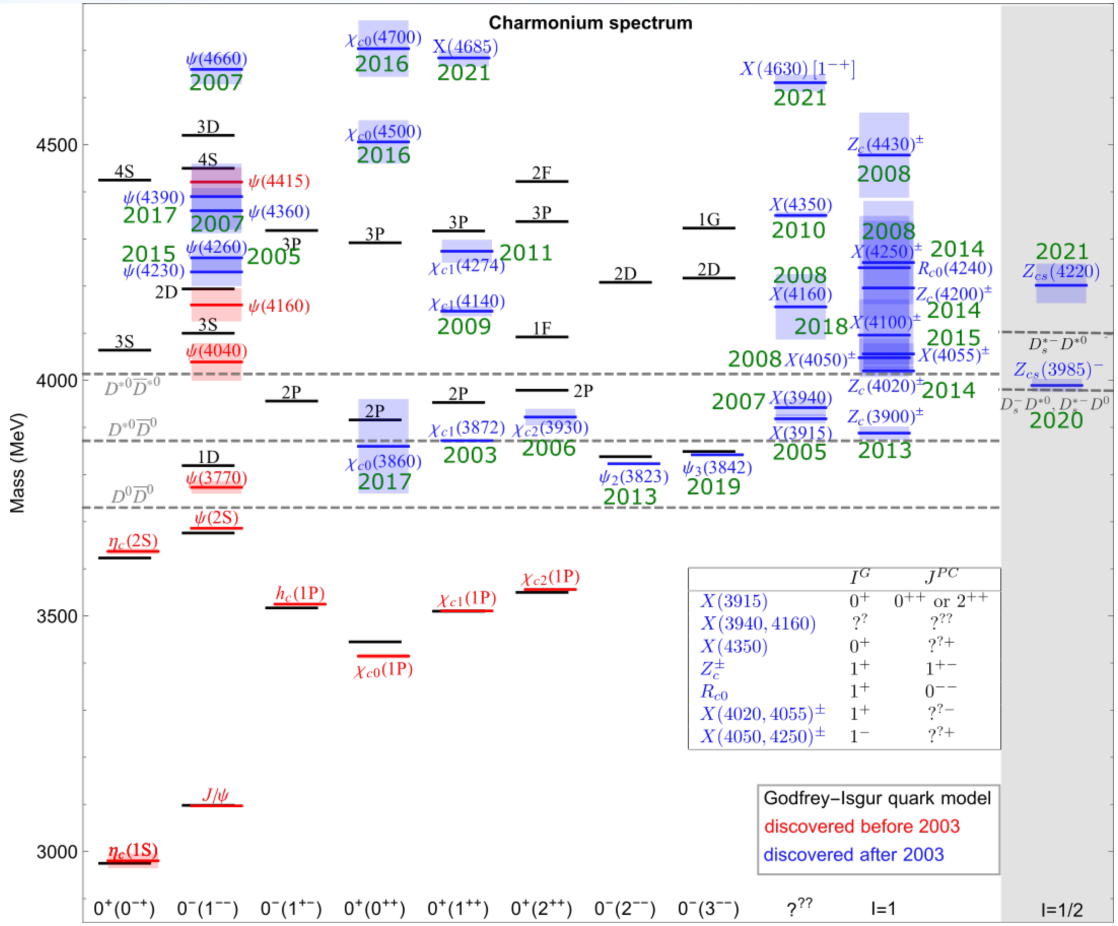
Feng-Kun Guo

Institute of Theoretical Physics, Chinese Academy of Sciences

Based on:

X.-K. Dong, FKG, B.-S. Zou, Phys. Rev. Lett. 126 (2021) 152001 [arXiv:2011.14517]; Progr. Phys. 41 (2021) 65 [arXiv:2101.01021]; Commun. Theor. Phys. 73 (2021) 125201 [arXiv:2108.02673]

# Charmonium(-like) structures




- $T_{cc}$
- $P_c$
- $J/\psi J/\psi$ spec.

- Many new structures are near thresholds of a pair of heavy hadrons.
- Is there any rule?

# Effective range expansion

$$f_0^{-1}(k) = \frac{1}{a_0} + \frac{1}{2} r_0 k^2 - ik + \mathcal{O}\left(\frac{k^4}{\beta^4}\right)$$

$a_0$: S-wave scattering length; negative for repulsion or attraction w/ a bound state

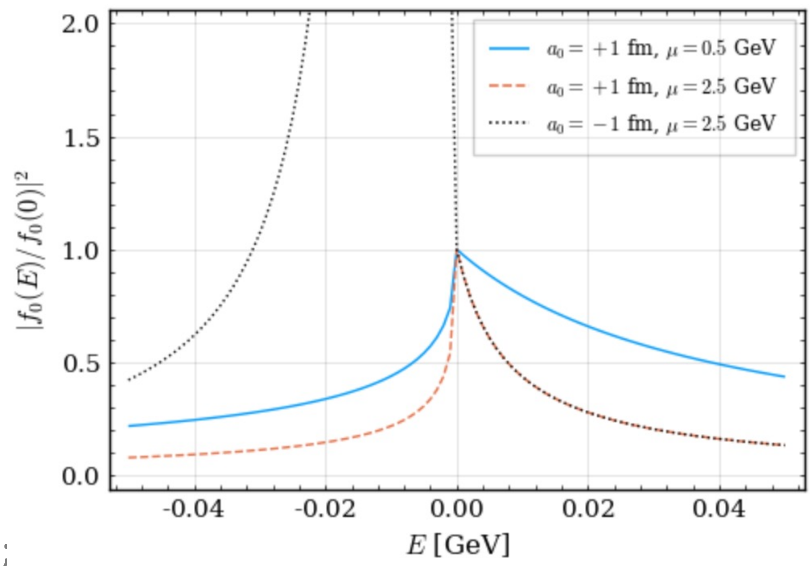
positive for attraction w/o bound state

Very close to threshold, then scattering length approximation: $f_0^{-1}(E) = \frac{1}{a_0} - i\sqrt{2\mu E}$

$$|f_0(E)|^2 = \begin{cases} \frac{1}{1/a_0^2 + 2\mu E} & \text{for } E \geq 0 \\ \frac{1}{\left(1/a_0 + \sqrt{-2\mu E}\right)^2} & \text{for } E < 0 \end{cases}$$

- Cusp at threshold (E=0)
- Maximal at threshold for positive $a_0$ (attraction)
- Half-maximum width: $\frac{2}{\mu a_0^2}$;

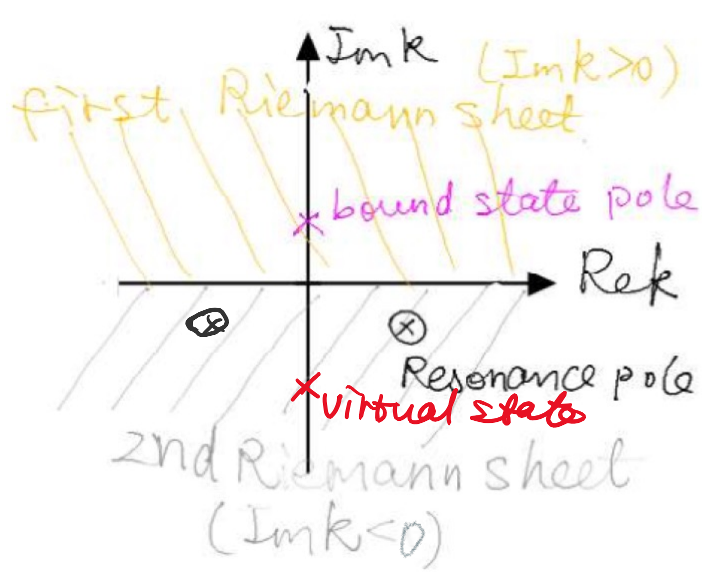
  virtual state pole at $E_{\text{virtual}} = -1/(2\mu a_0^2)$
- Strong interaction, $a_0$ becomes negative, pole below threshold, peak below threshold

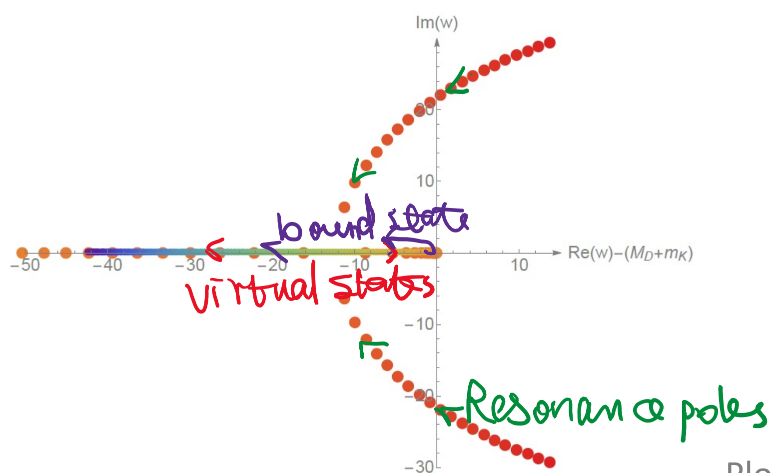



see also, e.g., Brambilla et al. Phys. Rept. 873, 1 (2020); Christoph's talk

# Bound state, virtual state and resonance

- Bound state: pole below threshold on real axis of the first Riemann sheet of complex energy plane
- Virtual state: pole below threshold on real axis of the second Riemann sheet
- Resonance: pole in the complex plane on the second Riemann sheet



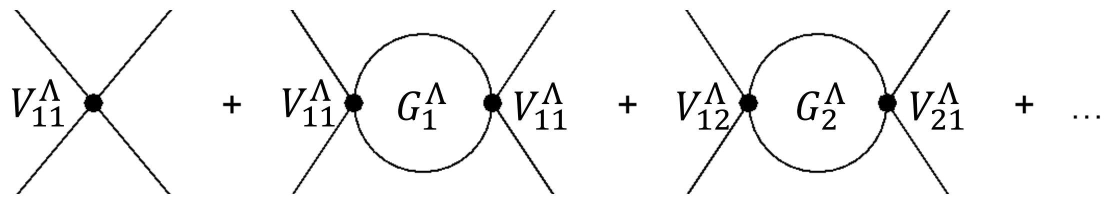




Plot from Matuschek, Baru, FKG, Hanhart, EPJA57(2021)101

For $\dfrac{1}{1/a_0 - i\,k}$, only bound or virtual state poles are possible

# Coupled channels

- Full threshold structure needs to be measured in a lower channel ⇨ coupled channels
- Consider a two-channel system, construct a nonrelativistic effective field theory (NREFT)
  - Energy region around the higher threshold, $\Sigma_2$
  - Expansion in powers of $E = \sqrt{s} - \Sigma_2$
  - Momentum in the lower channel can also be expanded




$$T(E) = V + VG(E)V + VG(E)VG(E)V + \cdots = \frac{1}{V^{-1} - G(E)}$$

$$G_1^{\Lambda}(E) = i\int^{\Lambda_1} \frac{d^4q}{(2\pi)^2} \frac{1}{(q^2 - m_{1,1}^2 + i\epsilon)[(P-q)^2 - m_{1,2}^2 + i\epsilon]} = R(\Lambda_1) - i\frac{k_1}{8\pi\sqrt{s}}$$

$$G_2^{\Lambda}(E) = -\frac{1}{4m_{2,1}m_{2,2}} \int^{\Lambda_2} \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{2\mu_2}{\mathbf{q}^2 - 2\mu_2 E - i\epsilon} = \frac{1}{8\pi\Sigma_2}\left[-\frac{2\Lambda_2}{\pi} + \boxed{\sqrt{-2\mu_2 E - i\epsilon}} + \mathcal{O}\left(\frac{k_2^2}{\Lambda_2}\right)\right],$$

