Endpoint singularities in μe backward scattering

(and the soft-overlap form factor for exclusive B decays)

Philipp Böer

based on: 2205.06021 with G. Bell and T. Feldmann, and PB PhD thesis 2018

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JOHANNES GUTENBERG UNIVERSITÄT MAINZ

Outline

1. Introduction
2. Endpoint singularities in muon-electron backward scattering
3. Soft overlap form factor for non-relativistic $B_c \to \eta_c$ transitions
4. Conclusion

1. Introduction

Introduction

Aim of soft-collinear factorization:

1. combined expansion in $\mu_{\text{low}}/\mu_{\text{high}}$ and α_s
   $\rightarrow$ requires rigorous power-counting scheme

2. separate dynamics related to momentum regions with different virtualities and/or rapidities
   $\rightarrow$ separation of perturbative and non-perturbative dynamics in hadronic processes

3. employ Renormalization Group Equations to sum large logarithms to all orders

$$\alpha_s^n \ln^m \left(\frac{\mu_{\text{low}}}{\mu_{\text{high}}} \right)$$

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Problem: standard procedure may lead to ill-defined convolution integrals

- they may only converge in $d = 4 - 2\varepsilon$ dimensions (SCET_{I})
  $\rightarrow$ standard renormalization program breaks down

- dim.-reg. insufficient to separate modes with equal virtuality but different rapidity (SCET_{II})
  $\rightarrow$ analytic regulators violate naive decoupling ("collinear anomaly") [Becher/Neubert]

- often related to soft fermions ($\rightarrow$ n.l.p.!)

Incomplete List of Examples

- off-diagonal channels in DIS [Beneke et al '20]
- bottom induced $h \to \gamma\gamma$ decay [Neubert et al. '19/20]
- off-diagonal gluon thrust [Beneke et al. '22]
- ...

In B-meson decays:

- power-corrections in $B \to h_1 h_2$ decays (e.g. weak annihilation) [BBNS '99/00]
- power-corrections in radiative $B \to \gamma\ell\nu$ decays [e.g. Beneke,Rohrwild '11]
- heavy-to-light form factors [Beneke,Feldmann '00]
- certain QED corrections in $B_s \to \mu^+\mu^-$ [Beneke,Bobeth,Szafron '19]
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Problem seems to arise generically in SCET at subleading power! A better understanding would constitute a major step in controlling the $1/E$ expansion.

$\Lambda_{\mathrm{QCD}}/m_B \gtrsim 0.1$ is not extremely small. Very relevant in B physics!

2. muon-electron backward scattering: a prime example for endpoint singularities

based on arXiv:2205.06021 with G. Bell and T. Feldmann

Why "prime example"?

Clean framework to study physics of endpoint singularities, because ...

- ✓ perturbative QED process from the textbook
- ✓ resummed double logarithms known for $>$ 50 years [Gorshkov et al. 1966]
- ✓ they arise from single scalar Integral at each order in $\alpha_{\rm em}$ ($\rightarrow$ playground for method-of-regions)
- ✓ bare factorization theorem can be reduced to a single term at the DL level
- ✓ most general structure of endpoint singularities, already for DLs at leading power!
  (more complicated than e.g. $h \to \gamma\gamma$, gluon thrust)
- ✓ mimics structure of endpoint singularities in exclusive B decays (& other hard-exclusive processes)

- :(phenomenologically not the most relevant process
- :(subleading logarithms way more complicated and unknown

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Backward Scattering

$$\boxed{2 \to 2 \text{ process:} \quad e^-(p)\mu^-(\bar{p}) \to e^-(\bar{p})\mu^-(p) \qquad \text{at} \qquad s \approx -t \gg m_\mu^2 \sim m_e^2 \gg u}$$

- consider common mass $m_\mu \simeq m_e \to m$ for simplicity (but distinguishable flavours)
- expansion parameter: $\lambda = m/\sqrt{s}$, and light-cone vectors

$$p^\mu = \frac{\sqrt{s}}{2} n_-^\mu + \frac{m^2}{2\sqrt{s}} n_+^\mu \qquad \bar{p}^\mu = \frac{\sqrt{s}}{2} n_+^\mu + \frac{m^2}{2\sqrt{s}} n_-^\mu$$

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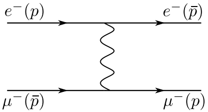
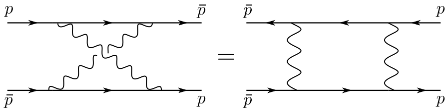




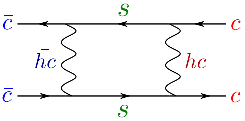
- high-energy limit: $\qquad \mathcal{M} = F_1(\lambda)\mathcal{M}^{(0)} + F_2(\lambda)\widetilde{\mathcal{M}}$

  $\to$ leading DLs in form factor $F_1(\lambda)$ that multiplies tree amplitude $\mathcal{M}^{(0)} \sim \alpha_{\rm em}$

- DL at NLO from twisted box: $\qquad F_1(\lambda) = 1 + \frac{\alpha_{\rm em}}{2\pi} \frac{1}{2} \ln^2 \lambda^2 + \dots$

Isolating the Double-Log at NLO

DLs arise from the kinem. configuration in which the virtual lepton propagators are soft, $k^\mu \sim \lambda$:



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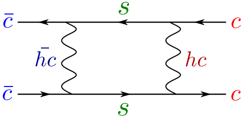

After some Dirac algebra, it is then easy to show that the DL is contained in the scalar integral

$$F_1^{(1)}(\lambda) \sim \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - m^2} \frac{1}{(k-p)^2} \frac{1}{(k-\bar{p})^2}$$

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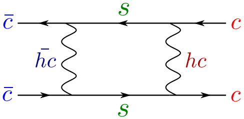

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$$(k-p)^2 + i0 \simeq -\sqrt{s}(n_- k) + i0\,, \qquad (k-\bar{p})^2 + i0 \simeq -\sqrt{s}(n_+ k) + i0$$

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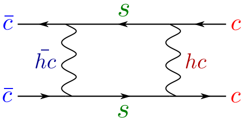

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Non-vanishing contribution from discontinuity of soft lepton propagator:

$$\int \frac{dk_\perp^2}{k^2 - m^2 + i0} \to -2\pi i \theta((n_+ k)(n_- k) - m^2)$$

