TMD factorization beyond the leading power

based on [2109.09771], [2204.03856]

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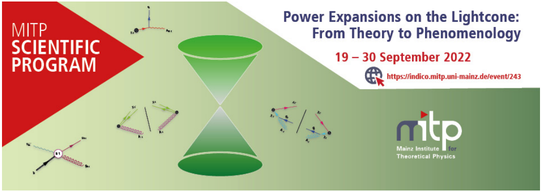
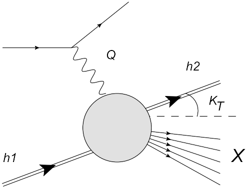


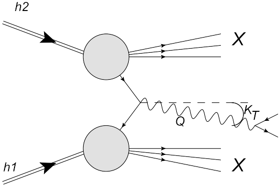
Transverse momentum dependent factorization

$$\frac{d\sigma}{dq_T} \quad \simeq \quad \sigma_0 \int \frac{d^2b}{(2\pi)^2} e^{-i(bq_T)} |C_V(Q)|^2 F_1(x_1, b; Q, Q^2) F_2(x_2, b; Q, Q^2)$$

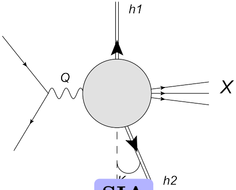
LP term is studied VERY WELL!



SIDIS



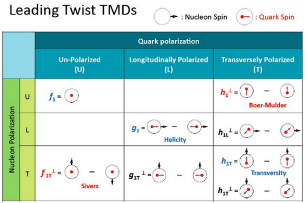
Drell-Yan



SIA

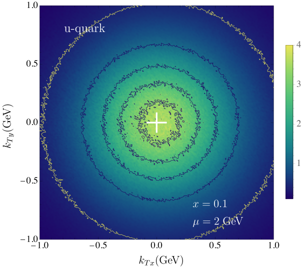
q is momentum of initiating EW-boson
$q^2 = \pm Q^2$
q_T^μ transverse component

$$\begin{cases} Q^2 \gg \Lambda_{QCD}^2 \\ Q^2 \gg q_T^2 \end{cases}$$

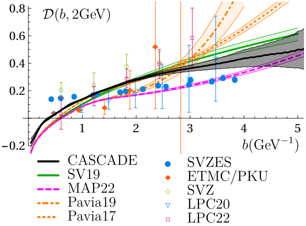


Leading Twist TMDs : Nucleon Spin : Quark Spin

		Quark polarization		
		Un-Polarized (U)	Longitudinally Polarized (L)	Transversely Polarized (T)
Nucleon Polarization	U	f_1 =		$h_1^{\perp}$ = − Boer-Mulder
	L		g_1 = − Helicity	$h_{1L}^{\perp}$ = −
	T	$f_{1T}^{\perp}$ = − Sivers	$g_{1T}^{\perp}$ = −	h_{1T} = − Transversity $h_{1T}^{\perp}$ = −



- Physics of hadron
- Multiple experiments
- Polarization
- Lattice
- ...



Transverse momentum dependent factorization

$$\begin{aligned}
\frac{d\sigma}{dq_T} \quad \simeq \quad & \sigma_0 \int \frac{d^2b}{(2\pi)^2} e^{-i(bq_T)} \Big\{ |C_V(Q)|^2 F_1(x_1, b; Q, Q^2) F_2(x_2, b; Q, Q^2) \qquad \longleftarrow \text{LP} \\
& + \left(\frac{q_T}{Q}; \frac{k_T}{Q}; \frac{M}{Q} \right) [C_2(Q) \otimes F_3(x, b; Q, Q^2) F_4(x, b; Q, Q^2)](x_1, x_2) \qquad \longleftarrow \text{NLP} \\
& + \left(\frac{q_T^2}{Q^2}; \frac{k_T q_T}{Q^2}; \ldots \right) [C_3(Q) \otimes F_5(x, b; Q, Q^2) F_6(x, b; Q, Q^2)](x_1, x_2) \qquad \longleftarrow \text{NNLP} \\
& + \ldots
\end{aligned}$$

Outline

- General approach to TMD factorization
- TMD factorization at NLP/NLO
- Systematics of power-suppressed TMD operators (distributions)

Disclaimer: so far, pure theory...