Nonanalyticity only from here

  - $\Lambda$ dependence absorbed by $V^{-1}$
  - Consider nonsingular $V$

# NREFT at LO

- Very close to the higher threshold, LO:

$$T(E) = 8\pi\Sigma_2 \begin{pmatrix} -\frac{1}{a_{11}} + ik_1 & \frac{1}{a_{12}} \\ \frac{1}{a_{12}} & -\frac{1}{a_{22}} - \sqrt{-2\mu_2 E - i\epsilon} \end{pmatrix}^{-1}$$

$$= -\frac{8\pi\Sigma_2}{\det} \begin{pmatrix} \frac{1}{a_{22}} + \sqrt{-2\mu_2 E - i\epsilon} & \frac{1}{a_{12}} \\ \frac{1}{a_{12}} & \frac{1}{a_{11}} - ik_1 \end{pmatrix},$$

$$\det = \left(\frac{1}{a_{11}} - i\, k_1\right)\left(\frac{1}{a_{22}} + \sqrt{-2\mu_2 E - i\epsilon}\right) - \frac{1}{a_{12}^2}$$

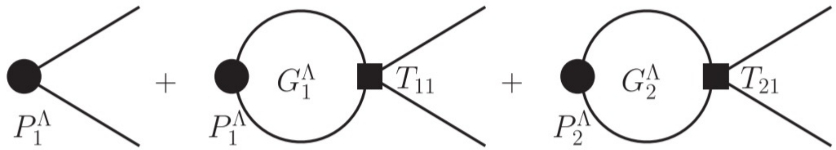
Effective scattering length with open-channel effects becomes complex, $\mathrm{Im}\frac{1}{a_{22,\mathrm{eff}}} \le 0$

$$T_{22}(E) = -\frac{8\pi}{\Sigma_2}\left[\frac{1}{a_{22,\mathrm{eff}}} - i\sqrt{2\mu_2 E} + \mathcal{O}(E)\right]^{-1}$$

$$\frac{1}{a_{22,\mathrm{eff}}} = \frac{1}{a_{22}} - \frac{a_{11}}{a_{12}^2(1 + a_{11}^2 k_1^2)} - i\frac{a_{11}^2 k_1}{a_{12}^2(1 + a_{11}^2 k_1^2)}.$$

# NREFT at LO

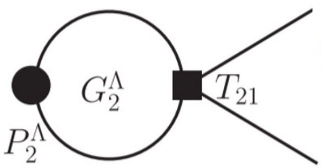
- Consider a production process, must go through final-state interaction (unitarity)




$$
\begin{aligned}
& P_1^\Lambda[1+G_1^\Lambda T_{11}(E)] + P_2^\Lambda G_2^\Lambda(E) T_{21}(E) \\
&= P_1^\Lambda (V_{11}^\Lambda)^{-1} T_{11}(E) + \left[P_1^\Lambda (V_{11}^\Lambda)^{-1} V_{12}^\Lambda + P_2^\Lambda\right] G_2^\Lambda T_{21}(E) \\
&\equiv P_1 T_{11}(E) + P_2 T_{21}(E)
\end{aligned}
$$

Around 2nd th.

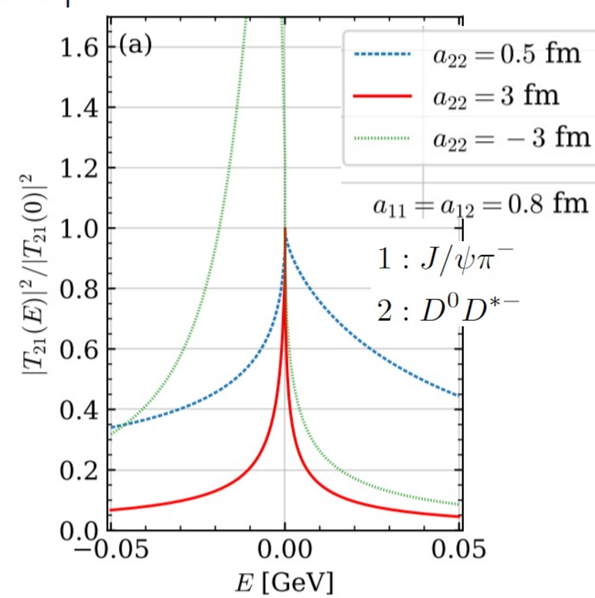
- All nontrivial energy dependence are contained in $T_{11}(E)$ and $T_{21}(E)$
- Case-1: dominated by $T_{21}(E)$,




$$
T_{21}(E) = \frac{-8\pi\Sigma_2}{a_{12}(1/a_{11} - ik_1)} \left[\frac{1}{a_{22,\text{eff}}} - i\sqrt{2\mu_2 E} + \mathcal{O}(E)\right]^{-1}.
$$

$$
|T_{21}(E)|^2 \propto |T_{22}(E)|^2 \propto
$$

$$
\begin{cases}
\left[\left(\text{Re}\,\frac{1}{a_{22,\text{eff}}}\right)^2 + \left(\text{Im}\,\frac{1}{a_{22,\text{eff}}} - \sqrt{2\mu E}\right)^2\right]^{-1} & \text{for } E \geq 0 \\
\left[\left(\text{Im}\,\frac{1}{a_{22,\text{eff}}}\right)^2 + \left(\text{Re}\,\frac{1}{a_{22,\text{eff}}} + \sqrt{-2\mu E}\right)^2\right]^{-1} & \text{for } E < 0
\end{cases}
$$

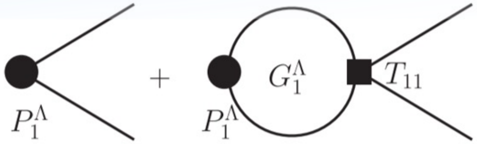



- ➢ Maximal at threshold for positive $\text{Re}(a_{22,\text{eff}})$ (attraction),

  FWHM $\propto \frac{1}{\mu}$: The heavier, the narrower!
- ➢ Peaking at pole for negative $\text{Re}(a_{22,\text{eff}})$ : "bound state" if near-th.

# NREFT at LO

- Case-2: dominated by $T_{11}(E)$

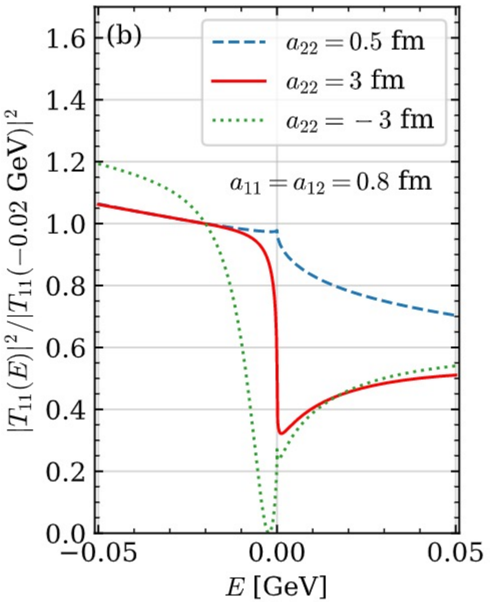



$$T_{11}(E) = \frac{-8\pi\Sigma_2\left(\frac{1}{a_{22}} - i\sqrt{2\mu_2 E}\right)}{\left(\frac{1}{a_{11}} - i\,k_1\right)\left[\frac{1}{a_{22,\mathrm{eff}}} - i\sqrt{2\mu_2 E} + \mathcal{O}(E)\right]}$$

➢ One pole and one zero

➢ For strongly interacting channel-2 (large $a_{22}$), there must be a dip around threshold (zero close to threshold)

$$\frac{1}{a_{22,\mathrm{eff}}} = \frac{1}{a_{22}} - \frac{a_{11}}{a_{12}^2(1+a_{11}^2k_1^2)} - i\frac{a_{11}^2k_1}{a_{12}^2(1+a_{11}^2k_1^2)}.$$




Poles in complex momentum plane:

$(0.37 - i0.08)$GeV

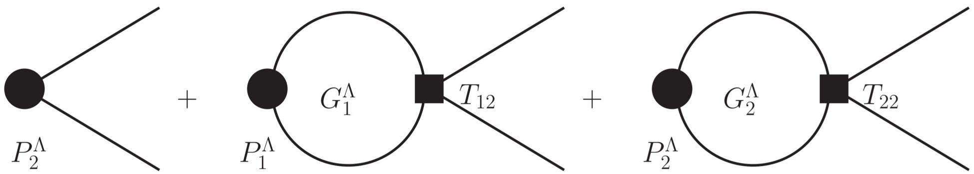
$(0.04 - i0.08)$GeV

$(-0.09 - i0.08)$GeV

- More complicated line shape if both channels are important for the production

# NREFT at LO

- Case-3: final states in channel-2



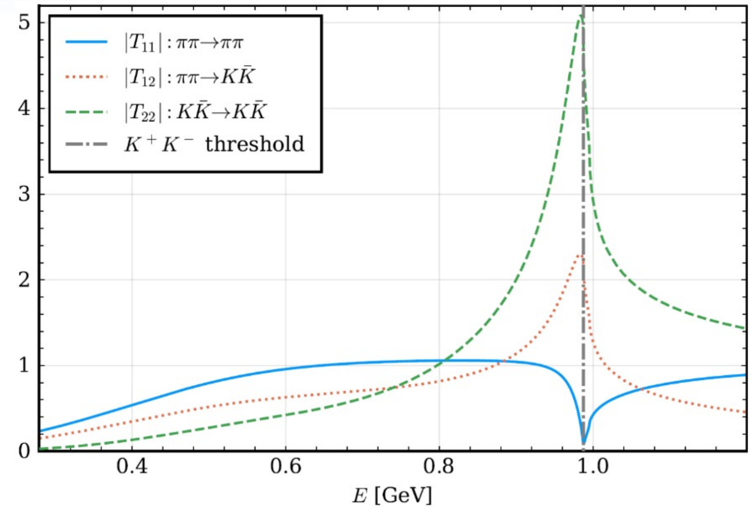

$$P_1 T_{12}(E) + P_2 T_{22}(E) \propto \left[\frac{1}{a_{22,\mathrm{eff}}} - i\sqrt{2\mu_2 E} + \mathcal{O}(E)\right]^{-1}$$

