

Endpoint singularities in μe backward scattering

(and the soft-overlap form factor for exclusive B decays)

Philipp B er

based on: 2205.06021 with G. Bell and T. Feldmann, and PB PhD thesis 2018

MITP program “Power Expansions on the Lightcone”
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1. Introduction
2. Endpoint singularities in muon-electron backward scattering
3. Soft overlap form factor for non-relativistic $B_c \rightarrow \eta_c$ transitions
4. Conclusion

1. Introduction

Introduction

Aim of soft-collinear factorization:

1. combined expansion in $\mu_{\text{low}}/\mu_{\text{high}}$ and α_s
→ requires rigorous **power-counting** scheme
2. separate dynamics related to momentum regions with **different virtualities** and/or **rapidities**
→ separation of perturbative and non-perturbative dynamics in hadronic processes
3. employ **Renormalization Group Equations** to sum large logarithms to all orders

$$\alpha_s^n \ln^m \left(\frac{\mu_{\text{low}}}{\mu_{\text{high}}} \right)$$

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Problem: standard procedure may lead to **ill-defined convolution integrals**

- they may only converge in $d = 4 - 2\varepsilon$ dimensions (SCET_I)
→ standard renormalization program breaks down
- dim.-reg. insufficient to separate modes with equal virtuality but different rapidity (SCET_{II})
→ **analytic regulators** violate naive decoupling (“collinear anomaly”) [Becher/Neubert]
- often related to soft fermions (→ n.l.p.!)

Incomplete List of Examples

- off-diagonal channels in DIS [Beneke et al '20]
- bottom induced $h \rightarrow \gamma\gamma$ decay [Neubert et al. '19/20]
- off-diagonal **gluon thrust** [Beneke et al. '22]
- ...

In B -meson decays:

- power-corrections in $B \rightarrow h_1 h_2$ decays (e.g. weak annihilation) [BBNS '99/00]
- power-corrections in radiative $B \rightarrow \gamma \ell \nu$ decays [e.g. Beneke, Rohrwild '11]
- **heavy-to-light form factors** [Beneke, Feldmann '00]
- certain QED corrections in $B_s \rightarrow \mu^+ \mu^-$ [Beneke, Bobeth, Szafron '19]
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Problem seems to arise **generically** in SCET at subleading power! A better understanding would constitute a major step in controlling the $1/E$ expansion.

$\Lambda_{\text{QCD}}/m_B \gtrsim 0.1$ is not extremely small. Very **relevant in B physics!**

2. muon-electron backward scattering: a prime example for endpoint singularities

based on arXiv:2205.06021 with G. Bell and T. Feldmann

Why “prime example”?

Clean framework to study physics of endpoint singularities, because ...

- ✓ perturbative QED process from the textbook
- ✓ resummed **double logarithms** known for > 50 years [Gorshkov et al. 1966]
- ✓ they arise from single scalar Integral at each order in α_{em} (\rightarrow playground for method-of-regions)
- ✓ bare factorization theorem can be reduced to a **single term** at the DL level
- ✓ **most general** structure of endpoint singularities, already for DLs at **leading power!**
(more complicated than e.g. $h \rightarrow \gamma\gamma$, gluon thrust)
- ✓ mimics structure of endpoint singularities in **exclusive B decays** (& other hard-exclusive processes)

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Backward Scattering

$2 \rightarrow 2$ process: $e^-(p)\mu^-(\bar{p}) \rightarrow e^-(\bar{p})\mu^-(p)$ at $s \approx -t \gg m_\mu^2 \sim m_e^2 \gg u$

- consider common mass $m_\mu \simeq m_e \rightarrow m$ for simplicity (but distinguishable flavours)
- expansion parameter: $\lambda = m/\sqrt{s}$, and light-cone vectors

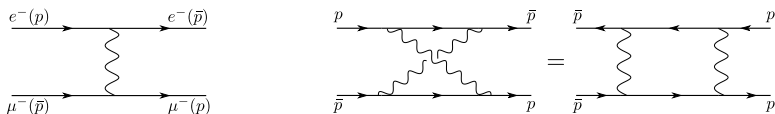
$$p^\mu = \frac{\sqrt{s}}{2} n_-^\mu + \frac{m^2}{2\sqrt{s}} n_+^\mu \qquad \bar{p}^\mu = \frac{\sqrt{s}}{2} n_+^\mu + \frac{m^2}{2\sqrt{s}} n_-^\mu$$

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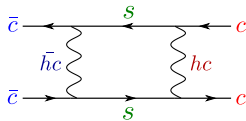
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- high-energy limit: $\mathcal{M} = F_1(\lambda)\mathcal{M}^{(0)} + F_2(\lambda)\widetilde{\mathcal{M}}$
 \rightarrow leading DLs in form factor $F_1(\lambda)$ that multiplies tree amplitude $\mathcal{M}^{(0)} \sim \alpha_{\text{em}}$
- DL at NLO from twisted box: $F_1(\lambda) = 1 + \frac{\alpha_{\text{em}}}{2\pi} \frac{1}{2} \ln^2 \lambda^2 + \dots$

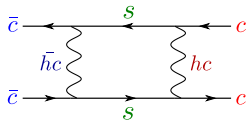
Isolating the Double-Log at NLO

DLs arise from the kinem. configuration in which the virtual lepton propagators are soft, $k^\mu \sim \lambda$:



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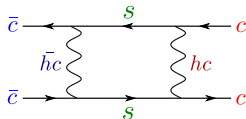


After some Dirac algebra, it is then easy to show that the DL is contained in the scalar integral

$$F_1^{(1)}(\lambda) \sim \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - m^2} \frac{1}{(k - p)^2} \frac{1}{(k - \bar{p})^2}$$

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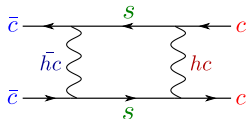
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Photon propagators become eikonal:

$$(k - p)^2 + i0 \simeq -\sqrt{s}(n_- k) + i0, \quad (k - \bar{p})^2 + i0 \simeq -\sqrt{s}(n_+ k) + i0$$

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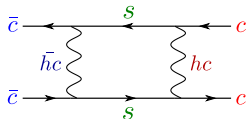
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Non-vanishing contribution from discontinuity of soft lepton propagator:

$$\int \frac{dk_\perp^2}{k^2 - m^2 + i0} \rightarrow -2\pi i \theta((n_+ k)(n_- k) - m^2)$$