Motivation

▶ Sub-leading power observables

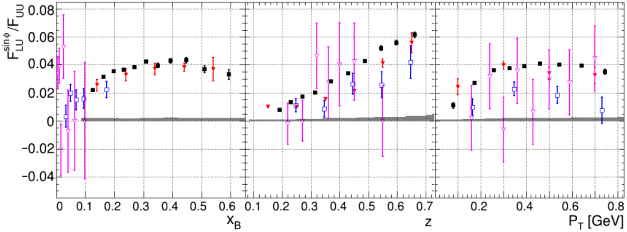
$$F_{LU}^{\sin\phi_h} = \frac{2M}{Q}\mathcal{C}\left[-\frac{\hat{h}\cdot k_T}{M_h}\left(x\overset{\text{twist-3 pdf}}{\boxed{e}}\underset{\text{Collins FF}}{\boxed{H_1^\perp}} + \frac{M_h}{M}\overset{\text{unpolarized PDF}}{\boxed{f_1}}\underset{\text{twist-3 FF}}{\frac{\boxed{\tilde{G}^\perp}}{z}}\right) + \frac{\hat{h}\cdot p_T}{M}\left(x\overset{\text{twist-3 t-odd PDF}}{\boxed{g^\perp}}\underset{\text{unpolarized FF}}{\boxed{D_1}} + \frac{M_h}{M}\overset{\text{Boer-Mulders}}{\boxed{h_1^\perp}}\underset{\text{twist-3 FF}}{\frac{\boxed{\tilde{E}}}{z}}\right)\right]$$

by Timothy B. Hayward at QCD-N

To describe it, one needs TMD factorization at NLP.

- ▶ **JLab**
- ▶ LHC

[CLAS, 2101.03544]



Motivation

- **Sub-leading power observables**

$$\frac{d\sigma}{dp_T^Z\, dy^Z\, dm^Z\, d\cos\theta\, d\phi} = \frac{3}{16\pi}\frac{d\sigma^{U+L}}{dp_T^Z\, dy^Z\, dm^Z}$$

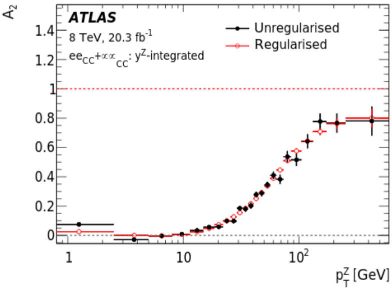
$$\Big\{(1+\cos^2\theta)+\frac{1}{2}\,A_0(1-3\cos^2\theta)+A_1\;\sin 2\theta\;\cos\phi$$

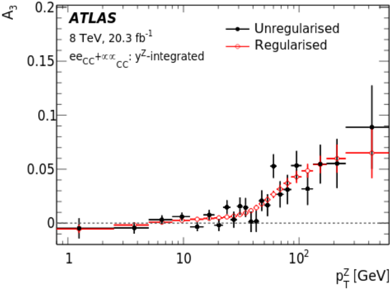
$$+\frac{1}{2}\,A_2\;\sin^2\theta\;\cos 2\phi+A_3\;\sin\theta\;\cos\phi+A_4\;\cos\theta$$

$$+A_5\;\sin^2\theta\;\sin 2\phi+A_6\;\sin 2\theta\;\sin\phi+A_7\;\sin\theta\;\sin\phi\Big\}.$$

To describe it, one needs TMD factorization at NNLP.

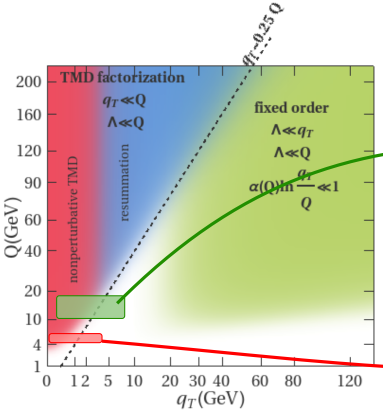
- JLab
- **LHC**



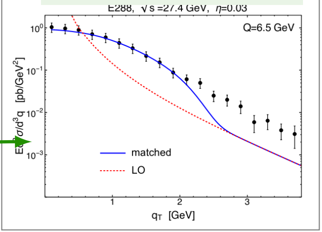


Motivation

- Sub-leading power observables
- **Increase of applicability domain**

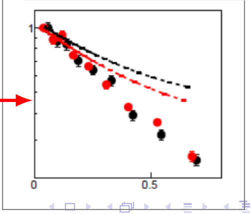


[Bacchetta,et al,1901.06916]