- Suppression due to phase space
- Narrow peak just above threshold would require an additional nearby singularity (pole or TS or both)

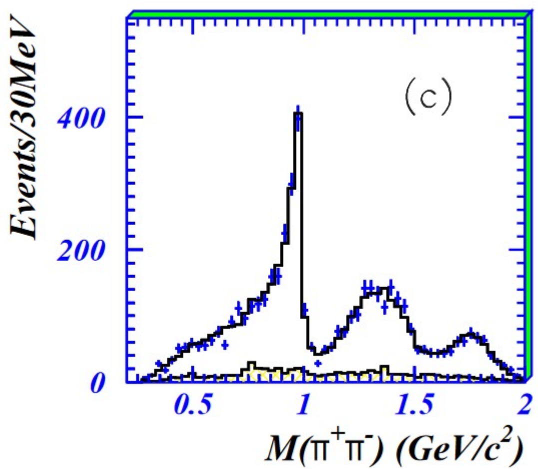
# Phenomenology

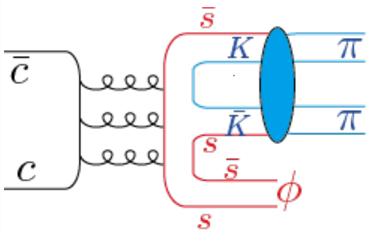
- $T$-matrix for $\pi\pi$ and $K\bar{K}$ coupled channels

  with the T-matrix from
  L.-Y. Dai, M. R. Pennington, PRD90(2014)036004




- $f_0(980)$ in $J/\psi \to \phi\pi^+\pi^-$ and $J/\psi \to \omega\pi^+\pi^-$ Channels: $\pi\pi$ and $K\bar{K}$

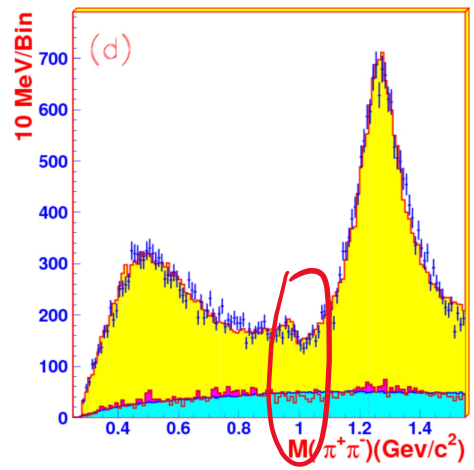


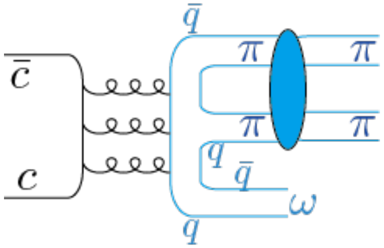



BES, PLB607(2005)243

Driving channel: $K\bar{K}$

$J/\psi \to \phi K\bar{K} \to \phi\pi^+\pi^-$



BES, PLB598(2004)149

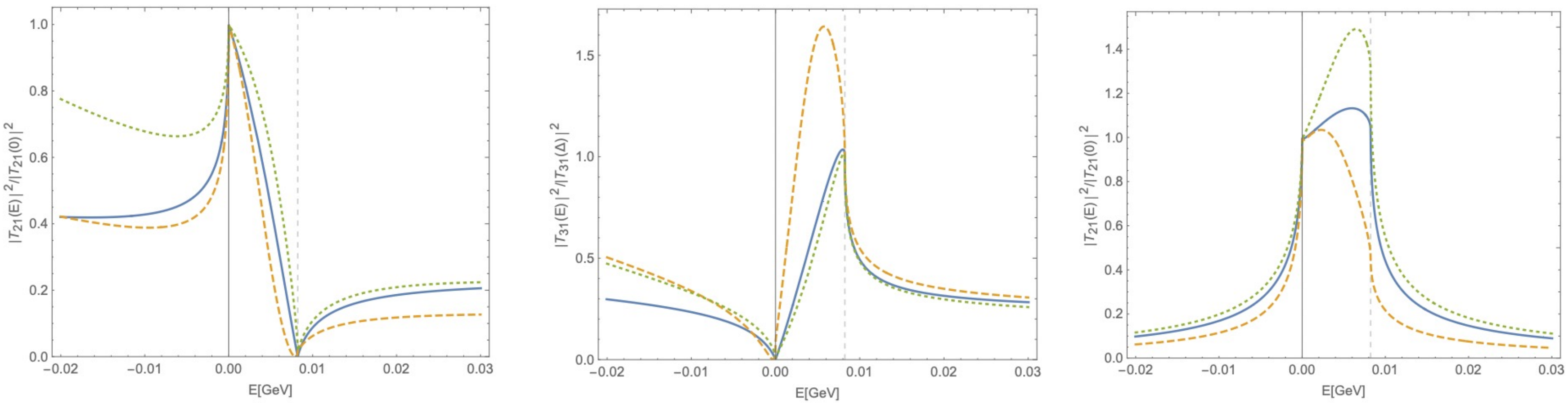
Driving channel: $\pi\pi$

$J/\psi \to \omega\pi\pi \to \omega\pi^+\pi^-$

# Phenomenology

- Open-flavor much easier produced than $Q\bar{Q}$ + light hadrons, peaks around threshold of a pair of open-flavor hadrons with attractive interaction;
- General pattern: the heavier, the more pronounced
- Complications due to more channels

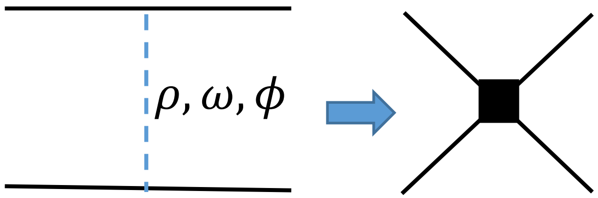
Zhang[2], FKG, in preparation




I do not mean that the near-threshold structures are just threshold cusps. Prominent near-threshold structures imply near-threshold singularities more singular than a threshold cusp!

# Interaction from VMD model

- Which pairs have short-range attraction? Many
- Approximations:
  - Constant contact terms ($V$) saturated by light-vector-meson exchange, similar to the vector-meson dominance in the resonance saturation of the low-energy constants in CHPT G. Ecker, J. Gasser, A. Pich, E. de Rafael, NPB321(1989)311
  - Single channels
  - Neglecting mixing with normal charmonia

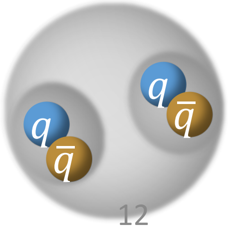



- The T-matrix:

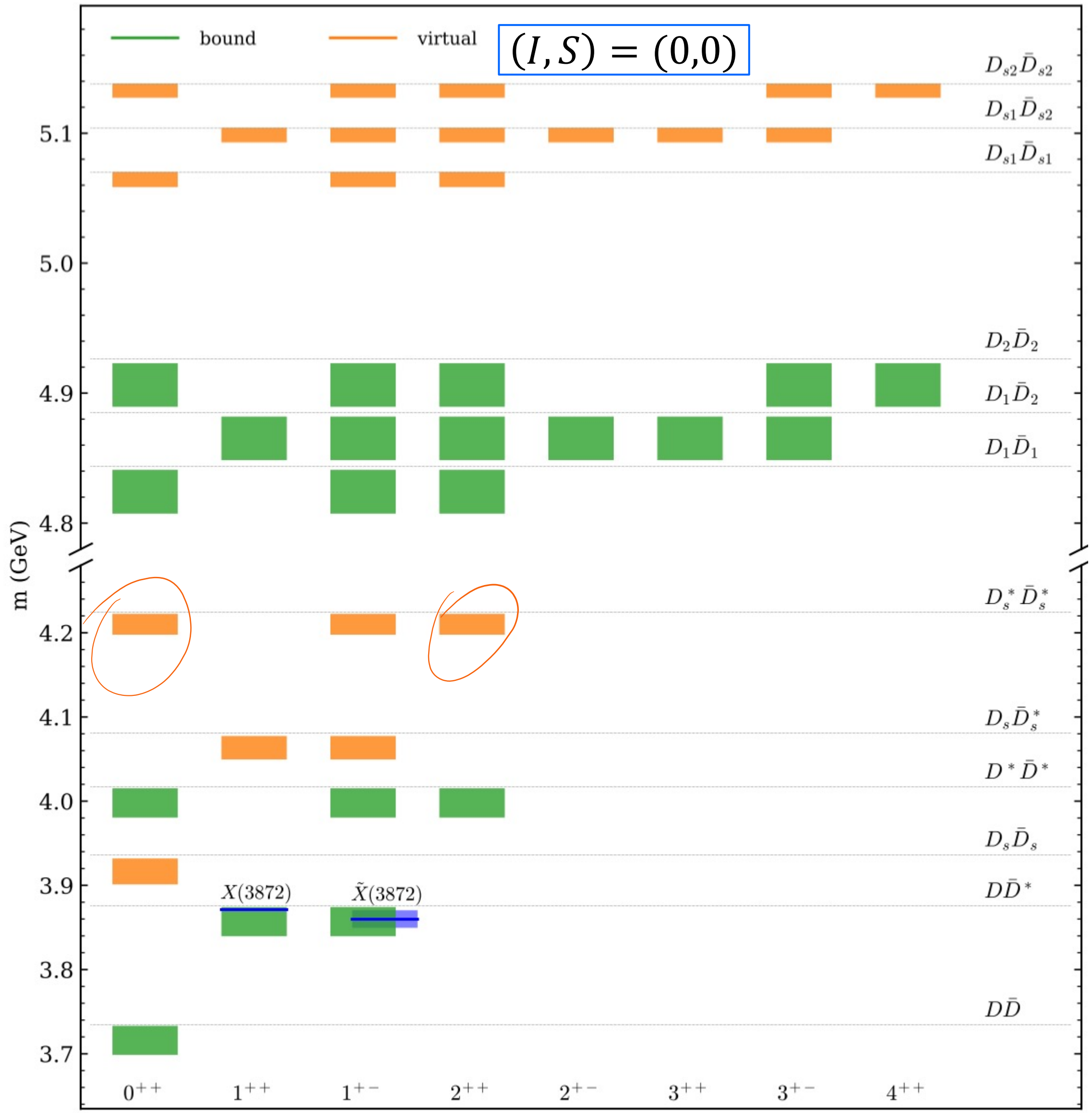
$$T = \frac{V}{1 - VG}$$

$G$: two-point scalar loop integral regularized using dim.reg. with a subtraction constant matched to a Gaussian regularized $G$ at threshold, with cutoff $\Lambda \in [0.5,1.0]$ GeV