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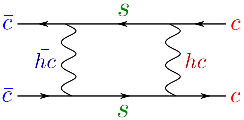

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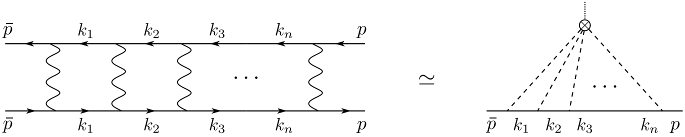
Traditional approach: put hard cut-offs on longitudinal momenta $(n_\pm k) \leq \sqrt{s}$

$$F_1^{(1)}(\lambda) \simeq \int_{\lambda^2}^1 \frac{dx}{x} \int_{\lambda^2/x}^1 \frac{dy}{y} = \frac{1}{2} \ln^2 \lambda^2 \quad \checkmark \qquad (n_+ k = x\sqrt{s}, n_- k = y\sqrt{s})$$

Isolating the Double-Log's at Higher Orders



all photon propagators eikonal: $\quad (k_i - k_{i-1})^2 + i0 \simeq -(n_+k_i)(n_-k_{i-1}) + i0$

strongly ordered longitudinal lepton momenta:

$$\frac{m^2}{\sqrt{s}} \approx n_+\bar{p} \ll n_+k_1 \ll \cdots \ll n_+k_n \ll n_+p \approx \sqrt{s}$$

$$\sqrt{s} \approx n_-\bar{p} \gg n_-k_1 \gg \cdots \gg n_-k_n \gg n_+p \approx \frac{m^2}{\sqrt{s}}$$

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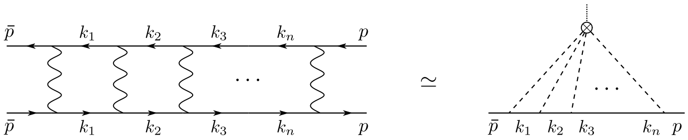

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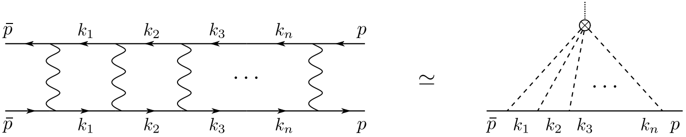
$$\sqrt{s} \approx n_-\bar{p} \gg n_-k_1 \gg \cdots \gg n_-k_n \gg n_+p \approx \frac{m^2}{\sqrt{s}}$$

yields nested integrals:

$$F_1^{(n)}(\lambda) \simeq \int_{\lambda^2}^1 \frac{dx_1}{x_1} \int_{x_1}^1 \frac{dx_2}{x_2} \cdots \int_{x_{n-1}}^1 \frac{dx_n}{x_n} \int_{\lambda^2/x_1}^1 \frac{dy_1}{y_1} \int_{\lambda^2/x_2}^{y_1} \frac{dy_2}{y_2} \cdots \int_{\lambda^2/x_n}^{y_{n-1}} \frac{dy_n}{y_n} = \frac{\ln^{2n}\lambda^2}{n!(n+1)!}$$

Isolating the Double-Log's at Higher Orders



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that sum up to modified Bessel function:

$$\boxed{F_1(\lambda) \simeq \sum_{n=0}^{\infty} \left(\frac{\alpha_{\rm em}}{2\pi}\right)^n F_1^{(n)}(\lambda) = \frac{I_1(2\sqrt{z})}{\sqrt{z}}\,, \qquad \text{with} \qquad z = \frac{\alpha_{\rm em}}{2\pi}\ln^2\lambda^2}$$

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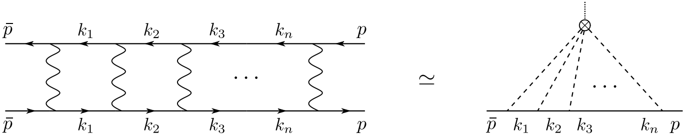

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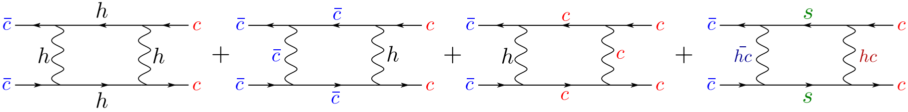
However, . . .

- only the leading double-logarithms identified in this way
- scale of running coupling undetermined
- factorize non-pert. physics from short-distance dynamics (in hadronic processes)

Goal: Formulate problem in SCET in terms of a renormalized factorization theorem!

. . . work in progress! Highly non-trivial endpoint singular convolutions!

Method-of-Regions Analysis



$$\mathcal{I}^{(\text{hard})} = \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\ln\frac{\mu^2}{s} + \frac{1}{2}\ln^2\frac{\mu^2}{s} - \frac{\pi^2}{12} + \mathcal{O}(\varepsilon)$$

- contains Sudakov-type double-logarithms involving the hard scale $\mu/\sqrt{s}$

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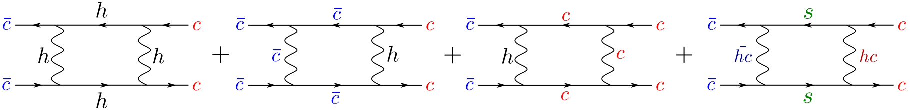

$$\mathcal{I}^{(c)} = e^{\varepsilon\gamma_E}\Gamma(\varepsilon)\left(\frac{\mu^2}{m^2}\right)^{\varepsilon}\int_0^1 \frac{dx}{x}(1-x)^{-2\varepsilon}\left(\frac{\nu}{x\sqrt{s}}\right)^{\alpha} = -\left(\frac{1}{\alpha}+\ln\frac{\nu}{\sqrt{s}}\right)\left(\frac{1}{\varepsilon}+\ln\frac{\mu^2}{m^2}\right)+\frac{\pi^2}{3}+\mathcal{O}(\alpha,\varepsilon)$$

- standard UV singularity $\Gamma(\varepsilon)$ from $k_\perp \to \infty$
- Endpoint-singularity for $n_+k = x\sqrt{s} \to 0$
  - $\to$ ill-defined in dim.-reg. due to lepton mass $m \neq 0$
  - $\to$ rapidity divergence! Fermion propagator overlaps between low-energy regions
  - $\to$ No IR-singularity in the conventional sense (no mode below $\mu \sim m$)
- requires additional (analytic) rapidity regulator (e.g. [Becher/Bell,Ebert et al.,Chiu et al.,Neill et al.,. . .])
  - $\to$ here: $(\nu/2k_0)^\alpha$ preserves symmetry, so $\mathcal{I}^{(c)} = \mathcal{I}^{(\bar{c})}$
  - $\to$ small virtuality $\mu \sim m$, large energy $\nu \sim \sqrt{s}$

