$$\mathcal{M}_1 = H_1\langle O_1\rangle + 2\int_0^1 dz\, H_2(z)\langle O_2(z)\rangle + H_2\langle \underline{O_3}\rangle$$

$$= H_1\langle O_1\rangle + 2\int_0^1 dz\, H_2(z)\langle O_2(z)\rangle + H_3\int_0^\infty \frac{d\ell_-}{\ell_-}\int_0^\infty \frac{d\ell_+}{\ell_+} J(-m_H\ell_-)$$
$$\times J(m_H\ell_+) \times S(\ell_-\ell_+)$$

Bare Elements

disclaimer: no refs in the "note/talk".

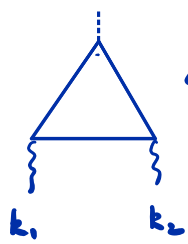
Hard Functions:

1. H_1

$$H_1 = \frac{\alpha_b}{\pi}\cdot\frac{y_b}{\sqrt{2}}\cdot C_F \times \left[H_1^{(1)} + \frac{\alpha_s}{4\pi} H_1^{(2)} + \cdots \right]$$

$$\left(\frac{\alpha_s}{\pi}\frac{y_b}{\sqrt{2}}\cdot T_F\,\delta_{ab} \times \left[H_1^{(0)} + \frac{\alpha_s}{4\pi} H_1^{(1)} + \cdots \right] \right.$$

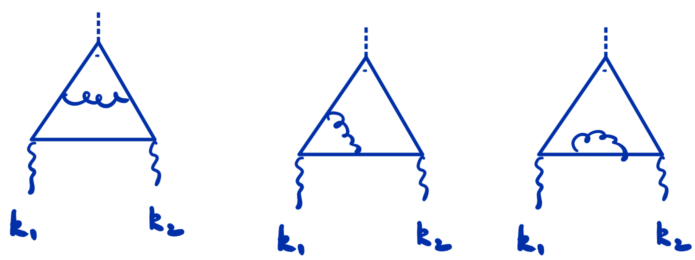
$H_1^{(0)}$:

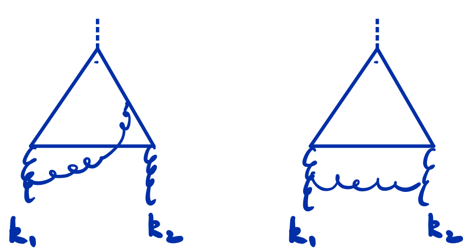


$k_1^2 = k_2^2 = 0, \quad k_1\cdot k_2 = \frac{1}{2} m_H^2$

$(m = 0)$

$H_1^{(1)}$:





$$F_1 = -(-m_H^2)^{1-2\epsilon} \times \frac{\Gamma^3(1-\epsilon)\,\Gamma(-1+2\epsilon)}{\Gamma(1-3\epsilon)}$$

$$F_2 = -(-m_H^2)^{-2\epsilon} \times \frac{\Gamma^4(1-\epsilon)\,\Gamma^2(\epsilon)}{\Gamma^2(2-2\epsilon)}$$

$H\to gg$

$$F_3 = -(-m_H^2)^{-2\epsilon} \times \frac{\Gamma^2(1-\epsilon)\,\Gamma(\epsilon)}{\Gamma(2-2\epsilon)} \quad \frac{\Gamma^2(1-2\epsilon)\,\Gamma(2\epsilon)}{\Gamma(2-3\epsilon)}$$

$$\left(F_4 = -(-m_H^2)^{-2-2\epsilon} \times \Gamma(1-\epsilon)\,\Gamma(1+\epsilon)\,\Gamma(1+2\epsilon) \times \left[\frac{1}{\epsilon^4} - \frac{3\pi^2}{2\epsilon^2} - \frac{25\zeta_3}{\epsilon} + \frac{\pi^4}{12} \right] \right)$$

Remark:

① H_1 is one-scale problem. Its dependence can always be factorized out by dimension analysis!

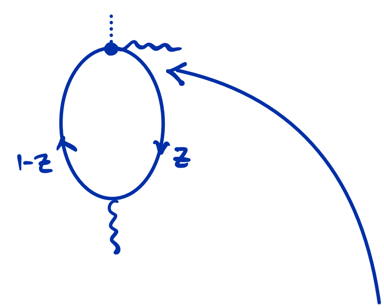
② It is "trivial" to evaluate in terms of Gamma functions or MZVs.

2. $H_2(z)$

Remark: Not only dependens on m_H, which is trivial, but always on momentum fraction z. → non-trivial.