- Hadronic molecules appear as bound or virtual state poles of the $T$ matrix

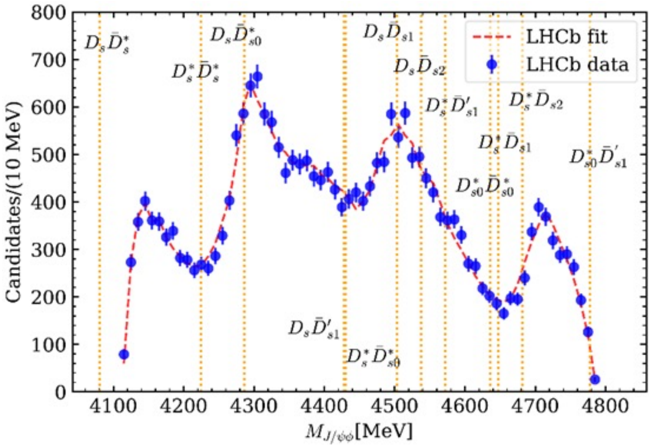


# X(3872) and related states



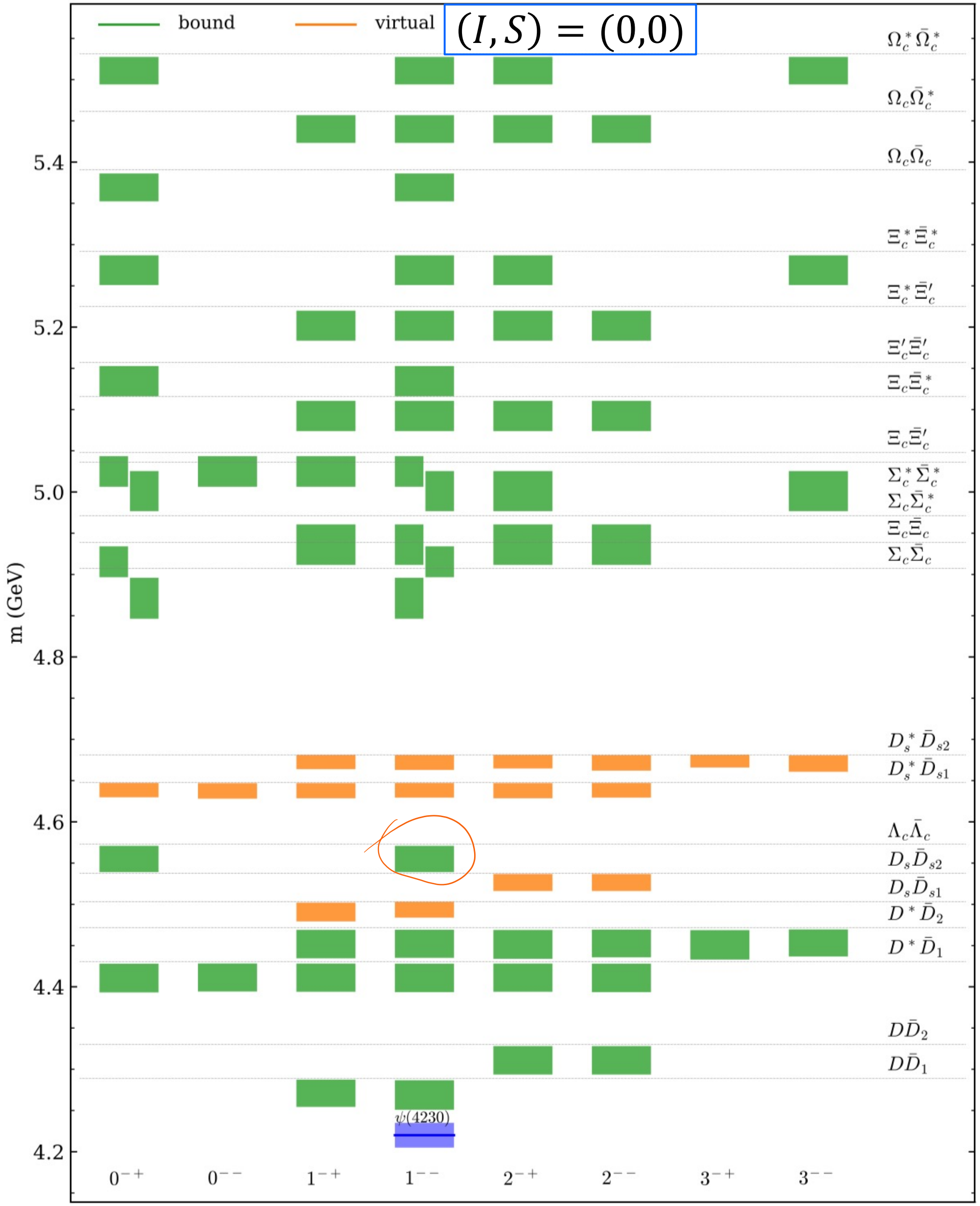

X.-K. Dong, FKG, B.-S. Zou, Progr.Phys. 41 (2021) 65

- ✓ $X(3872)$ as a $\bar{D}D^*$ bound state
- ✓ Negative-C parity partner observed by COMPASS PLB783(2018)334
- ✓ $\bar{D}D$ bound state predicted with lattice Prelovsek et al., JHEP2106,035 and other models
  e.g., Wong, PRC69, 055202; Zhang et al., PRD74, 014013; Gamermann et al., PRD76, 074016; Nieves et al., PRD86, 056004; …
- ✓ Evidence for a $D_s^*\bar{D}_s^*$ virtual state in LHCb data?

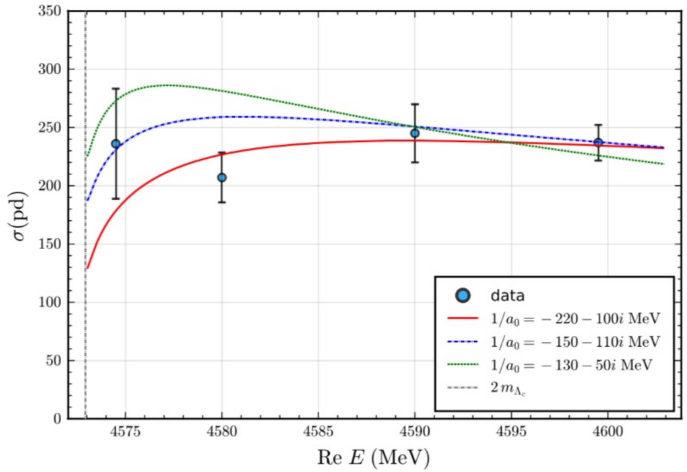



LHCb data: PRL127, 082001

# Isoscalar vectors and related states



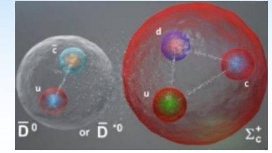

- ✓ $Y(4260)/\psi(4230)$ as a $\bar{D}D_1$ bound state
- ✓ Vector charmonia around 4.4 GeV unclear
- ✓ Evidence for $1^{--}$ $\Lambda_c\bar{\Lambda}_c$ mol. state in BESIII data
  - Sommerfeld factor
  - Near-threshold pole
  - Different from $Y(4630/4660)$



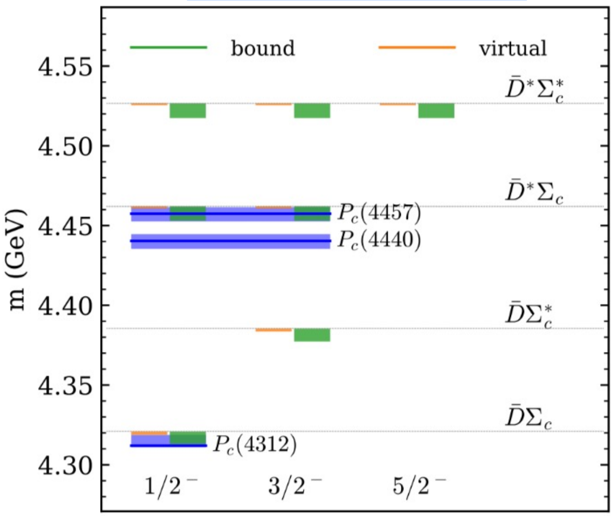

Data taken from BESIII, PRL120(2018)132001

- ✓ Many $1^{--}$ states above 4.8 GeV: Belle-II, BEPC-II-Upgrade, PANDA, STCF(?)