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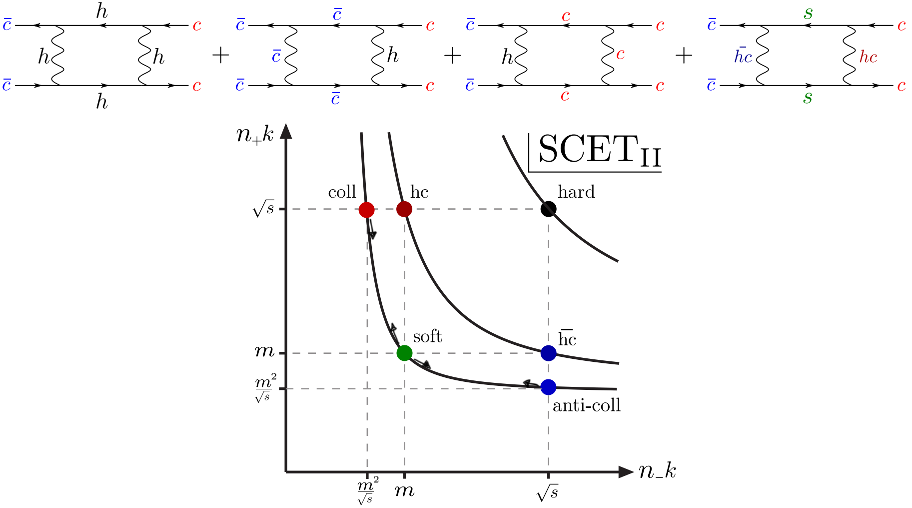

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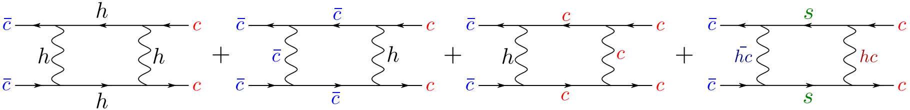

$$\mathcal{I}^{(s)} = 2\left(\frac{1}{\alpha} + \ln\frac{\nu}{m}\right)\left(\frac{1}{\varepsilon} + \ln\frac{\mu^2}{m^2}\right) - \frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\ln\frac{\mu^2}{m^2} - \frac{1}{2}\ln\frac{\mu^2}{m^2} + \frac{\pi^2}{12} + \mathcal{O}(\alpha,\varepsilon)$$

- again ill-defined in dim.-reg.
  - $\rightarrow$ $1/\alpha$ singularity from both limits $n_+k \to \infty$ and $n_-k \to \infty$

- symmetric regulator remains unexpanded: $2k_0 = n_+k + n_-k = (x+y)\sqrt{s}$
  - $\rightarrow$ small virtuality $\mu \sim m$, small energy $\nu \sim m$
  - $\rightarrow$ can be made scaleless by choosing an asymmetric regulator, e.g. $(\nu/n_+k)^\alpha$

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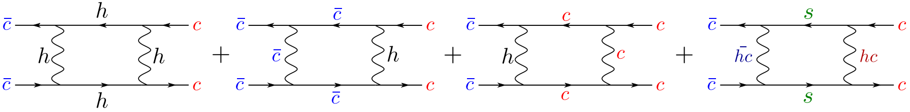


- Sum of regions:

$$\mathcal{I}^{(\text{hard})} + \mathcal{I}^{(c)} + \mathcal{I}^{(\bar{c})} + \mathcal{I}^{(s)} = \frac{1}{2}\ln^2\lambda^2 + \frac{2\pi^2}{3}$$

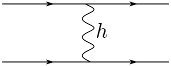
  ✓ dimensional and analytic regulator drop out
  ✓ leading DL recovered

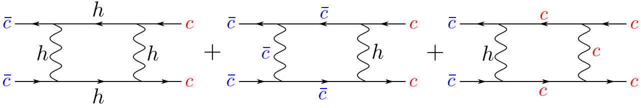
- remaining one-loop graphs standard
  $\rightarrow$ no endpoint-singularity, no analytic regulator, DL cancels

Formulation in SCET

Two-step matching: QED $\stackrel{s\to\infty}{\to}$ SCET$_{\mathrm{I}}$ $\stackrel{\sqrt{s}m\to\infty}{\to}$ SCET$_{\mathrm{II}}$:

(boring diagrams not shown)





Schematic form of bare factorization formula

$$F_1(\lambda) \simeq f_{\bar{c}} \otimes H \otimes f_c$$

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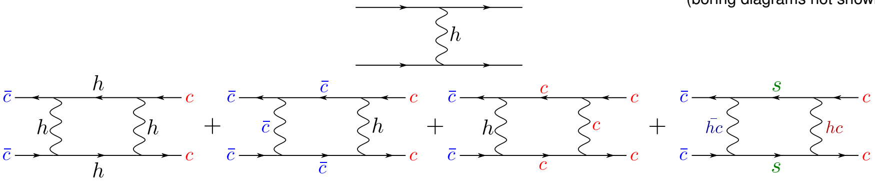

Schematic form of bare factorization formula

$$F_1(\lambda) \simeq f_{\bar{c}} \otimes H \otimes f_c \quad + \quad f_{\bar{c}} \otimes J_{\bar{hc}} \otimes S \otimes J_{hc} \otimes f_c$$

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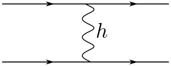

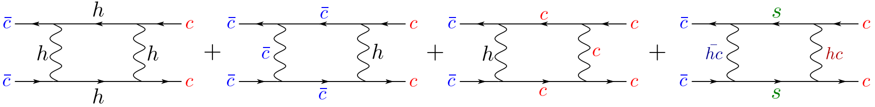

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- soft contribution leading power due to specific *soft-enhancement* mechanism ($\to$ backup)
- ✓ individual bare soft and coll. fct's defined as SCET operator matrix elements, e.g.