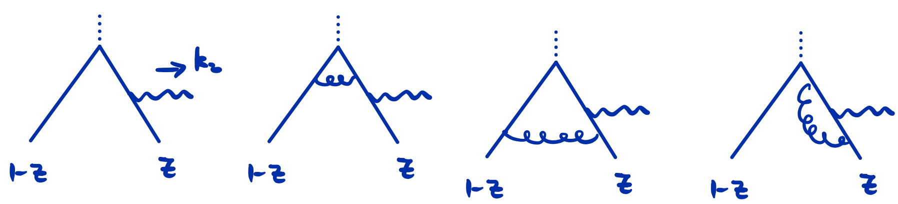
↳ <u>kind of like splitting function.</u>

$$\int_0^1 dz\, H_2(z)\,\langle O_2(z)\rangle \;=\; H_2 \otimes$$



To calculate $H_2(z)$, we "open up" the shrinked vertex:

$H_2(z)$:



To be more precise, calculate on-shell amplitudes of

$h \to \gamma(k_2)\, b(\bar z k_1)\, \bar b(z k_1)$ $\qquad \bar z = 1 - z$

$$\mathcal{M}(h \to \gamma b \bar b) = \int_0^1 dz'\, H_2(z')\, \underline{\langle b(z k_1)\, \bar b(\bar z k_1)\, \gamma(k_2)|\, O_2(z')\, |h\rangle}$$

↳ on-shell → tree level

$$= \frac{y_b}{m_H}\cdot H_2(z)\, \bar u(\bar z k_1)\, \not\varepsilon_\perp^*(k_2)\, \frac{\not n_1}{2}\, v(z k_1)$$

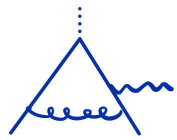
$$H_2 = \frac{y_b}{\sqrt 2}\left[H_2^{(0)} + \frac{\alpha_s}{4\pi} H_2^{(1)} + \dots \right]$$

$$H_2^{(0)} = \frac{1}{z} + \frac{1}{1-z} \qquad\qquad \frac{1}{(z k_1 + k_2)^2} = \frac{1}{z}\,\frac{1}{m_H^2}.$$

i) $[H_2(z)] = - H_3 * J(z m_H^2)$

$$\Downarrow$$
$$[\![H_2(z,\mu)]\!] = H_3(\mu)\times J(z m_H^2,\mu)$$

At NLO, an example:



$1-z \qquad z$

$$= \frac{y_b}{\sqrt{2}}\frac{\alpha_s}{4\pi}(m_H^2)^{-\epsilon}\times C_F\times\frac{\Gamma(1+\epsilon)\Gamma^2(-\epsilon)}{\Gamma(2-2\epsilon)}\times 4\epsilon^2\times\frac{1-z^{-\epsilon}}{1-z}$$

$$H_2(z) = \frac{y_b}{\sqrt{2}}\times\Big\{\frac{1}{z}+\frac{C_F\alpha_s}{4\pi}\times(m_H^2)^{-\epsilon}\times e^{\epsilon\gamma_E}\times\frac{\Gamma(1+\epsilon)\Gamma^2(-\epsilon)}{\Gamma(2-2\epsilon)}\times\Big[\frac{2-4\epsilon-\epsilon^2}{z^{1+\epsilon}}-\frac{2(1-\epsilon)^2}{z}$$
$$-2(1-2\epsilon-\epsilon^2)\frac{1-z^{-\epsilon}}{1-z}\Big] + (z\leftrightarrow 1-z)\Big\}$$

$$\downarrow$$

$$[\![H_2(z)]\!] = \frac{y_b}{\sqrt{2}}\times\Big\{\frac{1}{z}+\frac{C_F\alpha_s}{4\pi}\times(-m_H^2)^{-\epsilon}\times e^{\epsilon\gamma_E}\frac{\Gamma(1+\epsilon)\Gamma^2(-\epsilon)}{\Gamma(2-2\epsilon)}\times\Big[\frac{2-4\epsilon-\epsilon^2}{z^{1+\epsilon}}-\frac{2(1-\epsilon)^2}{z}$$
$$-2(1-2\epsilon-\epsilon^2)(1-z^{-\epsilon})\Big]\Big\}$$

↳ check re-factorization explicitly later on.

Jet function: as a matching coefficient for matching $SCET_I \to SCET_{II}$.