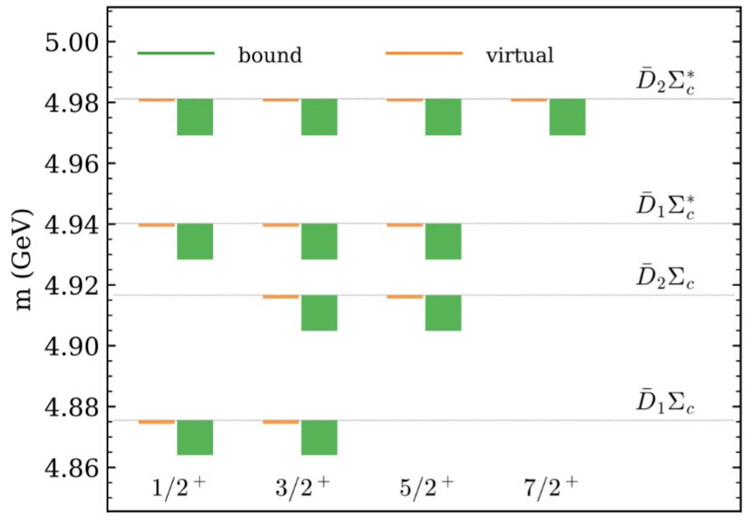
# Hidden-charm pentaquarks



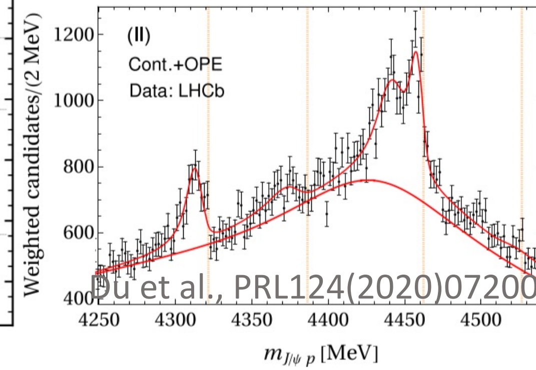
$(I,S) = (1/2,0)$




$(I,S) = (1/2,0)$

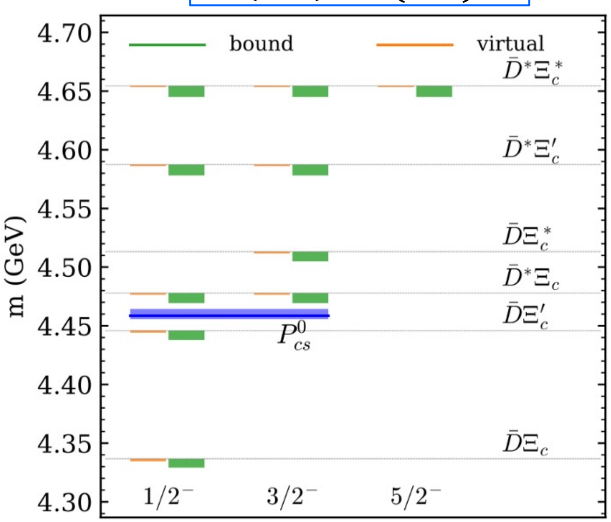



- ✓ The LHCb $P_c$ states as $\bar{D}^{(*)}\Sigma_c$ molecules
- ✓ $\bar{D}\Sigma_c^*$ molecule: hint in the LHCb data

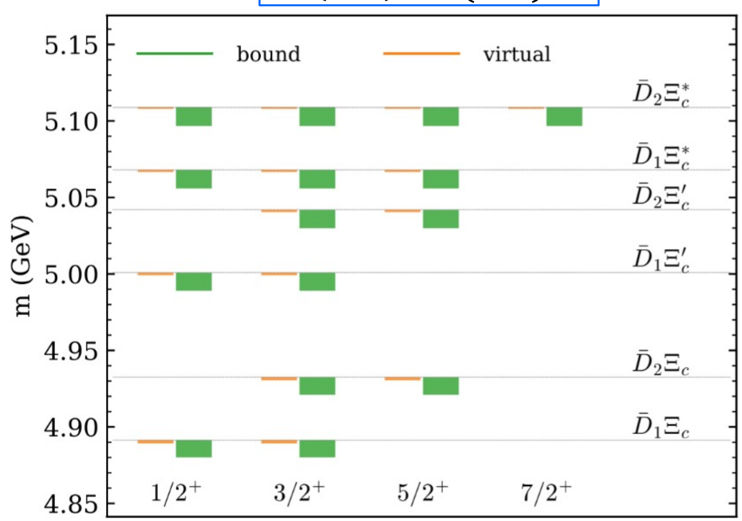



Du et al., PRL124(2020)072001

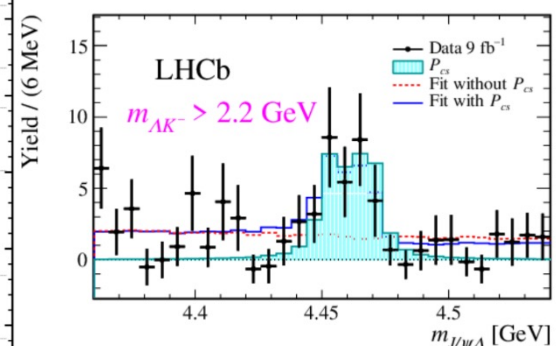
$(I,S) = (0,1)$




$(I,S) = (0,1)$




- ✓ The $P_{cs}(4459)$ could be two $\bar{D}^*\Xi_c$ molecules

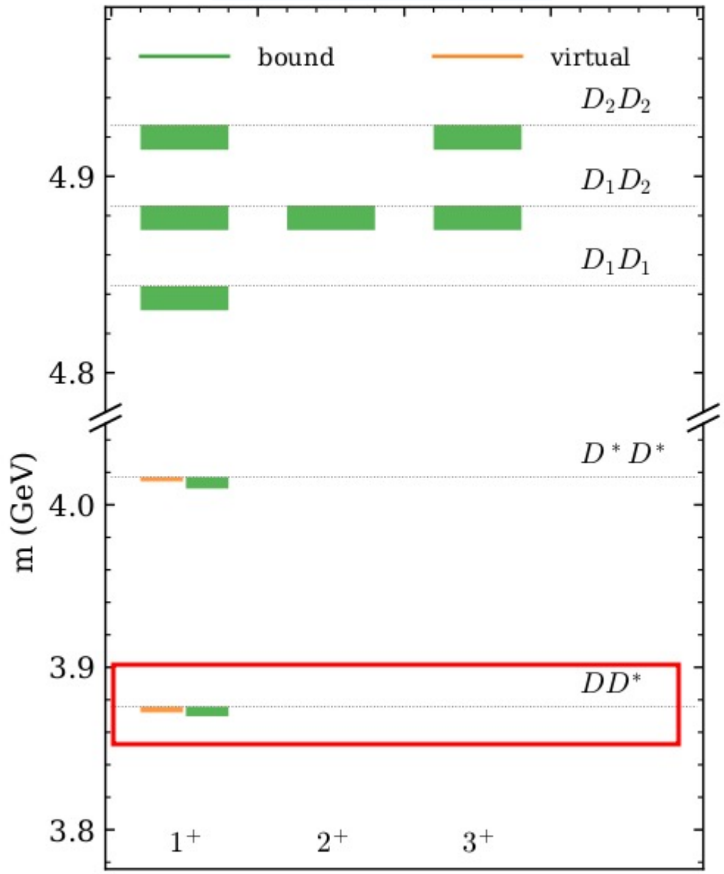



LHCb, Sci.Bull.66,1278

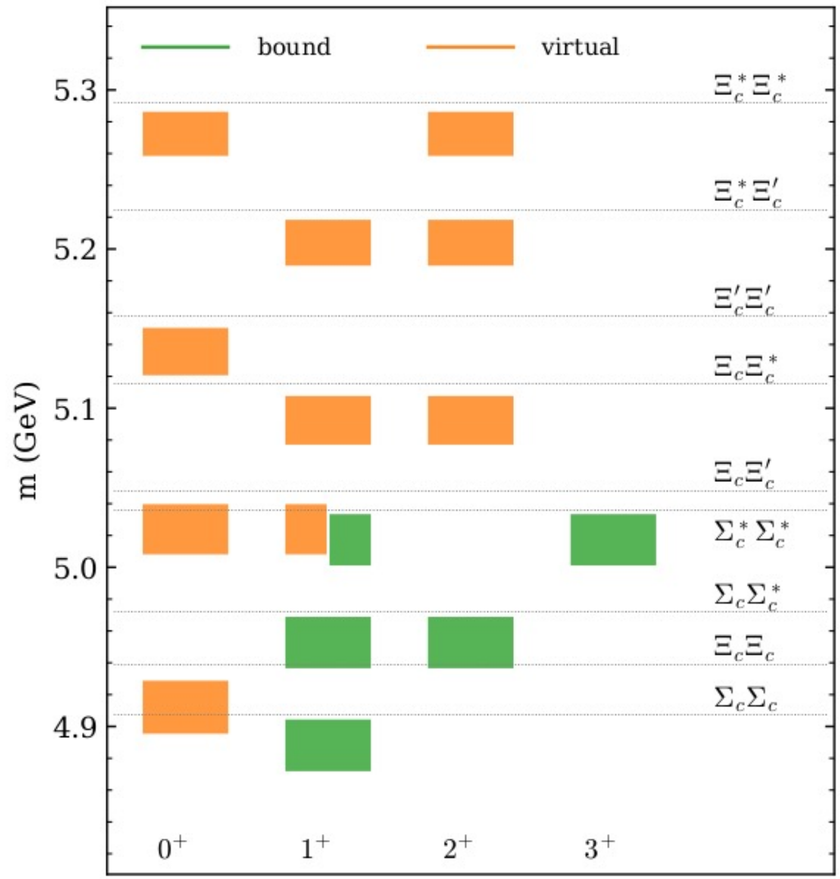
- ✓ Many more baryon-antibaryon molecular states above 4.7 GeV

# Double-charm

$(I,S,B) = (0,0,0)$




$(I,B) = (0,2)$




- ✓ There is an isoscalar $DD^*$ molecular state
- ✓ It has a spin partner $1^+$ $D^*D^*$ state
- ✓ Many other similar double-charm molecular states in other sectors

X.-K. Dong, FKG, B.-S. Zou, CTP73(2021)125201

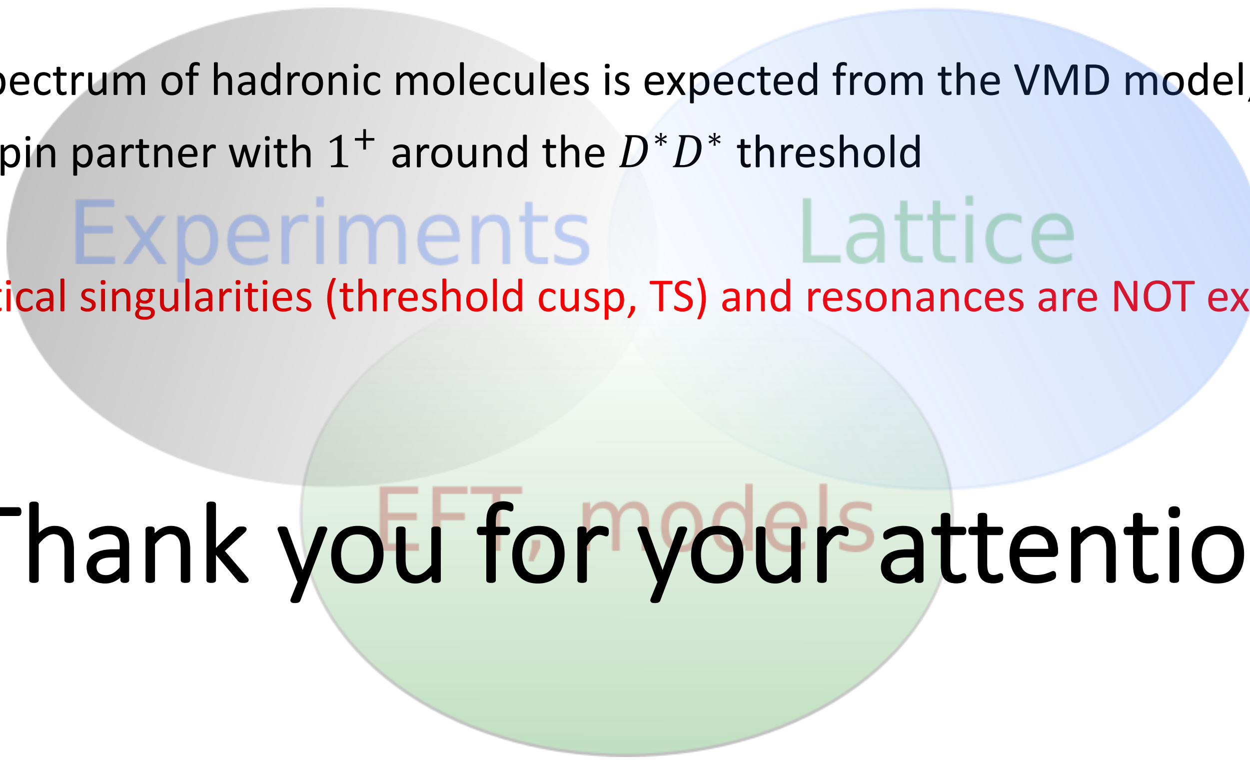
# Conclusion

- General rule for (near-)threshold structures: S-wave attraction, more prominent for heavier particles and stronger attraction
- Strong attraction, then hadronic molecules below threshold, otherwise threshold cusps (and virtual state poles)