$$\langle \mu^-(p)|\, \bar{\chi}_c^{(\mu)}(\tau n_+)\frac{\not{n}_+}{2}P_{R(L)}\chi_c^{(e)}(0)\,|e^-(p)\rangle = \int dx e^{ix\tau n_+ p}\left\{ f_c(x)[\bar{u}_\xi^{(\mu)}\frac{\not{n}_+}{2}P_{R(L)}u_\xi^{(e)}] + \tilde{f}_c(x)[\bar{u}_\xi^{(\mu)}\frac{\not{n}_+}{2}P_{L(R)}u_\xi^{(e)}]\right\}$$

$\to$ generalized parton distributions (forward, but flavour-non-diagonal)

$\to$ helicity-flipping functions $\tilde{f}_c(x)$ and $\tilde{f}_{\bar{c}}(y)$ do not contribute to leading DLs

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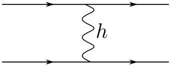

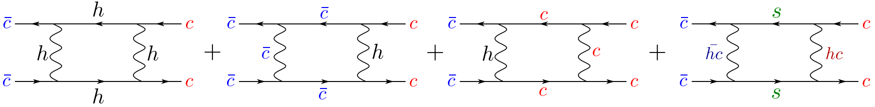

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- At one-loop level

$$f_c(x) \simeq \delta(1-x) + \frac{\alpha_{\mathrm{em}}}{2\pi}\theta(x)\theta(1-x)\left(\frac{\mu^2}{m^2}\right)^{\varepsilon}\Gamma(\varepsilon)\Big(1+\mathcal{O}(x)\Big)$$

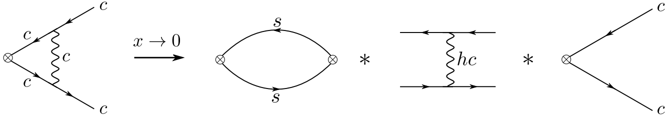
$\rightarrow$ convolution integrals $\int \frac{dx}{x} f_c(x)$ require rapidity regulator!

?? How to renormalize functions *before* performing the convolutions ??

Endpoint-Refactorization

Q: Can we understand the $x \to 0$ asymptotics of the bare functions $f_c(x)$ to all orders in α_{em}?
Can we isolate and subtract the divergences?

Recall: Rapidity diverences arise from the soft limit of the coll. fermion propagators!
$\to$ interpret $f_c(x)$ for $x \to 0$ as multi-scale object



$$f_c(x \to 0) \simeq \int \frac{dx'}{x'} f_c(x') \int \frac{d\rho}{\rho} J_{hc}(\rho x') S(\rho, x)$$

[PB '18]

✓ reflects structure of the second term $f \otimes J \otimes S \otimes J \otimes f$ as $1/\alpha$ poles must cancel

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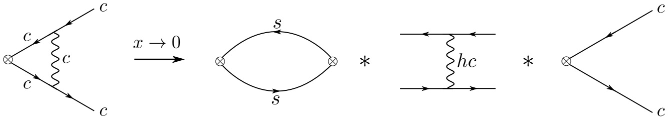

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Implications: (*before* expansion in ε!)

1.) collinear functions receive *positive* powers of x^ε from J_{hc} $\Rightarrow$ $\langle x^{-1-n\varepsilon} \rangle_{f_c} \sim 1/\alpha, \ \forall n$
2.) $1/\alpha$ cancel within $f_c(x \to 0)$ and generate powers of $\ln x$ $\Rightarrow$ higher powers in $1/\alpha$
3.) peculiar structure as $f_c(x')$ arises on the RHS $\Rightarrow$ non-additive problem

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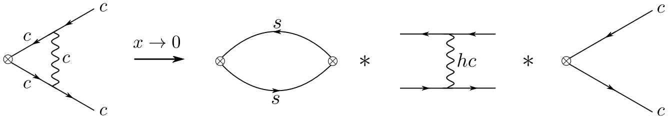

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Example: at four-loop $f_c(x)$ has the following asymptotic structure:

$$f_c^{(4)}(x \to 0) \simeq \left(\frac{\mu^2}{m^2}\right)^{4\varepsilon} \left\{ \frac{x^{3\varepsilon} - 15x^{2\varepsilon} + 339x^{\varepsilon} - 325}{144\varepsilon^7} - \frac{(3x^{\varepsilon} + 23)\ln x}{12\varepsilon^6} - \frac{3\ln^2 x}{4\varepsilon^5} - \frac{\ln^3 x}{6\varepsilon^4} \right\}$$

Resummation from Consistency Relations

Despite the complexity of the problem, the DL series is completely determined by

(i) scale separation (of bare quantities)

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1. Use asymmetric regulator that makes soft contribution scaleless: (clear scale separation ✓)

$$F_1(\lambda) \simeq \int_0^1 \frac{dx}{x} f_c\left(x; \frac{\mu}{m}, \frac{\nu}{\sqrt{s}}\right) \int_0^1 \frac{dy}{y} f_{\bar{c}}\left(y; \frac{\mu}{m}, \frac{\nu\sqrt{s}}{m^2}\right) H\left(\frac{\mu^2}{xys}\right)$$

$\rightarrow$ single term that involves only leading-twist projections

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$\rightarrow$ single term that involves only leading-twist projections

2. Insert perturbative expansion of hard function at double-log level:

$$H\left(\frac{\mu^2}{xys}\right) \simeq \sum_{n=0}^{\infty} z_h^n h^{(n)} (xy)^{-n\varepsilon} \qquad \text{with} \qquad z_h = \frac{\alpha_{\rm em}}{2\pi} \frac{1}{\varepsilon^2} \left(\frac{\mu^2}{s}\right)^{\varepsilon}$$

Form factor expressed as infinite sum of products of divergent moments:

$$F_1(\lambda) = \sum_{n=0}^{\infty} z_h^n h^{(n)} \langle x^{-1-n\varepsilon} \rangle_{f_c} \left(\frac{\mu}{m}, \frac{\nu}{\sqrt{s}}\right) \langle y^{-1-n\varepsilon} \rangle_{f_{\bar{c}}} \left(\frac{\mu}{m}, \frac{\nu\sqrt{s}}{m^2}\right)$$