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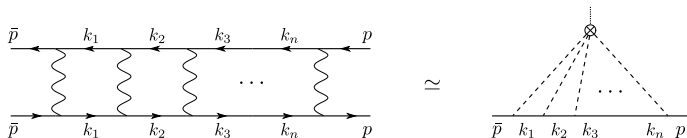
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Traditional approach: put hard cut-offs on longitudinal momenta $(n_\pm k) \leq \sqrt{s}$

$$F_1^{(1)}(\lambda) \simeq \int_{\lambda^2}^1 \frac{dx}{x} \int_{\lambda^2/x}^1 \frac{dy}{y} = \frac{1}{2} \ln^2 \lambda^2 \quad \checkmark \quad (n_+ k = x\sqrt{s}, n_- k = y\sqrt{s})$$

Isolating the Double-Log's at Higher Orders



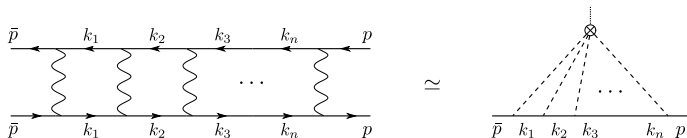
all photon propagators eikonal: $(k_i - k_{i-1})^2 + i0 \simeq -(n_+ k_i)(n_- k_{i-1}) + i0$

strongly ordered longitudinal lepton momenta:

$$\frac{m^2}{\sqrt{s}} \approx n_+ \bar{p} \ll n_+ k_1 \ll \dots \ll n_+ k_n \ll n_+ p \approx \sqrt{s}$$

$$\sqrt{s} \approx n_- \bar{p} \gg n_- k_1 \gg \dots \gg n_- k_n \gg n_- p \approx \frac{m^2}{\sqrt{s}}$$

Isolating the Double-Log's at Higher Orders



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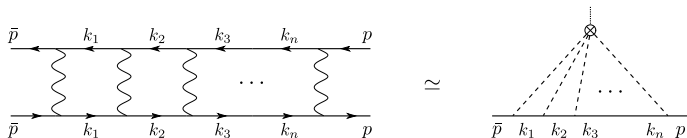
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$$\sqrt{s} \approx n_- \bar{p} \gg n_- k_1 \gg \dots \gg n_- k_n \gg n_- p \approx \frac{m^2}{\sqrt{s}}$$

yields nested integrals:

$$F_1^{(n)}(\lambda) \simeq \int_{\lambda^2}^1 \frac{dx_1}{x_1} \int_{x_1}^1 \frac{dx_2}{x_2} \dots \int_{x_{n-1}}^1 \frac{dx_n}{x_n} \int_{\lambda^2/x_1}^1 \frac{dy_1}{y_1} \int_{\lambda^2/x_2}^{y_1} \frac{dy_2}{y_2} \dots \int_{\lambda^2/x_n}^{y_{n-1}} \frac{dy_n}{y_n} = \frac{\ln^{2n} \lambda^2}{n!(n+1)!}$$

Isolating the Double-Log's at Higher Orders



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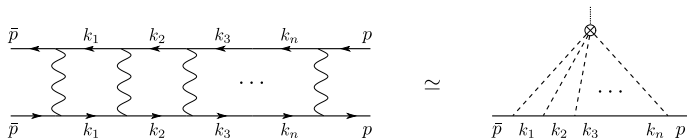
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$$\sqrt{s} \approx n_- \bar{p} \gg n_- k_1 \gg \dots \gg n_- k_n \gg n_- p \approx \frac{m^2}{\sqrt{s}}$$

that sum up to **modified Bessel function**:

$$F_1(\lambda) \simeq \sum_{n=0}^{\infty} \left(\frac{\alpha_{\text{em}}}{2\pi} \right)^n F_1^{(n)}(\lambda) = \frac{I_1(2\sqrt{z})}{\sqrt{z}}, \quad \text{with} \quad z = \frac{\alpha_{\text{em}}}{2\pi} \ln^2 \lambda^2$$

Isolating the Double-Log's at Higher Orders



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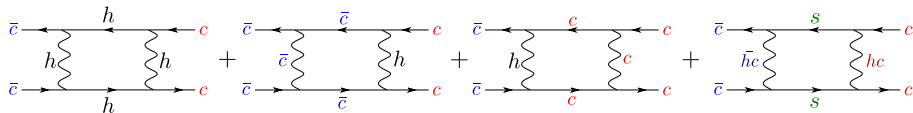
However, ...

- only the leading double-logarithms identified in this way
- scale of running coupling undetermined
- factorize non-pert. physics from short-distance dynamics (in hadronic processes)

Goal: Formulate problem in SCET in terms of a [renormalized factorization theorem!](#)

... work in progress! Highly non-trivial endpoint singular convolutions!

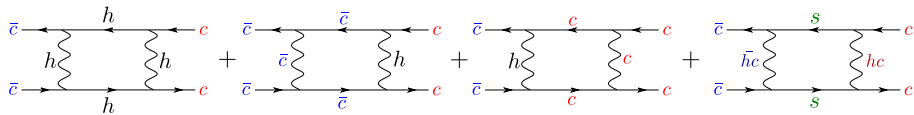
Method-of-Regions Analysis



$$\mathcal{I}^{(\text{hard})} = \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \ln \frac{\mu^2}{s} + \frac{1}{2} \ln^2 \frac{\mu^2}{s} - \frac{\pi^2}{12} + \mathcal{O}(\varepsilon)$$

- contains Sudakov-type double-logarithms involving the **hard scale** μ/\sqrt{s}