- Threshold structures should be more prominent in bottom than in charm

- A rich spectrum of hadronic molecules is expected from the VMD model; $T_{cc}^+$ would have a spin partner with $1^+$ around the $D^*D^*$ threshold

- Kinematical singularities (threshold cusp, TS) and resonances are NOT exclusive




Thank you for your attention!

# Interactions from VMD

X.-K. Dong, FKG, B.-S. Zou, arXiv:2108.02673

Attraction: $F > 0$

| System | $I$ | $S$ | Thresholds [MeV] | Exchanged particles | $F$ |
|---|---|---|---|---|---|
| $D^{(*)}\bar{D}^{(*)}/D^{(*)}D^{(*)}$ | 1 | 0/0 | (3734, 3876, 4017) | $\rho$, $\omega$ | $-\frac{1}{2}, \frac{1}{2}/-\frac{1}{2}, -\frac{1}{2}$ |
| | 0 | | | | $\frac{3}{2}, \frac{1}{2}/\frac{3}{2}, -\frac{1}{2}$ |
| $D_s^{(*)}\bar{D}^{(*)}/D_s^{(*)}D^{(*)}$ | $\frac{1}{2}$ | 1/1 | (3836, 3977, 3979, 4121) | $K^*$ | $0/-1$ |
| $D_s^{(*)}\bar{D}_s^{(*)}/D_s^{(*)}D_s^{(*)}$ | 0 | 0/2 | (3937, 4081, 4224) | $\phi$ | $1/-1$ |
| $\bar{D}^{(*)}\Lambda_c/D^{(*)}\Lambda_c$ | $\frac{1}{2}$ | 0/0 | (4154, 4295) | $\omega$ | $-1/1$ |
| $\bar{D}_s^{(*)}\Lambda_c/D_s^{(*)}\Lambda_c$ | 0 | $-1/1$ | (4255, 4399) | – | 0/0 |
| $\bar{D}^{(*)}\Xi_c/D^{(*)}\Xi_c$ | 1 | $-1/-1$ | (4337, 4478) | $\rho$, $\omega$ | $-\frac{1}{2}, -\frac{1}{2}/-\frac{1}{2}, \frac{1}{2}$ |
| | 0 | | | | $\frac{3}{2}, -\frac{1}{2}/\frac{3}{2}, \frac{1}{2}$ |
| $\bar{D}_s^{(*)}\Xi_c/D_s^{(*)}\Xi_c$ | $\frac{1}{2}$ | $-2/0$ | (4438, 4582) | $\phi$ | $-1/1$ |
| $\bar{D}^{(*)}\Sigma_c^{(*)}/D^{(*)}\Sigma_c^{(*)}$ | $\frac{3}{2}$ | 0/0 | (4321, 4385, 4462, 4527) | $\rho$, $\omega$ | $-1, -1/-1, 1$ |
| | $\frac{1}{2}$ | | | | $2, -1/2, 1$ |
| $\bar{D}_s^{(*)}\Sigma_c^{(*)}/D_s^{(*)}\Sigma_c^{(*)}$ | 1 | $-1/1$ | (4422, 4486, 4566, 4630) | – | 0/0 |
| $\bar{D}^{(*)}\Xi_c'^{(*)}/D^{(*)}\Xi_c'^{(*)}$ | 1 | $-1/-1$ | (4446, 4513, 4587, 4655) | $\rho$, $\omega$ | $-\frac{1}{2}, -\frac{1}{2}/-\frac{1}{2}, \frac{1}{2}$ |
| | 0 | | | | $\frac{3}{2}, -\frac{1}{2}/\frac{3}{2}, \frac{1}{2}$ |
| $\bar{D}_s^{(*)}\Xi_c'^{(*)}/D_s^{(*)}\Xi_c'^{(*)}$ | $\frac{1}{2}$ | $-2/0$ | (4547, 4614, 4691, 4758) | $\phi$ | $-1/1$ |
| $\bar{D}^{(*)}\Omega_c^{(*)}/D^{(*)}\Omega_c^{(*)}$ | $\frac{1}{2}$ | $-2/0$ | (4562, 4633, 4704, 4774) | – | 0/0 |
| $\bar{D}_s^{(*)}\Omega_c^{(*)}/D_s^{(*)}\Omega_c^{(*)}$ | 0 | $-3/-1$ | (4664, 4734, 4807, 4878) | $\phi$ | $-2/2$ |
| $\Lambda_c\bar{\Lambda}_c/\Lambda_c\Lambda_c$ | 0 | 0/0 | (4573) | $\omega$ | $2/-2$ |
| $\Lambda_c\bar{\Xi}_c/\Lambda_c\Xi_c$ | $\frac{1}{2}$ | $1/-1$ | (4756) | $\omega/K^*$ | $1, 0/-1, -1$ |
| $\Xi_c\bar{\Xi}_c/\Xi_c\Xi_c$ | 1 | $0/-2$ | (4939) | $\rho$, $\omega$, $\phi$ | $-\frac{1}{2}, \frac{1}{2}, 1/-\frac{1}{2}, -\frac{1}{2}, -1$ |
| | 0 | | | | $\frac{3}{2}, \frac{1}{2}, 1/\frac{3}{2}, -\frac{1}{2}, -1$ |
| $\Lambda_c\bar{\Sigma}_c^{(*)}/\Lambda_c\Sigma_c^{(*)}$ | 1 | 0/0 | (4740, 4805) | $\omega/K^*$ | $1, 0/-1, -1$ |

# Interactions from VMD

Attraction: $F > 0$

| System | $I$ | $S$ | Thresholds [MeV] | Exchanged particles | $F$ |
|---|---|---|---|---|---|
| $\Lambda_c\bar{\Xi}_c^{\prime(*)}/\Lambda_c\Xi_c^{\prime(*)}$ | $\frac{1}{2}$ | $1/-1$ | (4865, 4932) | $\omega$ | $1/-1$ |
| $\Lambda_c\bar{\Omega}_c^{(*)}/\Lambda_c\Omega_c^{(*)}$ | 0 | $2/-2$ | (4982, 5052) | – | $0/0$ |
| $\Sigma_c^{(*)}\bar{\Xi}_c/\Sigma_c^{(*)}\Xi_c$ | $\frac{3}{2}$ | $1/-1$ | (4923, 4988) | $\rho,\ \omega,\ K^*$ | $-1,\ 1,\ 0/-1,\ -1,\ -2$ |
| | $\frac{1}{2}$ | | | | $2,\ 1,\ 0/2,\ -1,\ -2$ |
| $\Xi_c\bar{\Xi}_c^{\prime(*)}/\Xi_c\Xi_c^{\prime(*)}$ | 1 | $0/-2$ | (5048, 5115) | $\rho,\ \omega,\ \phi$ | $-\frac{1}{2},\ \frac{1}{2},\ 1/-\frac{1}{2},\ -\frac{1}{2},\ -1$ |
| | 0 | | | | $\frac{3}{2},\ \frac{1}{2},\ 1/\frac{3}{2},\ -\frac{1}{2},\ -1$ |
| $\Xi_c\bar{\Omega}_c^{(*)}/\Xi_c\Omega_c^{(*)}$ | $\frac{1}{2}$ | $1/-3$ | (5165, 5235) | $\phi,\ K^*$ | $2,\ 0/-2,\ -2$ |
| $\Sigma_c^{(*)}\bar{\Sigma}_c^{(*)}/\Sigma_c^{(*)}\Sigma_c^{(*)}$ | 2 | $0/0$ | (4907, 4972, 5036) | $\rho,\ \omega$ | $-2,\ 2/-2,\ -2$ |
| | 1 | | | | $2,\ 2/2,\ -2$ |
| | 0 | | | | $4,\ 2/4,\ -2$ |
| $\Sigma_c^{(*)}\bar{\Xi}_c^{\prime(*)}/\Sigma_c^{(*)}\Xi_c^{\prime(*)}$ | $\frac{3}{2}$ | $1/-1$ | (5032, 5097, 5100, 5164) | $\rho,\ \omega,\ K^*$ | $-1,\ 1,\ 0/-1,\ -1-2$ |
| | $\frac{1}{2}$ | | | | $2,\ 1,\ 0/2,\ -1,\ -2$ |
| $\Sigma_c^{(*)}\bar{\Omega}_c^{(*)}/\Sigma_c^{(*)}\Omega_c^{(*)}$ | 0 | $2/-2$ | (5149, 5213, 5219, 5284) | – | $0/0$ |
| $\Xi_c^{\prime(*)}\bar{\Xi}_c^{\prime(*)}/\Xi_c^{\prime(*)}\Xi_c^{\prime(*)}$ | 1 | $0/-2$ | (5158, 5225, 5292) | $\rho,\ \omega,\ \phi$ | $-\frac{1}{2},\ \frac{1}{2},\ 1/-\frac{1}{2},\ -\frac{1}{2},\ -1$ |
| | 0 | | | | $\frac{3}{2},\ \frac{1}{2},\ 1/\frac{3}{2},\ -\frac{1}{2},\ -1$ |
| $\Xi_c^{\prime(*)}\bar{\Omega}_c^{(*)}/\Xi_c^{\prime(*)}\Omega_c^{(*)}$ | $\frac{1}{2}$ | $1/-3$ | (5272, 5341, 5345, 5412) | $\phi,\ K^*$ | $2,\ 0/-2,\ -2$ |
| $\Omega_c^{(*)}\bar{\Omega}_c^{(*)}/\Omega_c^{(*)}\Omega_c^{(*)}$ | 0 | $0/-4$ | (5390, 5461, 5532) | $\phi$ | $4/-4$ |