Resummation from Consistency Relations

3. Rapidity poles must cancel at each order in the hard-matching
   - $\rightarrow$ Collinear Anomaly: large rapidity log's exponentiate in products [Becher,Bell,Neubert '11]
   - $\rightarrow$ F_1 expressed as infinite sum of anomaly exponents $\mathcal{F}_n$ and "remainder functions" r_n

$$F_1(\lambda) = \sum_{n=0}^{\infty} z_h^n h^{(n)} \, r_n(\mu/m) \cdot \left(\frac{m^2}{s}\right)^{\mathcal{F}_n(\mu/m)}$$

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$$r_n(\mu/m) = \sum_{k=0}^{\infty} \left(\frac{\alpha_{\rm em}}{2\pi}\right)^k \left(\frac{\mu^2}{m^2}\right)^{k\varepsilon} \frac{r_n^{(k)}}{\varepsilon^{2k}}\,, \qquad \mathcal{F}_n(\mu/m) = \sum_{l=n+1}^{\infty} \left(\frac{\alpha_{\rm em}}{2\pi}\right)^l \left(\frac{\mu^2}{m^2}\right)^{l\varepsilon} \frac{\mathcal{F}_n^{(l)}}{\varepsilon^{2l-1}}$$

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5. Form factor finite for $\varepsilon \to 0$ gives consistency relations between $(h^{(n)}, r_n^{(k)}, \mathcal{F}_n^{(k)})$
   - ✓ reproduce known result order-by-order:

$$F_1(\lambda) \simeq \frac{I_1\left(2\sqrt{h^{(1)} z}\right)}{\sqrt{h^{(1)} z}}$$

   - ✓ single unknown coefficient $h^{(1)} = 1$ determined from one-loop calculation
   - **!** need infinite perturbative series of anomaly exponents

Comparison to $h \to \gamma\gamma$

At first sight, the two processes seem to be very similar at the technical level:
SCET$_{\mathrm{II}}$, same modes, massive fermion propagators, analytic regulators, . . . **but**:

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- endpoint-div. cancel in products of inv. moments $\to$ exponentiation of rapidity poles
- iterative refactorization condition: $f_c(x \to 0) \sim f_c \otimes J_{hc} \otimes S$
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$h \to \gamma\gamma$ bare factorization theorem can be written as

$$H_1 \cdot \langle\gamma\gamma|\, O_1 \,|h\rangle + H_2 \otimes \langle\gamma\gamma|\, O_2 \,|h\rangle + H_2 \otimes \langle\gamma\gamma|\, \bar{O}_2 \,|h\rangle + H_3 \cdot J_{\bar{hc}} \otimes S \otimes J_{hc}$$

- endpoint-div. cancel in sum of inv. moments $\to$ linear rapidity pole to all orders
- refactorization condition takes simpler form: $\langle\gamma\gamma|\, O_2 \,|h\rangle\big|_{x\to 0} \sim J_{hc} \otimes S$
- soft function vanishes for zero argument

Reason: Collinear and soft function in $h \to \gamma\gamma$ both helicity suppressed in m_b/m_H.
But this is *not* the case in the $2 \to 2$ scattering process $e^-\mu^- \to e^-\mu^-$.

Summary: Muon-Electron Backward Scattering

- ✓ simple $2 \to 2$ textbook process in QED
- ✓ leading log's resum to modified Bessel function (known for > 50 years)
- ✓ Bessel function in SCET recovered by iterative pattern of endpoint singularities
  - $\to$ infinite tower of collinear-anomaly exponents
  - $\to$ leading-power DLs already more complicated than other examples in the literature

  1.) "scale-separation"
  2.) consistency (pole-cancellation, "collinear anomaly")
  3.) re-factorization

- So far we did not derive a renormalized factorization theorem
  - $\to$ need to figure out whether rearrangement (in spirit of $h \to \gamma\gamma$) can be generalized

- **next**: mimics the endpoint structure in exclusive B_c decays, but in a much simpler setup

3. The soft-overlap form factor in non-relativistic $B_c \to \eta_c$ transitions

based on: PB PhD thesis 2018

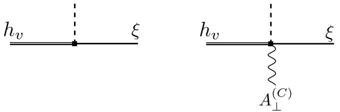
The Soft-Overlap Form Factor ξ_π

Form factor = non-perturbative input in exclusive semi-leptonic B decays, e.g. $B \to \pi \ell \nu$:

$$\langle \pi(p) | \, \bar{q} \gamma^\mu b \, | B(p_B) \rangle = F_+(q^2)(p_B^\mu + p^\mu) + F_-(q^2) q^\mu$$

At large pion energies (small q^2) $\to$ use SCET to factorize **hard**, *hc*, *coll.*, *soft*

- $\mathbf{m_b} \to \infty$: two $\mathrm{SCET_I}$ operators



$$J_A = \bar{\chi}_{hc} h_v \,, \qquad J_B = \bar{\chi}_{hc} \not{A}_\perp h_v$$

- $\sqrt{m_b \Lambda_{\mathrm{QCD}}} \to \infty$: J_B factorizes into convergent convolutions of LCDAs $\phi_B^+(\omega)$ and $\phi_\pi(u)$ ✓
  J_A does not factorize due to endpoint-divergent convolutions

However, A-type contribution spin-symmetry preserving: [Beneke,Feldmann '00]

$$F_i(q^2) = H_i(q^2, \mu) \cdot \xi_\pi(q^2, \mu) \quad + \quad (\text{factorizable})_i$$

with the soft-overlap form factor ξ_π defined as a $\mathrm{SCET_I}$ hadronic matrix element

$$2 E_\pi \xi_\pi = \langle \pi(p) | \, \bar{\chi}_{hc} h_v \, | \bar{B}_c \rangle$$

ξ_π: Tree-Level Matching

- SCET$_{\text{II}}$ operator basis: see [Lange,Neubert '03] for massless quarks

$$\mathcal{O}_1 = \left[\bar{\chi}(0)\frac{\not{\bar{n}}}{2}\gamma_5\chi(s\bar{n})\right]\left[\bar{\mathcal{Q}}_s(\tau n)\frac{\not{\bar{n}}\not{n}}{4}\gamma_5\mathcal{H}_v(0)\right] \longrightarrow \phi_B^- \, \phi_\pi$$

$$\mathcal{O}_2 = \left[\bar{\chi}(0)\frac{\not{\bar{n}}}{2}\gamma_5 i\not{\partial}_\perp\chi(s\bar{n})\right]\left[\bar{\mathcal{Q}}_s(\tau n)\frac{\not{n}}{2}\gamma_5\mathcal{H}_v(0)\right] \longrightarrow \phi_B^+ \, \{\phi_p, \phi_\sigma, \phi_3\}$$