$\begin{cases} k_1 \sim n_1 \\ k_2 \sim n_2 \end{cases}$

It is important for $gg\to H$

p_s: soft-quark

$$\int dx_+ d^{D-2}x_\perp \hat{T}\Big\{\big[\mathcal{G}^\perp_{n_1}(x)\chi_{n_1}(x)\big]^{\alpha i}\bar{\chi}^{\beta j}_{n_1}(0)\Big\}$$

m_H

$$= 2T^a_{ij}\,\Big[\gamma^\perp_\mu\frac{\not{n}_1}{2}\Big]^{\alpha\beta}\int\frac{dp_s^+}{2\pi}e^{-ip_s^+ x_-}\,\frac{iJ(p_s^+ n_1\cdot P)}{p_s^+ + i0}\,\mathcal{G}^{\mu,a}_{n_1,\perp}(0)$$

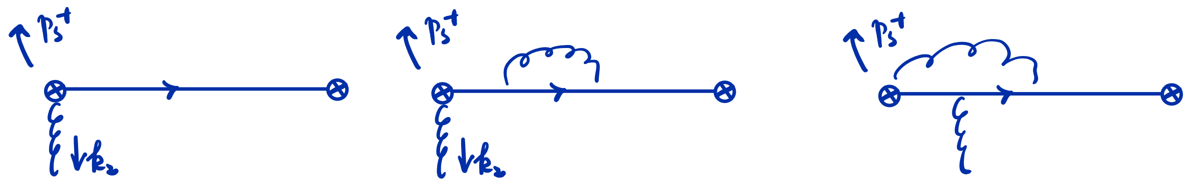
In reality, the most efficient way to calculate the radiative jet function is to evalute on-shell matrix element again.

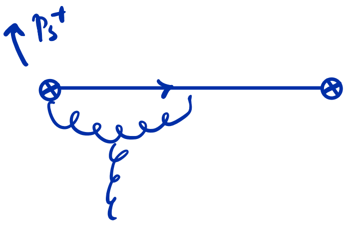
$$\boxed{\begin{aligned}&\int d^Dx\, e^{-ip_s^+\cdot x}\langle 0|\hat{T}\Big\{\big[\mathcal{G}^\perp_{n_1}(x)\chi_{n_1}(x)\big]^{\alpha i}\bar{\chi}^{\beta j}_{n_1}(0)\Big\}|g(k_2,a)\rangle\\ &= g_s T^a_{ij}\,\Big[\not{\varepsilon}_\perp(k_2)\frac{\not{n}}{2}\Big]^{\alpha\beta}\cdot\frac{iJ((p_s^+ + k_2)^2)}{p_s^+ + i0}\,.\qquad \begin{aligned}&\alpha,\beta:\ \text{Dirac}\\ &i,j:\ \text{color}\end{aligned}\end{aligned}}$$

$$J^{(n)}(p^2) \sim (-p^2)^{-n\epsilon}$$

$$J(p^2) = J^{(0)}(p^2) + \frac{\alpha_s}{4\pi} J^{(1)}(p^2) + \cdots$$

Remark: projection $\mathrm{Tr}\left[\frac{\not{n}_1}{2}\gamma_\perp^\mu \gamma_\perp^\nu \frac{\not{\bar{n}}_1}{2}\right] = 2 g_\perp^{\mu\nu}$ makes life easier...





2-loop: 10 MIs

$$\Rightarrow J(p^2) = 1 + \frac{\alpha_s}{4\pi}(-p^2)^{-\epsilon} e^{\epsilon\gamma_E}\cdot \frac{\Gamma(1+\epsilon)\Gamma^2(-\epsilon)}{\Gamma(2-2\epsilon)}\left((C_F - C_A)(2-4\epsilon-\epsilon^2)\right) + O(\alpha_s^2)$$

two-loop see 2112.00018.

Remark: One-scale problem, but its renormalization is highly non-trivial.

Re-factorization 1: $[\![\bar{H}_2(z)]\!] = -H_3 \cdot J(z m_H^2)$

$$H_3 = -\frac{y_b}{\sqrt{2}}\left[1 - \frac{\alpha_s}{4\pi} C_F (-m_H^2)^{-\epsilon}\cdot e^{\epsilon\gamma_E}\cdot \frac{\Gamma(1+\epsilon)\Gamma^2(-\epsilon)}{\Gamma(2-2\epsilon)}\, 2(1-\epsilon)^2\right]$$

Operators and matrix elements.