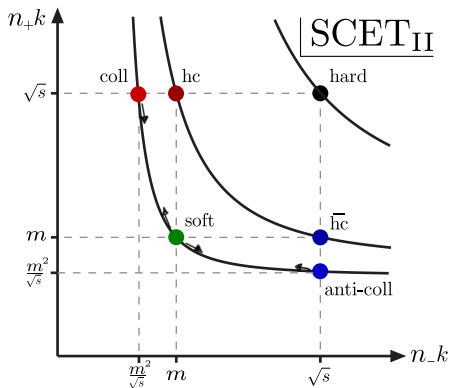
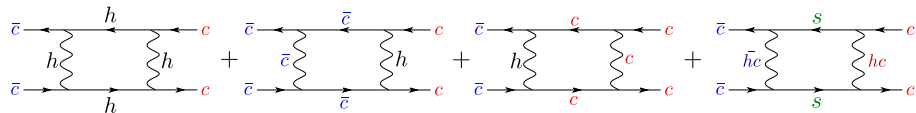
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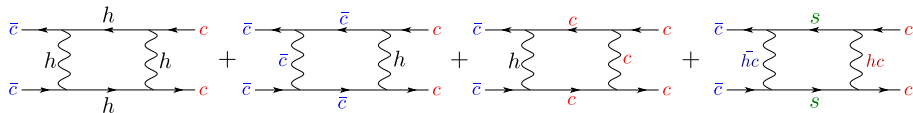
$$\mathcal{I}^{(c)} = e^{\varepsilon\gamma_E} \Gamma(\varepsilon) \left(\frac{\mu^2}{m^2} \right)^\varepsilon \int_0^1 \frac{dx}{x} (1-x)^{-2\varepsilon} \left(\frac{\nu}{x\sqrt{s}} \right)^\alpha = - \left(\frac{1}{\alpha} + \ln \frac{\nu}{\sqrt{s}} \right) \left(\frac{1}{\varepsilon} + \ln \frac{\mu^2}{m^2} \right) + \frac{\pi^2}{3} + \mathcal{O}(\alpha, \varepsilon)$$

- standard UV singularity $\Gamma(\varepsilon)$ from $k_\perp \rightarrow \infty$
- **Endpoint-singularity** for $n_+ k = x\sqrt{s} \rightarrow 0$
 - **ill-defined** in dim.-reg. due to lepton mass $m \neq 0$
 - **rapidity divergence!** Fermion propagator overlaps between low-energy regions
 - No IR-singularity in the conventional sense (no mode below $\mu \sim m$)
- requires additional (analytic) **rapidity regulator** (e.g. [Becher/Bell, Ebert et al., Chiu et al., Neill et al., ...])
 - here: $(\nu/2k_0)^\alpha$ preserves symmetry, so $\mathcal{I}^{(c)} = \mathcal{I}^{(\bar{c})}$
 - small virtuality $\mu \sim m$, large energy $\nu \sim \sqrt{s}$

Method-of-Regions Analysis



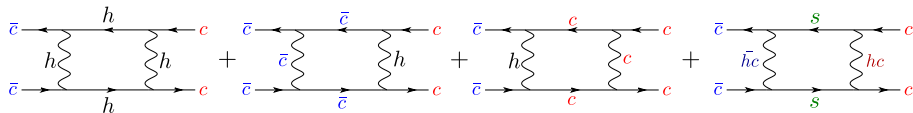
Method-of-Regions Analysis



$$\mathcal{I}^{(s)} = 2 \left(\frac{1}{\alpha} + \ln \frac{\nu}{m} \right) \left(\frac{1}{\varepsilon} + \ln \frac{\mu^2}{m^2} \right) - \frac{1}{\varepsilon^2} - \frac{1}{\varepsilon} \ln \frac{\mu^2}{m^2} - \frac{1}{2} \ln \frac{\mu^2}{m^2} + \frac{\pi^2}{12} + \mathcal{O}(\alpha, \varepsilon)$$

- again **ill-defined** in dim.-reg.
 - $1/\alpha$ singularity from both limits $n_+ k \rightarrow \infty$ and $n_- k \rightarrow \infty$
- symmetric regulator remains unexpanded: $2k_0 = n_+ k + n_- k = (x+y)\sqrt{s}$
 - small virtuality $\mu \sim m$, small energy $\nu \sim m$
 - **can be made scaleless** by choosing an asymmetric regulator, e.g. $(\nu/n_+ k)^\alpha$

Method-of-Regions Analysis



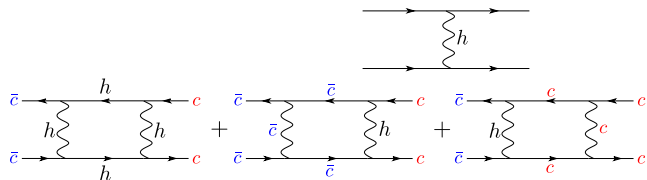
- Sum of regions:

$$\mathcal{I}^{(\text{hard})} + \mathcal{I}^{(c)} + \mathcal{I}^{(\bar{c})} + \mathcal{I}^{(s)} = \frac{1}{2} \ln^2 \lambda^2 + \frac{2\pi^2}{3}$$

- ✓ dimensional and analytic regulator drop out
- ✓ leading DL recovered
- remaining one-loop graphs standard
 - no endpoint-singularity, no analytic regulator, DL cancels

Formulation in SCET

Two-step matching: QED $\xrightarrow{s \rightarrow \infty}$ SCET_I $\xrightarrow{\sqrt{sm} \rightarrow \infty}$ SCET_{II}:



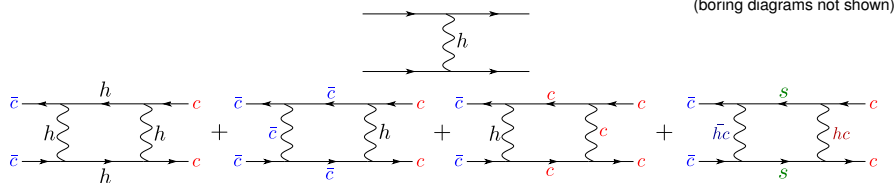
(boring diagrams not shown)

Schematic form of bare factorization formula

$$F_1(\lambda) \simeq f_{\bar{c}} \otimes H \otimes f_c$$

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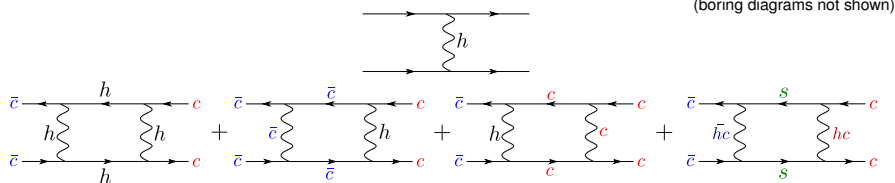


Schematic form of bare factorization formula

$$F_1(\lambda) \simeq f_{\bar{c}} \otimes H \otimes f_c + f_{\bar{c}} \otimes J_{\bar{h}c} \otimes S \otimes J_{hc} \otimes f_c$$

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Schematic form of bare factorization formula