# Interactions from VMD

Attraction: $F > 0$

| System | $I$ | $S$ | Thresholds [MeV] | Exchanged particles | $F$ |
|---|---|---|---|---|---|
| $D^{(*)}\bar{D}_{1,2}/D^{(*)}D_{1,2}$ | 0 | 0/0 | (4289, 4330, 4431, 4472) | $\rho,\ \omega$ | $\frac{3}{2},\ \frac{1}{2}/\frac{3}{2},\ -\frac{1}{2}$ |
| | 1 | 0/0 | | | $-\frac{1}{2},\ \frac{1}{2}/-\frac{1}{2},\ -\frac{1}{2}$ |
| $D^{(*)}\bar{D}_{s1,s2}/D^{(*)}D_{s1,s2}$ | $\frac{1}{2}$ | $1/-1$ | (4390, 4431, 4534, 4575) | – | 0/0 |
| $D_s^{(*)}\bar{D}_{1,2}/D_s^{(*)}D_{1,2}$ | $\frac{1}{2}$ | $-1/1$ | (4402, 4436, 4544, 4578) | – | 0/0 |
| $D_s^{(*)}\bar{D}_{s1,s2}/D_s^{(*)}D_{s1,s2}$ | 0 | $0/-2$ | (4503, 4537, 4647, 4681) | $\phi$ | $1/-1$ |
| $D_{1,2}\bar{D}_{1,2}/D_{1,2}D_{1,2}$ | 0 | 0/0 | (4844, 4885, 4926) | $\rho,\ \omega$ | $\frac{3}{2},\ \frac{1}{2}/\frac{3}{2},\ -\frac{1}{2}$ |
| | 1 | | | | $-\frac{1}{2},\ \frac{1}{2}/-\frac{1}{2},\ -\frac{1}{2}$ |
| $D_{s1,s2}\bar{D}_{1,2}/D_{s1,s2}D_{1,2}$ | $\frac{1}{2}$ | 1/1 | (4957, 4991, 4998, 5032) | | 0/0 |
| $D_{s1,s2}\bar{D}_{s1,s2}/D_{s1,s2}D_{s1,s2}$ | 0 | $0/-2$ | (5070, 5104, 5138) | $\phi$ | 1/1 |
| $\Lambda_c\bar{D}_{1,2}/\Lambda_c D_{1,2}$ | $\frac{1}{2}$ | 0/0 | (4708, 4750) | $\omega$ | $-1/1$ |
| $\Lambda_c\bar{D}_{s1,s2}/\Lambda_c D_{s1,s2}$ | 0 | $-1/1$ | (4822, 4856) | – | 0/0 |
| $\Xi_c\bar{D}_{1,2}/\Xi_c D_{1,2}$ | 1 | $-1/-1$ | (4891, 4932) | $\rho,\ \omega$ | $-\frac{1}{2},\ -\frac{1}{2}/-\frac{1}{2},\ \frac{1}{2}$ |
| | 0 | | | | $\frac{3}{2},\ -\frac{1}{2}/\frac{3}{2},\ \frac{1}{2}$ |
| $\Xi_c\bar{D}_{s1,s2}/\Xi_c D_{s1,s2}$ | $\frac{1}{2}$ | $-2/0$ | (5005, 5039) | $\phi$ | $-1/1$ |
| $\Sigma_c^{(*)}\bar{D}_{1,2}/\Sigma_c^{(*)}D_{1,2}$ | $\frac{3}{2}$ | 0/0 | (4876, 4917, 4940, 4981) | $\rho,\ \omega$ | $-1,\ -1/-1,\ 1$ |
| | $\frac{1}{2}$ | | | | $2,\ -1/2,1$ |
| $\Sigma_c^{(*)}\bar{D}_{s1,s2}/\Sigma_c^{(*)}D_{s1,s2}$ | 1 | $1/-1$ | (4989, 5023, 5053, 5087) | – | 0/0 |
| $\Xi_c'^{(*)}\bar{D}_{1,2}/\Xi_c'^{(*)}D_{1,2}$ | 1 | $-1/-1$ | (5001, 5042, 5068, 5109) | $\rho,\ \omega$ | $-\frac{1}{2},\ -\frac{1}{2}/-\frac{1}{2},\ \frac{1}{2}$ |
| | 0 | | | | $\frac{3}{2},\ -\frac{1}{2}/\frac{3}{2},\ \frac{1}{2}$ |
| $\Xi_c'^{(*)}\bar{D}_{s1,s2}/\Xi_c'^{(*)}D_{s1,s2}$ | $\frac{1}{2}$ | $-2/0$ | (5114, 5148, 5181, 5215) | $\phi$ | $-1/1$ |
| $\Omega_c^{(*)}\bar{D}_{1,2}/\Omega_c^{(*)}D_{1,2}$ | $\frac{1}{2}$ | $-2/-2$ | (5117, 5158, 5188, 5229) | – | 0/0 |
| $\Omega_c^{(*)}\bar{D}_{s1,s2}/\Omega_c^{(*)}D_{s1,s2}$ | 0 | $-3/-1$ | (5230, 5264, 5301, 5335) | $\phi$ | $-2/2$ |

# Weinberg's compositeness relations

- Compositeness for S-wave shallow bound state as derived in Weinberg's paper, $X_W$, expressed in terms of scattering length and effective range

$$a = -\frac{2X_W}{1+X_W}R + O(m_\pi^{-1})$$
$$r = -\frac{1-X_W}{X_W}R + O(m_\pi^{-1})$$

$$R \equiv \frac{1}{\sqrt{2\mu|E_B|}}$$

Binding energy

- Effective coupling: $$g^2 = \frac{8\pi^2}{\mu^2 R}X_W$$

- Applied to the deuteron case

($E_B = -2.22$ MeV, $R = 4.31$ fm, $a = -5.42$ fm, $r = 1.77$ fm), $X_W = 1.68 > 1$

- Assumptions used in the derivations
  - ☐ Neglecting the non-pole term from the Low equation
  - ☐ Approximating the form factor by a constant

$$T_{p,k} = V_{p,k} + \frac{g(p)\,g^*(k)}{h_k - E_B} + \int_0^\infty \frac{q^2dq}{(2\pi)^3}\frac{T_{p,q}T^*_{k,q}}{h_k + i\varepsilon - h_q} \qquad \text{w/ } h_k \equiv k^2/(2\mu)$$

Question: for ERE up to $\mathcal{O}(p^2)$, is a constant $g(p)$ a consistent approximation?
Inconsistency already pointed out in I. Matuschek et al., EPJA57, 101 (2021); see Christoph's talk

# Generalization

Y. Li, FKG, J.-Y. Pang, J.-J. Wu, arXiv:2110.02766

- The constant form factor assumption can be replaced by a more general separable ansatz

$$T_{p,k} = t_k\, g(p)\, g^*(k)$$

Twice-subtracted dispersion relation ⇒

$$t^{-1}(W) = (W - E_B) + (W - E_B)^2 \int_0^\infty \frac{q^2 dq}{(2\pi)^3} \frac{|g(q)|^2}{(h_q - E_B)^2 (h_q - W)}$$

Then, we get

$$t(W) = \frac{1}{1 - F(W)} \frac{1}{W - E_B}, \quad F(W) \equiv (W - E_B) \int_0^\infty \frac{q^2 dq}{(2\pi)^3} \frac{|g(q)|^2}{(h_q - E_B)^2 (W - h_q)}$$

- Compositeness emerges

$$F(\infty) = \int_0^\infty \frac{q^2 dq}{(2\pi)^3} \frac{|\langle q|\hat{V}|B\rangle|^2}{(h_q - E_B)^2} = \int_0^\infty \frac{q^2 dq}{(2\pi)^3} |\langle q|B\rangle|^2 = X$$

- Introducing

$$F_1(W) \equiv \frac{\ln[1 - F(W)]}{W - E_B}, \qquad \mathrm{Im}\, F_1(E + i\varepsilon) = -\frac{\delta_B(E)}{E - E_B}\theta(E)$$

here $\delta_B$ is the phase of the $T$-matrix with the nonpole term neglected (convention: $\delta_B(0) = 0$)

$$\delta_B(E = h_p) \equiv \arg T_{p,p} = -\arg\left(1 - F(E + i\varepsilon)\right) \qquad \delta_B \in [-\pi, 0]$$

↑

$F(0) \leq 0$, $\mathrm{Im}\, F(E + i\varepsilon) \leq 0$ for $E \geq 0$

# Generalization

Y. Li, FKG, J.-Y. Pang, J.-J. Wu, arXiv:2110.02766

- From the dispersion relation for $F_1(W)$, we obtain a solution:

$$F(W) = 1 - \exp\left(\frac{W - E_B}{\pi}\int_0^\infty dE \frac{-\delta_B(E)}{(E-W)(E-E_B)}\right)$$

and an expression for the compositeness

$$X = 1 - \exp\left(\frac{1}{\pi}\int_0^\infty dE \frac{\delta_B(E)}{E-E_B}\right) \in [0,1]$$

- Using $\mathrm{Im}\, F(h_p + i\epsilon) = -\frac{\pi p \mu}{(2\pi)^3}\frac{|g(p)|^2}{h_p - E_B}$, we get

$$|g(p)|^2 = -\frac{(2\pi)^3}{\pi\, p\mu}(h_p - E_B)\sin\delta_B(E)\exp\left[\frac{h_p - E_B}{\pi}\fint_0^\infty dE \frac{-\delta_B(E)}{(E-h_p)(E-E_B)}\right]$$

- Consider ERE $p\cot\delta_B \approx -\frac{8\pi^2}{\mu}\mathrm{Re}T^{-1}(h_p) = \frac{1}{a} + \frac{r}{2}p^2 + \mathcal{O}(p^4)$ , we finally get

$$g^2(p) = \frac{8\pi^2}{\mu^2 R} \times \begin{cases} X_W + \mathcal{O}(p^4) & \text{for } a \in [-R, 0]\ \&\ r \le 0 \\ \frac{a^2}{R^2}\frac{1}{1+(a+R)^2p^2} + \mathcal{O}(p^4) & \text{for } a < -R\ \&\ r > 0 \end{cases}$$

constant

contains $\mathcal{O}(p^2)$ terms, thus not self-consistent if using a constant $g^2$ but still work up to $\mathcal{O}(p^2)$ in ERE. Weinberg's relations do not hold in this case