1. $O_2(z)$: $\begin{cases} m_b \text{ dependence} \leftarrow \text{massive propagator} \\ z \text{ dependence} \end{cases}$

most convenient in momentum space

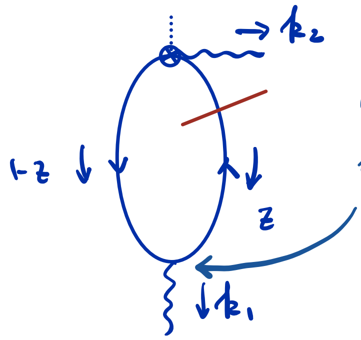
$\bar n_1 = n_2$

cut

$$O_{2,n_1}(z) = h(0)\left[\bar\chi_{n_1}(0)\,\gamma_\mu^\perp \frac{\bar{n}\!\!\!/_1}{2}\,\delta(z m_H + i\bar n_1\cdot\partial)\,\chi_{n_1}(0)\right]\mathcal{A}^\mu_{n_2,\perp}(0)$$

$$\mathcal{A}_n^\mu(x) = e_b\int_{-\infty}^0 ds\,\bar n_\nu F_n^{\nu\mu}(x+s\bar n) \overset{\text{light-cone gauge}}{=} e_b A_n^\mu(x).$$

LO =



already one-loop, needs a $\mathcal{L}_{int} = -ie_b\int d^4x\;\bar\chi\,\mathcal{A}\!\!\!/\,\chi$.

$$\Rightarrow \langle O_2^{(0)}(z)\rangle = e^{\epsilon\gamma_E}\cdot\int\frac{d^D\ell}{(2\pi)^D}\cdot\frac{Num}{[\ell^2-m_b^2]\cdot[(\ell-k_1)^2-m_b^2]}$$

$$Num = -ie_b\cdot e_b\cdot i^2\cdot\delta(zm_H - n\cdot\ell)\times(-1)\times \mathrm{Tr}\left[(\ell\!\!\!/+m_b)\gamma^\nu(\ell\!\!\!/-k\!\!\!/_1-m_b)\cdot\gamma_\perp^\mu\frac{n\!\!\!/}{2}\right]$$

$$= -ie_b^2\cdot(-2m_b m_H^2)\,g_\perp^{\mu\nu}\cdot\delta(zm_H^2 - 2k_2\cdot\ell)$$

$$\int d^D\ell\;\frac{\delta(zm_H^2-2k_2\cdot\ell)}{[\ell^2-m_b^2]\cdot[(\ell-k_1)^2-m_b^2]} = \int_0^1 d\alpha\int d^D\ell\;\frac{\delta(zm_H^2-2k_2\cdot\ell)}{[(\ell-\alpha k_1)^2-m_b^2]^2}$$

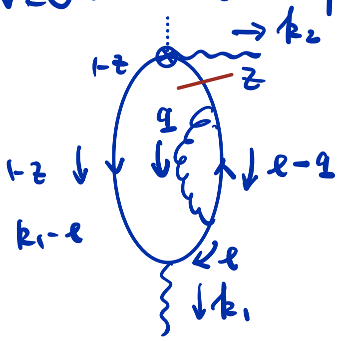
$$= \int_0^1 d\alpha\int d^D\ell\;\frac{\delta\big(2k_2\cdot\ell+(\alpha-z)m_H^2\big)}{[\ell^2-m_b^2]^2}$$

$$\int d^D\ell\;\frac{f(k_2\cdot\ell)}{(\ell^2-m_b^2)^n} = f(0)\int d^D\ell\;\frac{1}{(\ell^2-m^2)^n}.$$

$$= \frac{1}{m_H}\cdot i\pi^{D/2}\cdot(m_b^2)^{-\epsilon}\cdot\Gamma(\epsilon).$$

At NLO, δ-function can be re-phrased by reverse unitarity using IBP. $\quad \delta(x) \propto \frac{1}{x+i\varepsilon} - \frac{1}{x-i\varepsilon}$

NLO: an example:



cut propagator

$$\{\ell^2-m^2,\ (\ell-k_1)^2-m^2,\ (\ell-q)^2-m^2,\ q^2,\ 2k_2\cdot\ell - z m_H^2,$$
$$\underline{2k_2\cdot q,\ 2k_1\cdot q}\ \}$$

$\hookrightarrow$ irreducible scalar products (ISPs)

$$F \sim \int \frac{d^D\ell}{(2\pi)^D}\frac{d^Dq}{(2\pi)^D}\ \frac{\text{Num}}{[\ell^2-m^2]^{\nu_1}[(\ell-k_1)^2-m^2]^{\nu_2}[(\ell-q)^2-m^2]^{\nu_3}(q^2)^{\nu_4}(2k_2\cdot\ell - z m_H^2)^{\nu_5}}$$
$$\times \frac{1}{(2k_1\cdot q)^{\nu_6}\ (2k_2\cdot q)^{\nu_7}}$$