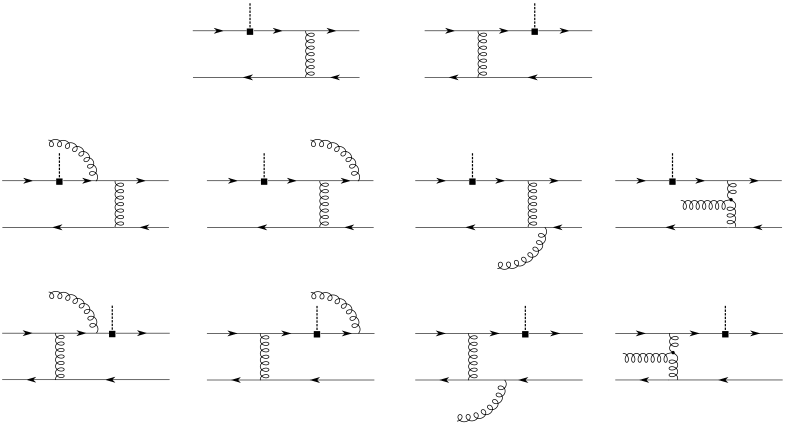
$$\mathcal{O}_3 = \left[\bar{\chi}(0)\frac{\not{\bar{n}}}{2}\gamma_5\mathcal{A}_{c,\perp}(r\bar{n})\chi(s\bar{n})\right]\left[\bar{\mathcal{Q}}_s(\tau n)\frac{\not{n}}{2}\gamma_5\mathcal{H}_v(0)\right] \longrightarrow \phi_B^+ \, \phi_3$$

$$\mathcal{O}_4 = \left[\bar{\chi}(0)\frac{\not{\bar{n}}}{2}\gamma_5\chi(s\bar{n})\right]\left[\bar{\mathcal{Q}}_s(\tau n)\mathcal{A}_{s,\perp}(\sigma n)\frac{\not{n}}{2}\gamma_5\mathcal{H}_v(0)\right] \longrightarrow \psi_{A-V} \, \phi_\pi$$

$$\mathcal{O}_m = \left[\bar{\chi}(0)\frac{\not{\bar{n}}}{2}\gamma_5\chi(s\bar{n})\right]\left[\bar{\mathcal{Q}}_s(\tau n)\frac{\not{n}}{2}\gamma_5\mathcal{H}_v(0)\right] \longrightarrow m \, \phi_B^+ \, \phi_\pi$$

- The operator $\mathcal{O}_m$ contributes only for non-vanishing light-quark masses

ξ_π: Tree-Level Matching



(+ symmetric diagrams with coll. gluon)

Remark: First non-vanishing contribution with correct quantum numbers for

$$J_A = \bar{\chi}_{hc} h_v \to \bar{\chi}_{hc}^{(5)} h_v \text{ in the notation of [Beneke/Feldmann 0311335].}$$

$\Psi_{hc}^{(5)} \sim \lambda^5$ describes splitting into one soft quark + two collinear quarks (+ soft and coll. gluons)

ξ_π: Tree-Level Matching

- Tree-level bare factorization formula:

$$
\begin{aligned}
\xi_\pi(E_\pi) \sim C_F \int_0^\infty \mathrm{d}\omega \int_0^1 \mathrm{d}u \Bigg[& \frac{\phi_B^-(\omega)}{\omega} \frac{1+\bar{u}}{\bar{u}^2} \phi_\pi(u) + \frac{\phi_B^+(\omega)}{\omega} \frac{u}{\bar{u}^2} \phi_\pi(u) \\
& + \frac{\phi_B^+(\omega)}{\omega^2} \left(- \frac{m_q \bar{u} + 2 m_{\bar{q}}}{\bar{u}^2} \phi_\pi(u) + 3 \frac{\mu_\pi \phi_P(u)}{\bar{u}} + \frac{\tilde{\mu}_\pi}{6} \frac{\phi_\sigma'(u)}{\bar{u}} \right) \Bigg] \\
& - 2\,(C_F - C_A/2)\, \frac{f_{3\pi}}{f_\pi} \int_0^\infty \mathrm{d}\omega\, \frac{\phi_B^+(\omega)}{\omega^2} \int \mathcal{D}\alpha\, \frac{\phi_{3\pi}(\{\alpha_i\})}{\alpha_g \alpha_{\bar{q}} (\alpha_g + \alpha_{\bar{q}})} \\
& + 2\,(C_F - C_A/2) \int_0^\infty \mathrm{d}\omega \int_0^\infty \mathrm{d}\xi\, \frac{\Psi_{A-V}(\omega,\xi)}{\omega\, \xi\, (\omega+\xi)} \int_0^1 \mathrm{d}u\, \frac{\phi_\pi(u)}{\bar{u}^2} \,.
\end{aligned}
$$

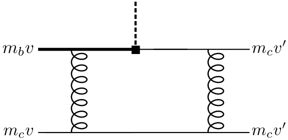
- Almost all convolutions endpoint-divergent for $\omega \to 0$ and $\bar{u} \to 0$!
- For example: $\phi_B^+ \sim \omega$, $\phi_B^- \sim$ const. for $\omega \to 0$, and $\phi_\pi \sim \bar{u}$, $\phi_P \sim \phi_\sigma' \sim$ const. for $\bar{u} \to 0$
- They appear in **products**!

Non-Relativistic Bound States

LCDAs are **non-perturbative** hadronic objects . . . How to approach the problem?