# Compositeness

Y. Li, FKG, J.-Y. Pang, J.-J. Wu, arXiv:2110.02766

- Poles of the $T$-matrix with ERE up to $\mathcal{O}(p^2)$: $\frac{1}{a} + \frac{r}{2}p^2 - i\,p = \frac{r}{2}(p - p_+)(p - p_-)$

$$p_- = \frac{i}{R}, \qquad p_+ = -\frac{i}{R+a}$$

with $R = \frac{1}{\sqrt{2\mu|E_B|}}$; $r$ is expressed as $r = \frac{2R}{a}(R+a)$

- For $a \in [-R, 0]$, then $r < 0$, one bound state and one virtual state pole

$$g^2(p) = \frac{8\pi^2}{\mu^2 R} X_W + \mathcal{O}(p^4), \qquad X = X_W \simeq \sqrt{\frac{1}{1+2r/a}}$$

- For $a < -R$, then $r > 0$, two bound state poles (the remote one $\sim i/\beta$ is unphysical)

$$g^2(p) = \frac{8\pi^2}{\mu^2 R}\frac{a^2}{R^2}\frac{1}{1+(a+R)^2 p^2} + \mathcal{O}(p^4), \qquad X \simeq 1 - e^{-\infty} = 1$$

For the deuteron, $R = 4.31$ fm, $a = -5.42$ fm, $a + R \sim \beta^{-1} \sim m_\pi^{-1}$

$$X = 1 - \exp\left(\frac{1}{\pi}\int_0^\infty dE\,\frac{\delta_B(E)}{E - E_B}\right) \in [0,1]$$

$$p\cot\delta_B = \frac{1}{a} + \frac{r}{2}p^2 \Rightarrow \delta_B(\infty) = 0 \text{ for } r < 0, \text{ and } \delta_B(\infty) = -\pi \text{ for } r > 0$$

- For extension of the Weinberg's relations to virtual state and near-threshold resonances, see Matuschek et al., EPJA57, 101 (2021); Christoph's talk
New extensions: Song, Dai, Oset, 2201.04414; Albaladejo, Nieves, 22203.04864

# Uncertainty of the relation

Y. Li, FKG, J.-Y. Pang, J.-J. Wu, arXiv:2110.02766

- The uncertainty was usually assumed to be $\mathcal{O}\left(\frac{\gamma}{\beta}\right)$, with $\gamma = \sqrt{2\mu|E_B|}$ the binding momentum. This comes from approximating the form factor by a constant $g(p^2) = 1 + \frac{p^2}{\Lambda^2} + \cdots$, $\Lambda \sim \beta$

$$\Delta X = \frac{1}{\Lambda^2} \int_0^{\Lambda} \frac{q^2 dq}{(2\pi)^3} \frac{q^2}{\left(h_q - E_B\right)^2} = \mathcal{O}\left(\frac{\gamma}{\Lambda}\right)$$

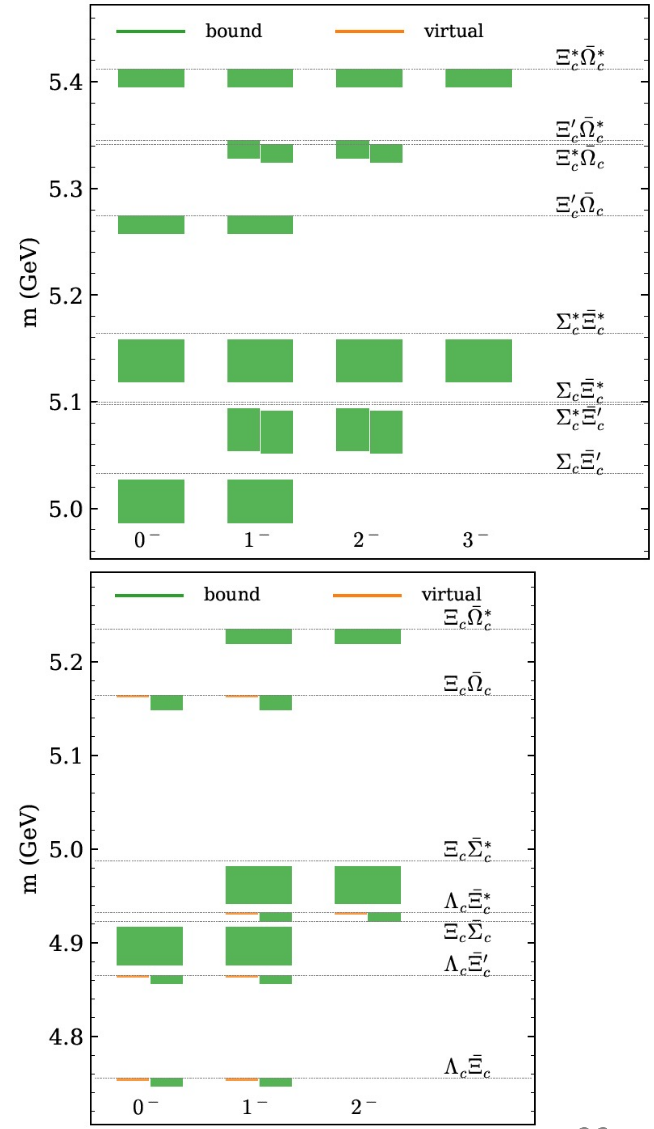
- Now this approximation has been lifted, then the uncertainty should be of $\mathcal{O}\left(\frac{\gamma^2}{\beta^2}\right)$!

I. Matuschek et al., EPJA57, 101 (2021); Christoph's talk

# More states with exotic quantum numbers

$(I,S) = (1,0)$

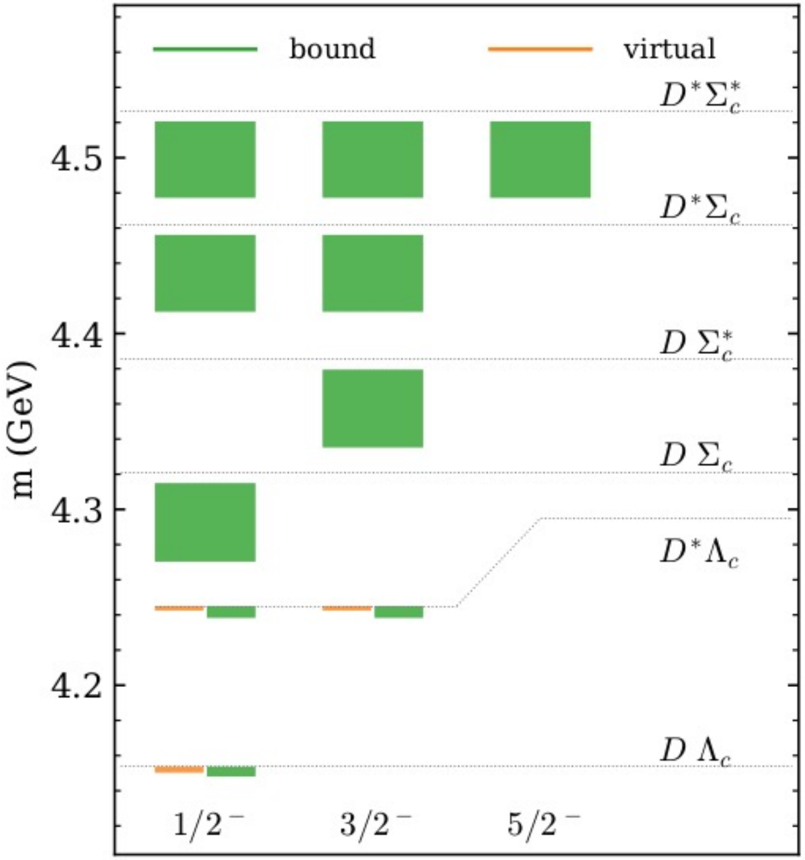

$(I,S) = (1/2,1)$



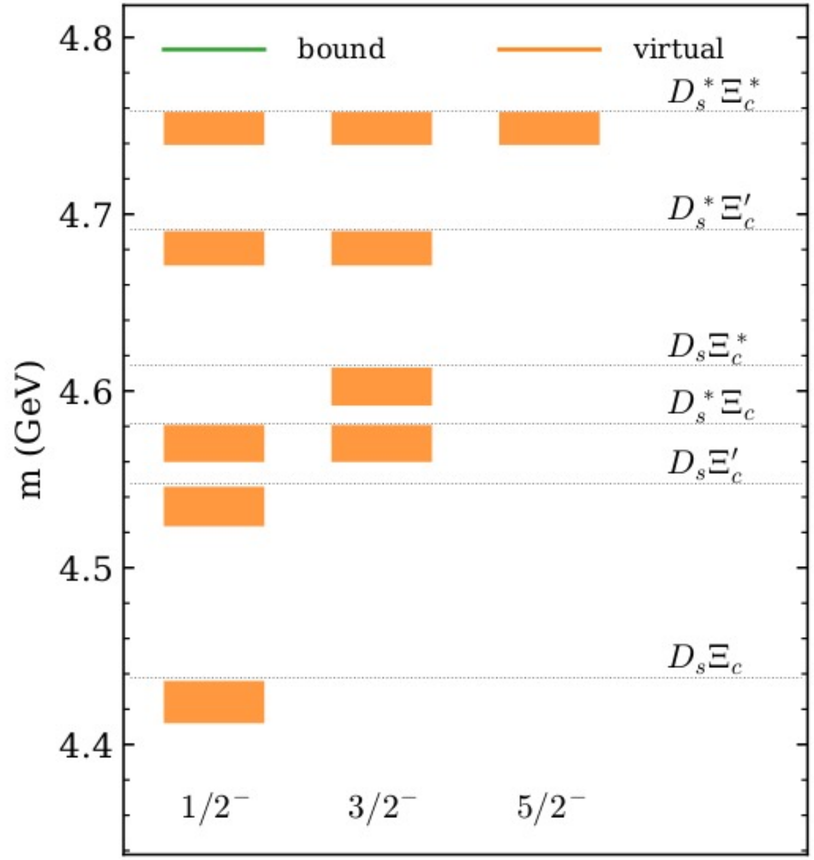

✓ Many baryon-antibaryon molecular states above 4.7 GeV, beyond the current exp. region

# Double-charm

$(I, S, B) = (1/2, -1, 1)$




$(I, S, B) = (0, 0, 1)$




X.-K. Dong, FKG, B.-S. Zou, arXiv:2108.02673

- ✓ The attractions for $D^{(*)}\Sigma_c^{(*)}$ are stronger than those for $\bar{D}^{(*)}\Sigma_c^{(*)}$
- ✓ However, the $D^{(*)}\Sigma_c^{(*)}$ states mix with normal double-charm baryons