MIs: $\begin{cases} (1\,1\,1\,0\,0\,1\,0) \\ (0\,1\,1\,1\,0\,1\,0) \leftarrow F_2 \\ (0\,2\,1\,1\,0\,1\,0) \end{cases}$

$\mathrm{Disc}(z-\bar{\alpha}) = \underbrace{\delta(z-\bar{\alpha})}$

$$F_2 = e^{2\epsilon\gamma_E}\int \frac{d^D\ell}{i\pi^{D/2}}\frac{d^Dq}{i\pi^{D/2}}\cdot\frac{\delta(2k_2\cdot\ell - z m_H^2)}{[(\ell-k_1)^2-m^2][(\ell-q)^2-m^2]\ q^2}$$

$\bar{\alpha} = 1-\alpha$

$$\Downarrow$$

$$\tilde{F}_2 = e^{2\epsilon\gamma_E}\int \frac{d^D\ell}{i\pi^{D/2}}\frac{d^Dq}{i\pi^{D/2}}\cdot\frac{1}{[(\ell-k_1)^2-m^2][(\ell-q)^2-m^2]\ q^2\ [z m_H^2 - 2k_2\cdot\ell]}$$
$$= \frac{(m^2)^{1-2\epsilon}}{m_H^2}\cdot\frac{\Gamma(\epsilon)\Gamma(2\epsilon)\Gamma(1-\epsilon)}{1-2\epsilon}\int_0^1 d\alpha\ \frac{\alpha^{-1+\epsilon}}{z-\bar{\alpha}}\ {}_2F_1(1-\epsilon, -1+2\epsilon, 1, 1-\alpha)$$

$\tilde{F}_2 \to F_2$ by taking discontinuity w.r.t. z.

$$\mathrm{Disc}\left(\frac{1}{z-\bar{\alpha}}\right) = \delta(z-\bar{\alpha}) \qquad \checkmark.$$

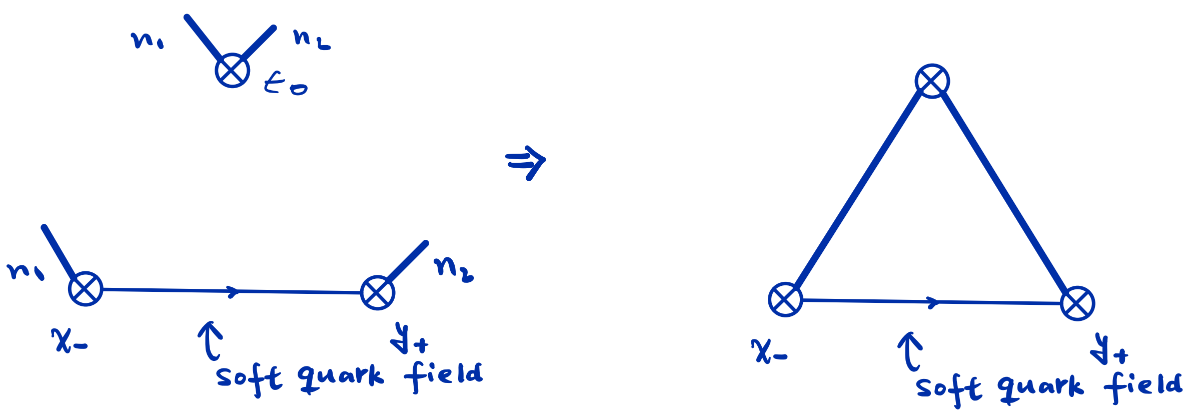
$$\underset{\sim}{\langle O_2(z)\rangle} = \frac{N_c\alpha_b}{2\pi}m_b\left[\underline{e^{\epsilon\gamma_E}\cdot\Gamma(\epsilon)\cdot(m^2)^{-\epsilon}} + \frac{\alpha_s}{4\pi}C_F\times(m^2)^{-2\epsilon}\times[\underset{\sim}{K}(z)+K(\bar{z})]\right]$$

$$\lim_{z\to 0}\left(K(z)+K(\bar z)\right)=\frac{e^{2\epsilon\gamma_E}}{1-2\epsilon}\Big[2(2-3\epsilon+2\epsilon^2)\Gamma^2(\epsilon)+2(1-\epsilon)\Gamma(\epsilon)\Gamma(2\epsilon)\Gamma(-\epsilon)$$
$$+z^{\epsilon}(2-4\epsilon-\epsilon^2)\frac{\Gamma(2\epsilon)\Gamma^2(-\epsilon)}{\Gamma(1-2\epsilon)}\Big]$$

2. Soft-quark soft function.

(vacuum matrix element of Wilson lines with soft-quark fields.)