$\rightarrow$ consider non-relativistic bound states: $B_c \to \eta_c$ in the limit $m_b \gg m_c \gg \Lambda_{\rm QCD}$

$\rightarrow$ LO in NR expansion: $2 \to 2$ scattering process of on-shell massive quarks
(with correct spin projections) e.g. [Bell/Feldmann '05+'08, Bell '06]



$(v \cdot v' \equiv \gamma \sim \mathcal{O}(m_B/m_\eta) \gg 1)$

✓ quark masses provide physical IR cut-off (they mimic $\Lambda_{\rm QCD}$)

✓ perturbative partonic calculation can be trusted down to the low scale m_c

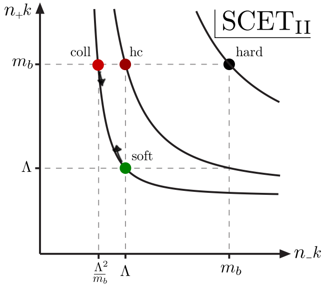
- $\rightarrow$ perturbative corrections to the LCDAs [Bell/Feldmann '08]
- $\rightarrow$ at tree-level: $\phi_B^+(\omega) = \phi_B^-(\omega) = \delta(\omega - m_c)$ and $\phi_\pi(u) = \phi_P(u) = \delta(u - 1/2)$

:(realistic quarks massless(?). quark masses complicate the analysis

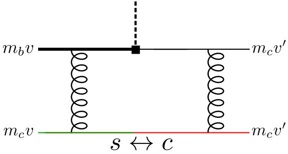
- $\rightarrow$ endpoint-div's show up as rapidity poles in rad. corr., requires rapidity regulator

Momentum Regions

- Standard modes for analytic regulators $(\nu/n_\pm k)^\alpha$, $(\nu/vk)^\alpha$
  - $\rightarrow$ no soft-collinear messenger modes with virtuality below m_c
  - $\rightarrow$ bare factorization theorem gives finite form factor

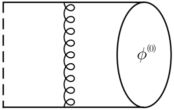


- Endpoint divergences arise from the soft limit of the collinear sector, and vice versa
  - $\rightarrow$ they cancel between inv. moments of the B_c and the η_c LCDAs
  - $\rightarrow$ different hadronic matrix elements have a common overlap in the endpoint region



Structure of Rapidity Divergences: NLO

…now start computing one-loop diagrams like



and find for the leading rapidity sing. of the soft moments (here used $(\nu/n_+k)^\alpha$, use EoM for 3-particle DA)

$$2m_{\bar{q}}\int_0^\infty \frac{d\omega}{\omega^2}\phi_B^+(\omega) \simeq \int_0^\infty \frac{d\omega}{\omega}\phi_B^-(\omega) \simeq \frac{\alpha_s C_F}{4\pi}\frac{1}{m_{\bar{q}}}\left(\frac{\mu^2}{m_{\bar{q}}^2}\right)^\varepsilon \left(\frac{\nu}{m_{\bar{q}}}\right)^\alpha \frac{4}{\alpha\varepsilon}$$

and for the collinear moments, e.g. ($u_0 = 1/2$ for $m_q = m_{\bar{q}} = m_c$)

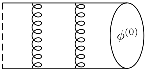
$$\int_0^1 \mathrm{d}u\, \frac{\phi_\eta^{(1)}(u)}{\bar{u}^2} \simeq \frac{\alpha_s C_F}{4\pi}\left(-\frac{2}{\alpha\varepsilon}\right)\left(\frac{\mu^2}{m_{\bar{q}}^2}\right)^\varepsilon \left(\frac{\nu}{2\gamma m_{\bar{q}}}\right)^\alpha \frac{1+\bar{u}_0}{\bar{u}_0^2}$$

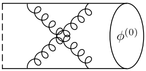
$\to$ plug into fact.-formula, add (hard, hc, cusp) to obtain finite result: (✓ with full-theory calculation)

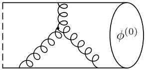
$$\xi_{\eta_c}^{(1)} \sim \frac{\alpha_s C_F}{4\pi}\left(\frac{2C_F}{\bar{u}_0^2} - \frac{C_{FA}}{\bar{u}_0^3}\right)\ln^2(2\gamma)$$

Structure of Rapidity Divergences: NNLO

. . . now start computing two-loop diagrams like







and find for the leading rapidity singularities of the soft moments

$$\int_0^\infty \frac{d\omega}{\omega^2}\,\phi_B^+(\omega) \simeq \frac{1}{m_{\bar q}^2}\left(\frac{\alpha_s}{4\pi}\right)^2\left(\frac{\mu^2}{m_{\bar q}^2}\right)^{2\varepsilon}\left(\frac{\nu}{m_{\bar q}}\right)^{2\alpha}\frac{2C_F^2}{\alpha^2\varepsilon^2}$$

$$\int_0^\infty \frac{d\omega}{\omega}\,\phi_B^-(\omega) \simeq \frac{1}{m_{\bar q}}\left(\frac{\alpha_s}{4\pi}\right)^2\left(\frac{\mu^2}{m_{\bar q}^2}\right)^{2\varepsilon}\left(\frac{\nu}{m_{\bar q}}\right)^{2\alpha}\frac{6C_F^2 - C_A C_F}{\alpha^2\varepsilon^2}$$

and some more complicated expressions in the coll. sector.

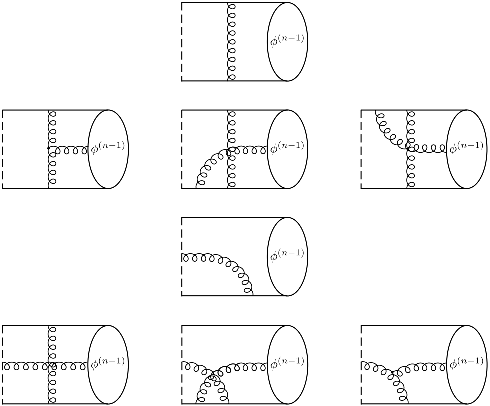
$\rightarrow$ again, all $1/\alpha^2$ poles drop out in the sum *ss* + *cc* + *sc* ✓

Note that the mixed soft-coll. contribution is $\sim 1/\alpha^2$! It contains products of divergent moments!

Structure of Rapidity Divergences: All-Orders

Q: Can we make some all order statements?

Yes! At least the leading rapidity poles are determined by recursion relations! (like a multipl. Z-factor)



Mixing of various two- and three-particle LCDAs at endpoint!

Structure of Rapidity Divergences: All-Orders

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- Gives exponentiation structures, e.g. (like in μe scattering!)

$$\int_0^\infty \frac{d\omega}{\omega^2}\phi_B^+(\omega) \simeq \frac{1}{m_{\bar{q}}^2}\exp\mathcal{E}$$

$$\int_0^\infty \frac{d\omega}{\omega}\phi_B^-(\omega) \simeq \frac{1}{m_{\bar{q}}}\left\{\exp\mathcal{E} - \frac{C_{FA}}{C_F}\mathcal{E}\exp\mathcal{E} + \frac{C_A}{2C_F}\left(\exp\mathcal{E} - 1\right)\right\}$$

- with leading contribution to the (bare) anomaly exponent:

$$\mathcal{E} = \frac{\mathcal{F}^{(1)}}{\alpha}\left(\frac{\nu}{m_{\bar{q}}}\right)^{\alpha}, \qquad \mathcal{F}^{(1)} = \frac{\alpha_s C_F}{4\pi}\frac{2}{\varepsilon}\left(\frac{\mu^2}{m_{\bar{q}}^2}\right)^{\varepsilon}$$

✓ With similar relations for the η_c, leading $1/\alpha$ poles cancel when inserted in fact.-theorem!