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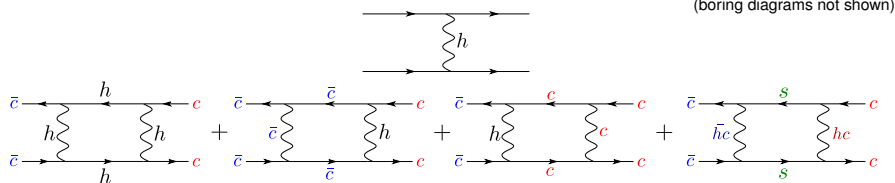
- soft contribution leading power due to specific *soft-enhancement* mechanism (\rightarrow backup)
- ✓ individual **bare** soft and coll. fct's defined as SCET operator matrix elements, e.g.

$$\langle \mu^- (p) | \bar{\chi}_c^{(\mu)}(\tau n_+) \frac{\not{n}_+}{2} P_{R(L)} \chi_c^{(e)}(0) | e^- (p) \rangle = \int dx e^{ix\tau n_+ p} \left\{ f_c(x) [\bar{u}_\xi^{(\mu)} \frac{\not{n}_+}{2} P_{R(L)} u_\xi^{(e)}] + \tilde{f}_c(x) [\bar{u}_\xi^{(\mu)} \frac{\not{n}_+}{2} P_{L(R)} u_\xi^{(e)}] \right\}$$

- \rightarrow generalized parton distributions (forward, but flavour-non-diagonal)
- \rightarrow helicity-flipping functions $\tilde{f}_c(x)$ and $\tilde{f}_{\bar{c}}(y)$ do not contribute to leading DLs

Formulation in SCET

Two-step matching: QED $\xrightarrow{s \rightarrow \infty}$ SCET_I $\xrightarrow{\sqrt{sm} \rightarrow \infty}$ SCET_{II}:



Schematic form of bare factorization formula

$$F_1(\lambda) \simeq f_{\bar{c}} \otimes H \otimes f_c + f_{\bar{c}} \otimes J_{\bar{h}c} \otimes S \otimes J_{hc} \otimes f_c$$

- At one-loop level

$$f_c(x) \simeq \delta(1-x) + \frac{\alpha_{\text{em}}}{2\pi} \theta(x)\theta(1-x) \left(\frac{\mu^2}{m^2}\right)^\epsilon \Gamma(\epsilon) (1 + \mathcal{O}(x))$$

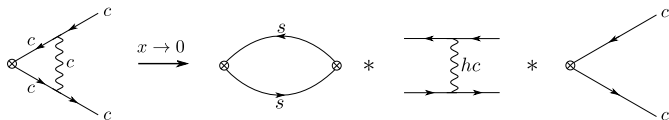
→ convolution integrals $\int \frac{dx}{x} f_c(x)$ require rapidity regulator!

?? How to renormalize functions *before* performing the convolutions ??

Endpoint-Refactorization

Q: Can we understand the $x \rightarrow 0$ asymptotics of the **bare** functions $f_c(x)$ to all orders in α_{em} ?
 Can we isolate and subtract the divergences?

Recall: Rapidity divergences arise from the **soft limit** of the coll. fermion propagators!
 \rightarrow interpret $f_c(x)$ for $x \rightarrow 0$ as multi-scale object



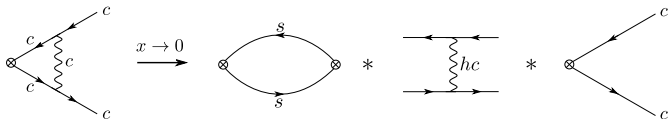
$$f_c(x \rightarrow 0) \simeq \int \frac{dx'}{x'} f_c(x') \int \frac{d\rho}{\rho} J_{hc}(\rho x') S(\rho, x)$$

✓ reflects structure of the second term $f \otimes J \otimes S \otimes J \otimes f$ as $1/\alpha$ poles must cancel [PB '18]

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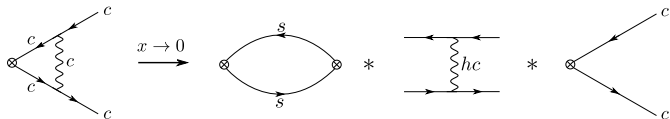
Implications: (before expansion in ϵ !)

- 1.) collinear functions receive *positive* powers of x^ϵ from J_{hc} $\Rightarrow \langle x^{-1-n\epsilon} \rangle_{f_c} \sim 1/\alpha, \forall n$
- 2.) $1/\alpha$ cancel within $f_c(x \rightarrow 0)$ and generate powers of $\ln x$ \Rightarrow higher powers in $1/\alpha$
- 3.) peculiar structure as $f_c(x')$ arises on the RHS \Rightarrow non-additive problem

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Example: at four-loop $f_c(x)$ has the following asymptotic structure:

$$f_c^{(4)}(x \rightarrow 0) \simeq \left(\frac{\mu^2}{m^2} \right)^{4\epsilon} \left\{ \frac{x^{3\epsilon} - 15x^{2\epsilon} + 339x^\epsilon - 325}{144\epsilon^7} - \frac{(3x^\epsilon + 23) \ln x}{12\epsilon^6} - \frac{3 \ln^2 x}{4\epsilon^5} - \frac{\ln^3 x}{6\epsilon^4} \right\}$$

Resummation from Consistency Relations

Despite the complexity of the problem, the DL series is completely determined by

- (i) scale separation (of bare quantities)
- (ii) consistency (i.e. pole cancellation in $1/\alpha$ and $1/\varepsilon$)
- (iii) refactorization

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1. Use **asymmetric regulator** that makes soft contribution scaleless: (clear scale separation ✓)

$$F_1(\lambda) \simeq \int_0^1 \frac{dx}{x} f_c \left(x; \frac{\mu}{m}, \frac{\nu}{\sqrt{s}} \right) \int_0^1 \frac{dy}{y} f_{\bar{c}} \left(y; \frac{\mu}{m}, \frac{\nu\sqrt{s}}{m^2} \right) H \left(\frac{\mu^2}{xys} \right)$$

→ single term that involves only leading-twist projections

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2. Insert perturbative expansion of hard function at double-log level:

$$H \left(\frac{\mu^2}{xys} \right) \simeq \sum_{n=0}^{\infty} z_h^n h^{(n)} (xy)^{-n\varepsilon} \quad \text{with} \quad z_h = \frac{\alpha_{\text{em}}}{2\pi} \frac{1}{\varepsilon^2} \left(\frac{\mu^2}{s} \right)^\varepsilon$$