Remark: There are three positions where Wilson lines locate. x_-, y_+ and 0.



finite segements of Wilson lines

↳ no rapidity divergence.

$$\frac{e_q^2}{\pi}\langle 0|\hat{T}\{\underbrace{S_{n_2}(0)S^{\dagger}_{n_2}(y_+)}_{\text{finite segement}}q_s^{\gamma}(y_+)\bar{q}_s^{\alpha}(x_-)\underbrace{S_{n_1}(x_-)S^{\dagger}_{n_1}(0)}_{\text{finite segement}}\}|0\rangle$$

The momentum space version enters in factorization:

$$\underset{\ell_-\ell_+}{S}(\ell_-\ell_+)=\int\frac{dx_-dy_+}{2}e^{-i\frac{\ell_+x_--\ell_-y_+}{2}}\times\langle 0|\hat{T}\{\cdots\}|0\rangle$$

Hence, ℓ_+ flows in at x_- and ℓ_- flows out at y_-

Remark: ① LO already occurs at one-loop ($\ell_\perp$);

② But light-cone components enter in a non-trivial way, resulting in non-locality !

③ Due to Dirac-structure prefactor in two jet functions only trivial Dirac spinor structure in $S(\ell_-\ell_+)$ enters in O_3;

④ Discontinuity over $\ell_-\ell_+$ has to be taken at last.

LO: $$S^{(0)}(\ell_-\ell_+) = \mathrm{Disc}\left[i \frac{e_b^2}{\pi} N_c \int \frac{d^{D-2}\ell_\perp}{(2\pi)^{D-2}} \frac{i m_a}{\ell^2 - m_a^2 + i\varepsilon}\right]$$

$$= -m \frac{\alpha_b}{\pi} N_c \cdot \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} (\ell_-\ell_+ - m^2)^{-\epsilon} \, \theta(\ell_-\ell_+ - m^2)$$

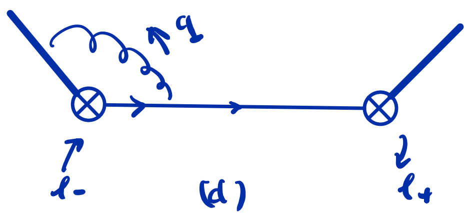
$\nearrow 0$

$\not\ell + m$

NLO : examples

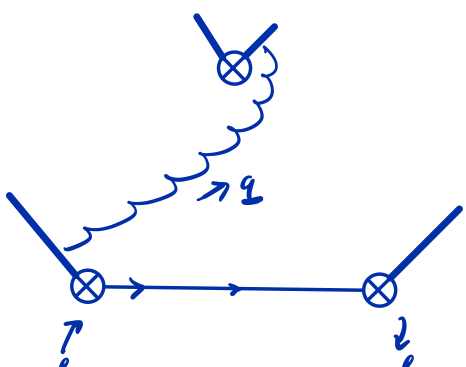
local in ℓ_- & ℓ_+

↳ easy for calculations.

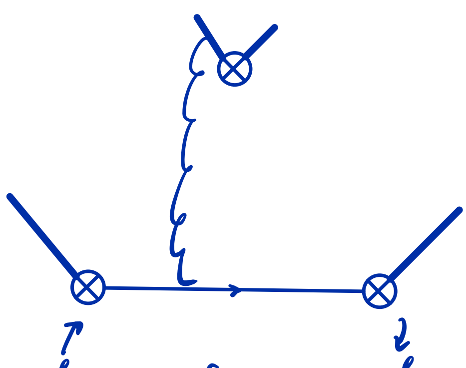


(d)

$$S_d^{(1)}(\ell_-\ell_+) = \mathrm{Disc}\left\{ i \frac{e_b^2 g_s^2}{\pi} C_F N_c \int \frac{d^{D-2}\ell_\perp}{(2\pi)^{D-2}} \int \frac{d^D q}{(2\pi)^D} \frac{\left[(\not\ell - \not q + m)\gamma_\perp (\not\ell + m)\right]^{\gamma\alpha}}{(\ell^2 - m^2)((\ell-q)^2 - m^2)\, q^2\, [n_+\cdot q]}\right\}$$



(f)