Structure of Rapidity Divergences: All-Orders

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Yes! At least the leading rapidity poles are determined by recursion relations! (like a multipl. Z-factor)

- For example, rapidity poles cancel in the product

$$\int_0^\infty \frac{d\omega}{\omega^2}\phi_B^+(\omega) \times \int_0^1 du\, \frac{1+\bar{u}}{\bar{u}^2}\phi_{\eta_c}(u) \simeq \frac{1}{m_{\bar{q}}^2}\frac{1+\bar{u}_0}{\bar{u}_0^2} \times (2\gamma)^{\mathcal{F}^{(1)}}$$

✓ large rapidity logarithms due to collinear anomaly resummed to all orders

- more complicated structures for ξ_{η_c} due to mixing

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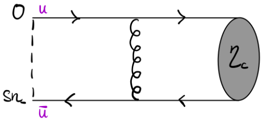
Remark: Results for inv. moments process-independent!
$\rightarrow$ similar structure for other hard-exclusive processes!

Endpoint-Refactorization

Q: What is the all-order $\bar{u} \to 0$ asymptotics of, for example, the bare η_c LCDA?

$\to$ take soft (or soft-collinear) limit of overlapping propagator that carries momentum fraction $\bar{u}$:

$$\langle \eta_c(p)| \, \bar{\chi}_c(0) \frac{\not{n}_+}{2} \gamma_5 \chi_c(sn_-) \, |0\rangle = -iE_{\eta_c} f_{\eta_c} \int_0^1 d\bar{u} \, e^{i\bar{u}s(n_- p)} \, \phi_{\eta_c}(u)$$



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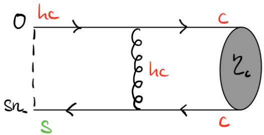

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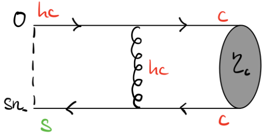
$\chi_{hc} \sim \lambda^5$ $\quad q_s \sim \lambda^3$ $\quad \lambda^2$ $\quad \sim \bar{u} \sim \lambda^2$ $\quad (\lambda^2 \sim \Lambda/m_c)$



- power-counting: $\phi_{\eta_c}(\bar{u} \to 0)$ involves the $\psi_{hc}^{(5)}$ splitting into a soft and two coll. quarks!
  - $\to$ endpoint described by a vacuum matrix element of soft (or soft-collinear) fermion fields
  - $\to$ same overlap matrix elements appear in $\phi_{B_c}(\omega \to 0) \Rightarrow$ common overlap ✓
  - $\to$ **but**: again iterative non-additive structure (like soft mode in $e^- \mu^-$ scattering)

4. Conclusion

Conclusion

Take-home messages:

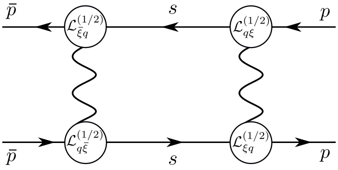
1. Endpoint-singularities longstanding problem that prevents a systematic study of power-corrections in SCET. General treatment still an open problem. Phenomenologically relevant in B physics.

2. Despite the recent progress, they can manifest is a more complicated non-additive way, in particular in $2 \to 2$ processes (or higher multiplicities).
   $\to$ exclusive charmless B decays

3. Muon-electron backward-scattering provides a well-defined perturbative playground for studying non-trivial aspects of soft-coll factorization in the presence of endpoint-div's.

Backup-Slides

Soft-Enhancement



- soft contribution leading power despite four insertions of $\mathcal{L}^{(1/2)}_{\xi q}$:

$$\mathcal{O}_2 = \int d^d x_1 \int d^d x_2 \int d^d x_3 \, T\left\{[\bar{\chi}^{(e)}_{\bar{hc}} \mathcal{A}^{\perp}_{\bar{hc}}](0)\,[\mathcal{A}^{\perp}_{\bar{hc}} \chi^{(\mu)}_{\bar{hc}}](x_1)\right\}$$
$$\times\, T\left\{[\bar{\chi}^{(\mu)}_{hc} \mathcal{A}^{\perp}_{hc}](x_2 + x_3)\,[\mathcal{A}^{\perp}_{hc} \chi^{(e)}_{hc}](x_3)\right\}$$
$$\times\, T\left\{\psi^{(e)}_s(0)\,\bar{\psi}^{(\mu)}_s(x_{1+})\,\psi^{(\mu)}_s(x_{2-} + x_3)\,\bar{\psi}^{(e)}_s(x_3)\right\}$$

- multipole expansion w.r.t. hc and $\bar{hc}$ fields at different space-time points $x = 0$ and $x = x_3$
  - $\rightarrow$ soft fluctuations $d^4 x_3 \sim 1/\lambda^4$ compensate suppression from $\mathcal{L}^{(1/2)}_{\xi q}$ insertions
  - $\rightarrow$ related to special backward kinematics
- In SCET$_{\mathrm{II}}$ the suppression is compensated by inverse soft derivatives

Some Backup Formulas

Comparison of the muon-electron scattering and $h \to \gamma\gamma$ amplitude at the DL level:

- for $h \to \gamma\gamma$ get standard $h \to b^* b^*$ Sudakov in integrand:

$$\mathscr{F}_b(z) = 2 \int_0^1 d\xi \int_0^1 d\eta \, \theta(1 - \xi - \eta) e^{-2\xi\eta z}$$

- for muon-electron backward-scattering the form factor itself appears in the integrand
  $\rightarrow$ nested structure

$$\mathscr{F}_1(z) = 1 + z \int_0^1 d\xi \int_0^1 d\eta \, \mathscr{F}_1(\xi^2 z)\theta(1 - \xi - \eta)\mathscr{F}_1(\eta^2 z)$$