Form factor expressed as infinite sum of **products** of divergent moments:

$$F_1(\lambda) = \sum_{n=0}^{\infty} z_h^n h^{(n)} \langle x^{-1-n\varepsilon} \rangle_{f_c} \left(\frac{\mu}{m}, \frac{\nu}{\sqrt{s}} \right) \langle y^{-1-n\varepsilon} \rangle_{f_{\bar{c}}} \left(\frac{\mu}{m}, \frac{\nu\sqrt{s}}{m^2} \right)$$

Resummation from Consistency Relations

3. Rapidity poles must cancel at each order in the hard-matching

→ **Collinear Anomaly**: large rapidity log's **exponentiate** in products

[Becher,Bell,Neubert '11]

→ F_1 expressed as infinite sum of anomaly exponents \mathcal{F}_n and “remainder functions” r_n

$$F_1(\lambda) = \sum_{n=0}^{\infty} z_h^n h^{(n)} r_n(\mu/m) \cdot \left(\frac{m^2}{s} \right)^{\mathcal{F}_n(\mu/m)}$$

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$$r_n(\mu/m) = \sum_{k=0}^{\infty} \left(\frac{\alpha_{\text{em}}}{2\pi} \right)^k \left(\frac{\mu^2}{m^2} \right)^{k\epsilon} \frac{r_n^{(k)}}{\epsilon^{2k}}, \quad \mathcal{F}_n(\mu/m) = \sum_{l=n+1}^{\infty} \left(\frac{\alpha_{\text{em}}}{2\pi} \right)^l \left(\frac{\mu^2}{m^2} \right)^{l\epsilon} \frac{\mathcal{F}_n^{(l)}}{\epsilon^{2l-1}}$$

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5. Form factor **finite for $\varepsilon \rightarrow 0$** gives consistency relations between $(h^{(n)}, r_n^{(k)}, \mathcal{F}_n^{(k)})$

✓ reproduce known result order-by-order:

$$F_1(\lambda) \simeq \frac{l_1 \left(2\sqrt{h^{(1)}} z \right)}{\sqrt{h^{(1)}} z}$$

✓ single unknown coefficient $h^{(1)} = 1$ determined from one-loop calculation

! need infinite perturbative series of anomaly exponents

At first sight, the two processes seem to be very similar at the technical level:
SCET_{II}, same modes, massive fermion propagators, analytic regulators, . . . **but**:

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Bare factorization theorem for form factor F_1 takes the schematic form

$$f_{\bar{c}} \otimes H \otimes f_c \quad + \quad f_{\bar{c}} \otimes J_{\bar{h}c} \otimes S \otimes J_{hc} \otimes f_c$$

- endpoint-div. cancel in **products** of inv. moments \rightarrow **exponentiation** of rapidity poles
- iterative refactorization condition: $f_c(x \rightarrow 0) \sim f_c \otimes J_{hc} \otimes S$
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$h \rightarrow \gamma\gamma$ bare factorization theorem can be written as

$$H_1 \cdot \langle \gamma\gamma | O_1 | h \rangle + H_2 \otimes \langle \gamma\gamma | O_2 | h \rangle + H_2 \otimes \langle \gamma\gamma | \bar{O}_2 | h \rangle + H_3 \cdot J_{\bar{h}c} \otimes S \otimes J_{hc}$$

- endpoint-div. cancel in **sum** of inv. moments \rightarrow **linear** rapidity pole to all orders
- refactorization condition takes simpler form: $\langle \gamma\gamma | O_2 | h \rangle|_{x \rightarrow 0} \sim J_{hc} \otimes S$
- soft function vanishes for zero argument

Reason: Collinear and soft function in $h \rightarrow \gamma\gamma$ both **helicity suppressed** in m_b/m_H .
 But this is *not* the case in the $2 \rightarrow 2$ scattering process $e^- \mu^- \rightarrow e^- \mu^-$.

Summary: Muon-Electron Backward Scattering

- ✓ simple $2 \rightarrow 2$ textbook process in QED
 - ✓ leading log's resum to modified Bessel function (known for > 50 years)
 - ✓ Bessel function in SCET recovered by **iterative pattern** of endpoint singularities
 - infinite tower of collinear-anomaly exponents
 - leading-power DLs already more complicated than other examples in the literature
 - 1.) “scale-separation”
 - 2.) consistency (pole-cancellation, “collinear anomaly”)
 - 3.) re-factorization
-
- So far we did not derive a **renormalized factorization theorem**
 - need to figure out whether rearrangement (in spirit of $h \rightarrow \gamma\gamma$) can be generalized
 - **next**: mimics the endpoint structure in **exclusive B_c decays**, but in a much simpler setup

3. The soft-overlap form factor in non-relativistic $B_c \rightarrow \eta_c$ transitions

based on: PB PhD thesis 2018

The Soft-Overlap Form Factor ξ_π

Form factor = non-perturbative input in exclusive semi-leptonic B decays, e.g. $B \rightarrow \pi \ell \nu$:

$$\langle \pi(p) | \bar{q} \gamma^\mu b | B(p_B) \rangle = F_+(q^2)(p_B^\mu + p^\mu) + F_-(q^2)q^\mu$$

At large pion energies (small q^2) \rightarrow use SCET to factorize **hard**, *hc*, *coll.*, *soft*

- $m_b \rightarrow \infty$: two SCET_I operators

$$J_A = \bar{\chi}_{hc} h_\nu, \quad J_B = \bar{\chi}_{hc} A_\perp h_\nu$$

- $\sqrt{m_b \Lambda_{\text{QCD}}} \rightarrow \infty$: J_B factorizes into **convergent** convolutions of LCDAs $\phi_B^+(\omega)$ and $\phi_\pi(u)$ ✓
 J_A does not factorize due to **endpoint-divergent** convolutions

However, A -type contribution spin-symmetry preserving:

[Beneke, Feldmann '00]

$$F_i(q^2) = H_i(q^2, \mu) \cdot \xi_\pi(q^2, \mu) + (\text{factorizable})_i$$

with the **soft-overlap form factor** ξ_π defined as a SCET_I hadronic matrix element

$$2E_\pi \xi_\pi = \langle \pi(p) | \bar{\chi}_{hc} h_\nu | \bar{B}_c \rangle$$

ξ_π : Tree-Level Matching

- SCET_{II} operator basis:

see [Lange,Neubert '03] for massless quarks

$$\mathcal{O}_1 = \left[\bar{\chi}(0) \frac{\vec{\not{n}}}{2} \gamma_5 \chi(s\bar{n}) \right] \left[\bar{Q}_s(\tau n) \frac{\vec{\not{n}} \not{n}}{4} \gamma_5 \mathcal{H}_V(0) \right]$$

$$\mathcal{O}_2 = \left[\bar{\chi}(0) \frac{\vec{\not{n}}}{2} \gamma_5 i \not{\phi}_\perp \chi(s\bar{n}) \right] \left[\bar{Q}_s(\tau n) \frac{\not{n}}{2} \gamma_5 \mathcal{H}_V(0) \right]$$

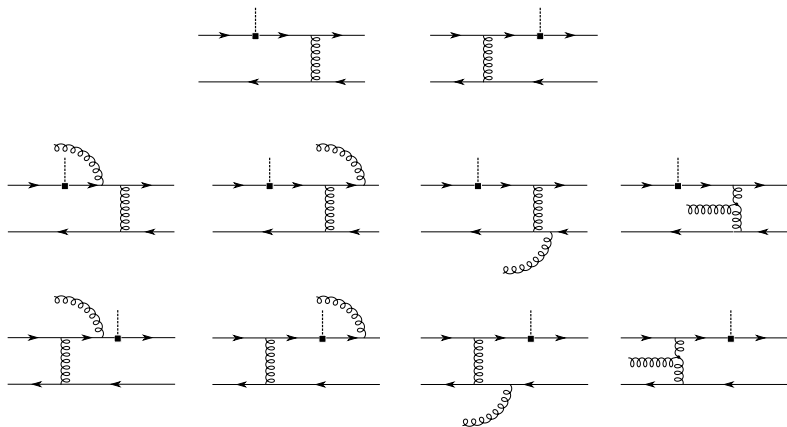
$$\mathcal{O}_3 = \left[\bar{\chi}(0) \frac{\vec{\not{n}}}{2} \gamma_5 \mathcal{A}_{c,\perp}(r\bar{n}) \chi(s\bar{n}) \right] \left[\bar{Q}_s(\tau n) \frac{\not{n}}{2} \gamma_5 \mathcal{H}_V(0) \right]$$

$$\mathcal{O}_4 = \left[\bar{\chi}(0) \frac{\vec{\not{n}}}{2} \gamma_5 \chi(s\bar{n}) \right] \left[\bar{Q}_s(\tau n) \mathcal{A}_{s,\perp}(\sigma n) \frac{\not{n}}{2} \gamma_5 \mathcal{H}_V(0) \right]$$

$$\mathcal{O}_m = \left[\bar{\chi}(0) \frac{\vec{\not{n}}}{2} \gamma_5 \chi(s\bar{n}) \right] \left[\bar{Q}_s(\tau n) \frac{\not{n}}{2} \gamma_5 \mathcal{H}_V(0) \right]$$

- The operator \mathcal{O}_m contributes only for non-vanishing light-quark masses

ξ_π : Tree-Level Matching



(+ symmetric diagrams with coll. gluon)

Remark: First non-vanishing contribution with correct quantum numbers for

$$J_A = \bar{\chi}_{hc} h_V \rightarrow \bar{\chi}_{hc}^{(5)} h_V \text{ in the notation of [Beneke/Feldmann 0311335].}$$

$\Psi_{hc}^{(5)} \sim \lambda^5$ describes splitting into one soft quark + two collinear quarks (+ soft and coll. gluons)

ξ_π : Tree-Level Matching

- Tree-level **bare** factorization formula:

$$\begin{aligned}\xi_\pi(E_\pi) \sim C_F \int_0^\infty d\omega \int_0^1 du \left[\frac{\phi_B^-(\omega)}{\omega} \frac{1+\bar{u}}{\bar{u}^2} \phi_\pi(u) + \frac{\phi_B^+(\omega)}{\omega} \frac{u}{\bar{u}^2} \phi_\pi(u) \right. \\ \left. + \frac{\phi_B^+(\omega)}{\omega^2} \left(-\frac{m_q \bar{u} + 2m_{\bar{q}}}{\bar{u}^2} \phi_\pi(u) + 3 \frac{\mu_\pi \phi_P(u)}{\bar{u}} + \frac{\tilde{\mu}_\pi \phi'_\sigma(u)}{6 \bar{u}} \right) \right] \\ - 2 (C_F - C_A/2) \frac{f_{3\pi}}{f_\pi} \int_0^\infty d\omega \frac{\phi_B^+(\omega)}{\omega^2} \int \mathcal{D}\alpha \frac{\phi_{3\pi}(\{\alpha_i\})}{\alpha_g \alpha_{\bar{q}} (\alpha_g + \alpha_{\bar{q}})} \\ + 2 (C_F - C_A/2) \int_0^\infty d\omega \int_0^\infty d\xi \frac{\Psi_{A-V}(\omega, \xi)}{\omega \xi (\omega + \xi)} \int_0^1 du \frac{\phi_\pi(u)}{\bar{u}^2}.\end{aligned}$$

- Almost all convolutions endpoint-divergent for $\omega \rightarrow 0$ and $\bar{u} \rightarrow 0$!
- For example: $\phi_B^+ \sim \omega$, $\phi_B^- \sim \text{const.}$ for $\omega \rightarrow 0$, and $\phi_\pi \sim \bar{u}$, $\phi_P \sim \phi'_\sigma \sim \text{const.}$ for $\bar{u} \rightarrow 0$
- They appear in **products**!

Non-Relativistic Bound States

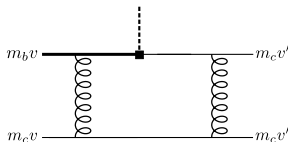
LCDAs are **non-perturbative** hadronic objects ... How to approach the problem?