(h)

scaless

$$S_f^{(1)} = \mathrm{Disc}\left\{ i\,\frac{e_b^2 g_s^2}{\pi} C_F N_c \int \frac{d^D p}{(2\pi)^{D-2}} \int \frac{d^D q}{(2\pi)^D} \frac{n_1\cdot n_2 \,[\not{p}+m]^{\gamma\alpha}}{[p^2-m^2][q^2][n_2\cdot q][-n_1\cdot q]} \times \delta(\ell_+ - p_+)\,\delta(\ell_- - p_- - q_-) \right\}$$

$$= \mathrm{Disc}\left\{ i\,\frac{e_b^2 g_s^2}{\pi} C_F N_c \int \frac{d^{D-2}\ell_\perp}{(2\pi)^{D-2}} \int \frac{d^D q}{(2\pi)^D} \frac{n_1\cdot n_2 \,[\not{\ell}+m]^{\gamma\alpha}}{[\ell^2-\ell_+ q_- - m^2][q^2][q_+][-q_-]} \right\}$$

$$S_n^{(1)} = \mathrm{Disc}\left\{ i\,\frac{e_b^2 g_s^2}{\pi} C_F N_c \int \frac{d^{D-2}\ell_\perp}{(2\pi)^{D-2}} \int \frac{d^D q}{(2\pi)^D} \frac{[(\not{\ell}+m)\gamma_\perp(\not{\ell}-\not{q}+m)]^{\gamma\alpha}}{[\ell^2-\ell_- q_+ - m^2][(\ell-q)^2+q_+(\ell_- - q_-)-m^2]} \times \frac{1}{q^2(-q_-)} \right\}$$

$$S_d^{(1)} \propto \{ {}_2F_1 ,\, T \cdots \} \times \theta(\ell_-\ell_+ - m^2)$$

$$S_n^{(1)} \propto \{ {}_2F_1 \cdots \} \times \theta(\ell_-\ell_+ - m^2) + \cdots \times \theta(m^2-\ell_-\ell_+)$$

non-locality $\Rightarrow$ $\theta(m^2-\ell_-\ell_+)$.

$$\Rightarrow\ S(w) = -m_b \frac{N_c \alpha_b}{\pi} \times \left[S_a(w)\,\theta(w-m_b^2) + S_b(w)\,\theta(m_b^2-w) \right]$$

$\Rightarrow$ Check the second re-factorization condition

$$[\![O_2(z)]\!] = -\frac{1}{2}\int_0^\infty \frac{d\ell_+}{\ell_+} J(-m_H \ell_+)\, S(z m_H \ell_+) .$$

Subtraction :

$$T_1 = (H_1 + \Delta H_1)\langle O_1 \rangle$$

$$T_2 = 2\int_0^1 dz \Big(H_2(z)\langle O_2(z)\rangle - [\![H_2(z)]\!]\, [\![\langle O_2(z)\rangle]\!] - [\![H_2(\bar z)]\!]\, [\![\langle O_2(\bar z)\rangle]\!] \Big)$$

$$T_3 = H_3 \int_0^{m_H} \frac{d\ell_-}{\ell_-} \int^{a m_H} \frac{d\ell_+}{\ell_+} J(-m_H \ell_-)\, J(m_H \ell_+)\, S(\ell_-\ell_+)$$

Renormalization

bootstrap with the help of re-factorization theorems.
(proved via SCET)

$$[\![H_2(z)]\!] = -H_3 J(zm_H^2) \times \frac{1}{z}$$

$$[\![H_2(z,\mu)]\!] = -H_3(\mu) J(zm_H^2,\mu)\frac{1}{z}$$

$$\hookrightarrow \int_0^\infty dz' [\![H_2(z')]\!][\![Z_{22}^{-1}(z',z)]\!] = -\frac{1}{z}H_3 Z_{33}^{-1}\int_0^\infty dx\, Z_J(zm_H^2,xm_H^2) \times J(xm_H^2)$$

$$Z_{22} \sim Z_{33} Z_J^{-1}$$

$$\hookrightarrow [\![Z_{22}^{-1}]\!] \sim Z_{33}^{-1} Z_J$$

"RG-invariance" of O_3 is (without cutoff)

$$Z_S = Z_{33}\, Z_J^{-1} \otimes Z_J^{-1}$$

$$\begin{cases} O_2(z,\mu) = \int_0^1 dz'\, Z_{22}(z,z')\,O_2(z') + Z_{21}(z)\,O_1 \\ [\![O_2(z,\mu)]\!] = \int_0^\infty dz' [\![Z_{22}(z,z')]\!]\,[\![O_2(z')]\!] + [\![Z_{21}(z)]\!]\,O_1 \end{cases}$$