→ consider **non-relativistic bound states**: $B_c \rightarrow \eta_c$ in the limit $m_b \gg m_c \gg \Lambda_{\text{QCD}}$

→ LO in NR expansion: $2 \rightarrow 2$ scattering process of on-shell massive quarks

(with correct spin projections)

e.g. [Bell/Feldmann '05+'08, Bell '06]



$$(v \cdot v' \equiv \gamma \sim \mathcal{O}(m_B/m_\eta) \gg 1)$$

✓ quark masses provide **physical IR cut-off** (they mimic Λ_{QCD})

✓ **perturbative** partonic calculation can be trusted down to the low scale m_c

→ perturbative corrections to the LCDAs

[Bell/Feldmann '08]

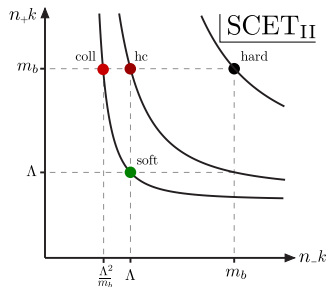
→ at tree-level: $\phi_B^+(\omega) = \phi_B^-(\omega) = \delta(\omega - m_c)$ and $\phi_\pi(u) = \phi_P(u) = \delta(u - 1/2)$

:(realistic quarks massless(?). quark masses complicate the analysis

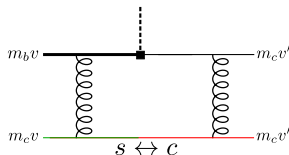
→ endpoint-div's show up as rapidity poles in rad. corr., requires **rapidity regulator**

Momentum Regions

- **Standard modes** for analytic regulators $(\nu/n_{\pm}k)^{\alpha}, (\nu/vk)^{\alpha}$
 - no soft-collinear messenger modes with virtuality below m_c
 - bare factorization theorem gives **finite** form factor

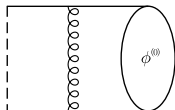


- **Endpoint divergences** arise from the soft limit of the collinear sector, and vice versa
 - they **cancel between inv. moments of the B_c and the η_c LCDAs**
 - different hadronic matrix elements have a **common overlap** in the endpoint region



Structure of Rapidity Divergences: NLO

... now start computing one-loop diagrams like



and find for the leading rapidity sing. of the soft moments (here used $(\nu/n+k)^\alpha$, use EoM for 3-particle DA)

$$2m_{\bar{q}} \int_0^\infty \frac{d\omega}{\omega^2} \phi_B^+(\omega) \simeq \int_0^\infty \frac{d\omega}{\omega} \phi_B^-(\omega) \simeq \frac{\alpha_s C_F}{4\pi} \frac{1}{m_{\bar{q}}} \left(\frac{\mu^2}{m_{\bar{q}}^2} \right)^\epsilon \left(\frac{\nu}{m_{\bar{q}}} \right)^\alpha \frac{4}{\alpha\epsilon}$$

and for the collinear moments, e.g. ($u_0 = 1/2$ for $m_q = m_{\bar{q}} = m_c$)

$$\int_0^1 du \frac{\phi_\eta^{(1)}(u)}{\bar{u}^2} \simeq \frac{\alpha_s C_F}{4\pi} \left(-\frac{2}{\alpha\epsilon} \right) \left(\frac{\mu^2}{m_{\bar{q}}^2} \right)^\epsilon \left(\frac{\nu}{2\gamma m_{\bar{q}}} \right)^\alpha \frac{1 + \bar{u}_0}{\bar{u}_0^2}$$

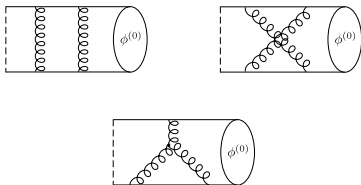
→ plug into fact.-formula, add (hard, hc, cusp) to obtain **finite** result:

(✓ with full-theory calculation)

$$\xi_{\eta c}^{(1)} \sim \frac{\alpha_s C_F}{4\pi} \left(\frac{2C_F}{\bar{u}_c^2} - \frac{C_{FA}}{\bar{u}_0^3} \right) \ln^2(2\gamma)$$

Structure of Rapidity Divergences: NNLO

... now start computing two-loop diagrams like



and find for the leading rapidity singularities of the soft moments

$$\int_0^\infty \frac{d\omega}{\omega^2} \phi_B^+(\omega) \simeq \frac{1}{m_{\bar{q}}^2} \left(\frac{\alpha_s}{4\pi} \right)^2 \left(\frac{\mu^2}{m_{\bar{q}}^2} \right)^{2\epsilon} \left(\frac{\nu}{m_{\bar{q}}} \right)^{2\alpha} \frac{2C_F^2}{\alpha^2 \epsilon^2}$$
$$\int_0^\infty \frac{d\omega}{\omega} \phi_B^-(\omega) \simeq \frac{1}{m_{\bar{q}}} \left(\frac{\alpha_s}{4\pi} \right)^2 \left(\frac{\mu^2}{m_{\bar{q}}^2} \right)^{2\epsilon} \left(\frac{\nu}{m_{\bar{q}}} \right)^{2\alpha} \frac{6C_F^2 - C_A C_F}{\alpha^2 \epsilon^2}$$

and some more complicated expressions in the coll. sector.

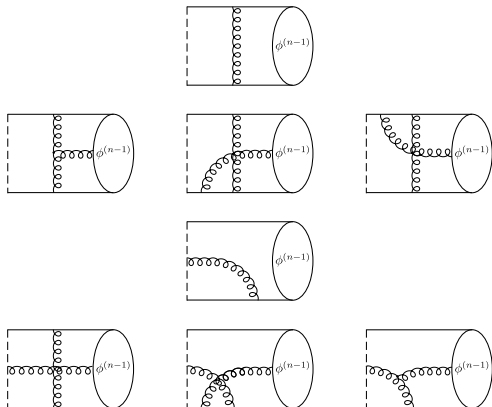
→ again, all $1/\alpha^2$ poles drop out in the sum $ss + cc + sc$ ✓

Note that the mixed soft-coll. contribution is $\sim 1/\alpha^2$! It contains **products** of divergent moments!

Structure of Rapidity Divergences: All-Orders

Q: Can we make some all order statements?

Yes! At least the leading rapidity poles are determined by **recursion relations!** (like a multipl. Z-factor)



Mixing of various two- and three-particle LCDAs at endpoint!

Structure of Rapidity Divergences: All-Orders

Q: Can we make some all order statements?

Yes! At least the leading rapidity poles are determined by **recursion relations!** (like a multipl. Z-factor)

- Gives **exponentiation structures**, e.g.