Cross-check: $Z_{22} \to [\![Z_{22}]\!]$

Example: how soft function renormalizes?

$$Z_S \propto \alpha_s^0 + \alpha_s^1 \ldots$$

$$S(w,\mu) = \int_0^\infty dw'\, Z_S(w,w')\,S(w') \overset{!}{=} \text{finite}$$

$$S(w) = -m_b\frac{\alpha_s \alpha_b}{\pi}\left[S_a(w)\,\theta(w-m_b^2) + S_b(w)\,\theta(m_b^2-w)\right]$$

$$S_a(w) = (w-m_b^2)^{-\epsilon}\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} + O(\alpha_s) \quad \to Z_S$$

$$S_b(w) = \frac{\alpha_s}{4\pi}C_F\cdot\left(\frac{m_b^2}{\mu^2}\right)^{-2\epsilon}\cdot\left[-\frac{4}{\epsilon}\ln\left(1-\frac{w}{m_b^2}\right) + 6\ln^2\left(1-\frac{w}{m_b^2}\right)\right]$$

$$\int_0^\infty dw' \,(w'-m_b^2)^{-\epsilon}\, \theta(w'-m_b^2) \times w \left[\frac{\theta(w-w')}{w(w-w')} + \frac{\theta(w'-w)}{w'(w'-w)} \right]_+$$

$$= \quad (\cdots) \times \theta(w-m_b^2)$$

$$\boxed{+ \ (\cdots) \times \theta(m_b^2-w)}$$

Resummation :

T_1 is hard

① $$T_3(\mu_h) = \mathcal{M}_0 \times \frac{L^2}{2} \begin{array}{c} \begin{array}{ccccc} \text{LL} & \text{NLL} & \text{NLL}' & & \end{array} \\ \left(\begin{array}{ccccc} 1 & & & & \\ \alpha_s L^2 & \alpha_s L & \alpha_s & & \\ \alpha_s^2 L^4 & \alpha_s^2 L^3 & \alpha_s^2 L^2 & \alpha_s^2 L & \alpha_s^2 \\ \alpha_s^3 L^6 & \alpha_s^3 L^5 & \cdots & & \\ \vdots & \vdots & \ddots & & \end{array} \right) \end{array} \begin{array}{l} \leftarrow \text{LO} \\ \leftarrow \text{NLO} \\ \leftarrow \text{NNLO} \end{array}$$

$m_s \to \mu_h$

NLL $\subseteq$ RG-impr. LO

Sudakov type (exponentiate)

$$\hookrightarrow \ \ln \frac{T_3(\mu_h)}{\mathcal{M}_0 \times L^2/2} = \left(\begin{array}{ccccc} \alpha_s L^2 & \alpha_s L & \alpha_s & & \\ \alpha_s^2 L^3 & \alpha_s^2 L^2 & \alpha_s^2 L & \alpha_s^2 & \\ \alpha_s^3 L^4 & \alpha_s^3 L^3 & \alpha_s^3 L^2 & \alpha_s^3 L & \alpha_s^3 \\ \vdots & \ddots & & & \\ \alpha_s^n L^{n+1} & \alpha_s^n L^n & & & \end{array} \right)$$

RG: LO NLO .. NNLO

② T_2 is not Sudakov-type, and very techinical due to mixing and subtraction.

$$T_2(\mu_h) = \mathcal{M}_0 \left(\begin{array}{ccccc} \alpha_s L^2 & \alpha_s L & \alpha_s & & \\ \alpha_s^2 L^3 & \alpha_s^2 L^2 & \alpha_s^2 L & \alpha_s^2 & \\ \alpha_s^3 L^4 & \alpha_s^3 L^3 & \alpha_s^3 L^2 & \alpha_s^3 L & \alpha_s^3 \\ \vdots & \ddots & & & \end{array} \right)$$

$n \geq 0$

$\mathcal{M}$:
- $L^2 \times (\alpha_s^n L^{2n}, \alpha_s^n L^{2n-1}, \alpha_s^n L^{2n-2})$ (NLL')
- RG-improved LO with NLO matching functions. (LO')

NLL' $\subseteq$ RG-improved LO'.

Summary and outlook :

① $H \xrightarrow{b} \gamma\gamma$ is one of the benchmarks of using NLP SCET very successfully.

② Re-factorizations now seem very crucial and fundamental for both SCET-1 & 2. In turn, it makes subtraction method work well (though may complicated)
i.e, not commuting with renor.

③ More evidence for the above.

④ We should push together NLP to precision era ...

⑤ Other generic feature, more anomalous dimensions?
(QFT & EFT)