(like in μe scattering!)

$$\int_0^\infty \frac{d\omega}{\omega^2} \phi_B^+(\omega) \simeq \frac{1}{m_{\bar{q}}^2} \exp \mathcal{E}$$
$$\int_0^\infty \frac{d\omega}{\omega} \phi_B^-(\omega) \simeq \frac{1}{m_{\bar{q}}} \left\{ \exp \mathcal{E} - \frac{C_{FA}}{C_F} \mathcal{E} \exp \mathcal{E} + \frac{C_A}{2C_F} (\exp \mathcal{E} - 1) \right\}$$

- with leading contribution to the (bare) **anomaly exponent**:

$$\mathcal{E} = \frac{\mathcal{F}^{(1)}}{\alpha} \left(\frac{\nu}{m_{\bar{q}}} \right)^\alpha, \quad \mathcal{F}^{(1)} = \frac{\alpha_s C_F}{4\pi} \frac{2}{\varepsilon} \left(\frac{\mu^2}{m_{\bar{q}}^2} \right)^\varepsilon$$

- ✓ With similar relations for the η_c , leading $1/\alpha$ poles cancel when inserted in fact.-theorem!

Structure of Rapidity Divergences: All-Orders

Q: Can we make some all order statements?

Yes! At least the leading rapidity poles are determined by **recursion relations!** (like a multipl. Z-factor)

- For example, rapidity poles cancel in the **product**

$$\int_0^\infty \frac{d\omega}{\omega^2} \phi_B^+(\omega) \times \int_0^1 du \frac{1+\bar{u}}{\bar{u}^2} \phi_{\eta_c}(u) \simeq \frac{1}{m_q^2} \frac{1+\bar{u}_0}{\bar{u}_0^2} \times (2\gamma)^{\mathcal{F}(1)}$$

- ✓ large rapidity logarithms due to collinear anomaly resummed to all orders
- more complicated structures for ξ_{η_c} due to **mixing**

Structure of Rapidity Divergences: All-Orders

Q: Can we make some all order statements?

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- ✓ large rapidity logarithms due to collinear anomaly resummed to all orders
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Remark: Results for inv. moments **process-independent!**

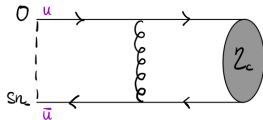
→ similar structure for other hard-exclusive processes!

Endpoint-Refactorization

Q: What is the all-order $\bar{u} \rightarrow 0$ asymptotics of, for example, the bare η_c LCDA?

→ take soft (or soft-collinear) limit of overlapping propagator that carries momentum fraction \bar{u} :

$$\langle \eta_c(p) | \bar{\chi}_c(0) \frac{\not{p}_+}{2} \gamma_5 \chi_c(s n_-) | 0 \rangle = -i E_{\eta_c} f_{\eta_c} \int_0^1 d\bar{u} e^{i\bar{u}s(n-p)} \phi_{\eta_c}(u)$$

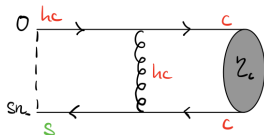


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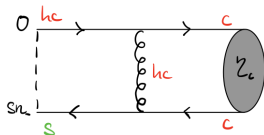


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- **power-counting:** $\phi_{\eta_c}(\bar{u} \rightarrow 0)$ involves the $\psi_{hc}^{(5)}$ splitting into a soft and two coll. quarks!
 - endpoint described by a vacuum matrix element of soft (or soft-collinear) fermion fields
 - same overlap matrix elements appear in $\phi_{B_c}(\omega \rightarrow 0) \Rightarrow$ common overlap ✓
 - **but:** again **iterative** non-additive structure (like soft mode in $e^- \mu^-$ scattering)

4. Conclusion

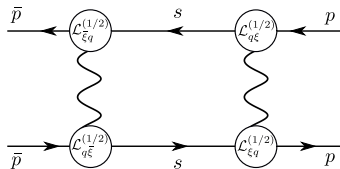
Conclusion

Take-home messages:

1. Endpoint-singularities longstanding problem that prevents a systematic study of **power-corrections** in SCET. General treatment still an **open problem**. Phenomenologically relevant in B physics.
2. Despite the recent progress, they can manifest in a more complicated **non-additive** way, in particular in $2 \rightarrow 2$ processes (or higher multiplicities).
→ exclusive charmless B decays
3. Muon-electron backward-scattering provides a **well-defined perturbative playground** for studying non-trivial aspects of soft-coll factorization in the presence of endpoint-div's.

Backup-Slides

Soft-Enhancement



- soft contribution leading power despite **four** insertions of $\mathcal{L}_{\xi q}^{(1/2)}$:

$$\begin{aligned} \mathcal{O}_2 = & \int d^d x_1 \int d^d x_2 \int d^d x_3 T \left\{ [\bar{\chi}_{hc}^{(e)} \mathcal{A}_{hc}^\perp(0)] [\mathcal{A}_{hc}^\perp \chi_{hc}^{(\mu)}](x_1) \right\} \\ & \times T \left\{ [\bar{\chi}_{hc}^{(\mu)} \mathcal{A}_{hc}^\perp](x_2 + x_3) [\mathcal{A}_{hc}^\perp \chi_{hc}^{(e)}](x_3) \right\} \\ & \times T \left\{ \psi_s^{(e)}(0) \bar{\psi}_s^{(\mu)}(x_{1+}) \psi_s^{(\mu)}(x_{2-} + x_3) \bar{\psi}_s^{(e)}(x_3) \right\} \end{aligned}$$

- multipole expansion w.r.t. hc and $\bar{h}\bar{c}$ fields at different space-time points $x = 0$ and $x = x_3$
 - soft fluctuations $d^4 x_3 \sim 1/\lambda^4$ compensate suppression from $\mathcal{L}_{\xi q}^{(1/2)}$ insertions
 - related to special backward kinematics
- In SCET_{II} the suppression is compensated by **inverse soft derivatives**

Some Backup Formulas

Comparison of the muon-electron scattering and $h \rightarrow \gamma\gamma$ amplitude at the DL level:

- for $h \rightarrow \gamma\gamma$ get standard $h \rightarrow b^*b^*$ Sudakov in integrand:

$$\mathcal{F}_b(z) = 2 \int_0^1 d\xi \int_0^1 d\eta \theta(1 - \xi - \eta) e^{-2\xi\eta z}$$

- for muon-electron backward-scattering the form factor itself appears in the integrand
→ nested structure

$$\mathcal{F}_1(z) = 1 + z \int_0^1 d\xi \int_0^1 d\eta \mathcal{F}_1(\xi^2 z) \theta(1 - \xi - \eta) \mathcal{F}_1(\eta^2 z)$$