$$\frac{q_T}{Q}$$

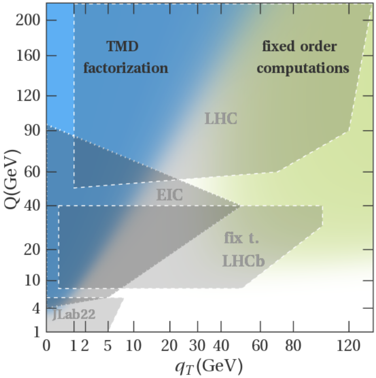
COMPASS preliminary



$$\frac{p_T}{zQ}$$

Motivation

- Sub-leading power observables
- **Increase of applicability domain**



Phase space of EIC is centered directly in the transition region

COMPASS, JLab
fixed target LHCb
have large contribution of power corrections

Motivation

- Sub-leading power observables
- Increase of applicability domain
- **Restoration of broken properties**

LP TMD factorization breaks EM-gauge invariance

$$W^{\mu\nu} = \int dy e^{iqy} \langle J^{\mu}(y) J^{\nu}(0) \rangle$$

$$q_\mu W^{\mu\nu} = 0$$

$$W^{\mu\nu}_{\text{LP}} = g_T^{\mu\nu} |C_V|^2 \mathcal{F}(F_1 F_2)$$

$$q_\mu W^{\mu\nu}_{\text{LP}} \sim q_T^{\nu}$$

- The violation is of the NLP
- Similar problem with frame-dependence (GTMD case)

Sources of power corrections

*(exact)=known at all powers

$$\frac{d\sigma}{dP.S.} = \sigma_{PS} L_{\mu\nu} W^{\mu\nu}$$

Phase space PC **(exact)**

e.g. SIDIS $\sigma_{PS} = \dfrac{\pi}{\sqrt{1+\gamma^2 \frac{\mathbf{p}_{h\perp}^2}{z^2Q^2}}}$

Leptonic tensor **(exact)**

e.g. un.DY with fid.cuts

$$L^{\mu\nu} \sim (l^\mu l'^\nu + l^\nu l'^\mu - g^{\mu\nu}(ll'))\mathcal{P}$$

- l, l' with transverse parts
- $\mathcal{P}$ fiducial part

Hadronic tensor (e.g. DY)

$$W^{\mu\nu} = \int \frac{d^4 y e^{i(yq)}}{(2\pi)^4} \langle p_1 p_2 | J^\mu(y) | X \rangle \langle X | J^\nu | p_1 p_2 \rangle$$

Factorized in powers of

$$\frac{q_T}{q^+}, \quad \frac{q_T}{q^-}$$

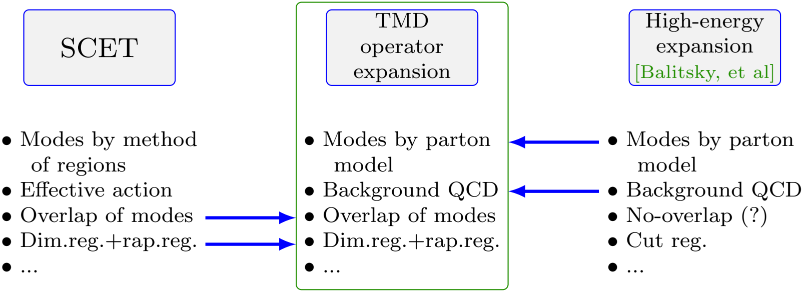
Power corrections due to frame choice **(exact)**

$$p_1^+ \gg p_1^-, \qquad p_2^- \gg p_2^+$$

e.g. SIDIS $q_T^2 = \dfrac{p_\perp^2}{z^2} \dfrac{1+\gamma^2}{1-\gamma^2 \frac{p_\perp^2}{z^2Q^2}}$

There are already computations of TMD factorization at NLP/NNLP

- Small-x-like
  - Balitksy [1712.09389],[2012.01588],...
  - Nefedov, Saleev, [1810.04061],[1906.08681]
- SCET
  - Ebert, et al [1812.08189] *fixed order at NNLP*
  - Ebert, et al [2112.07680] *NLP at LO*
  - Inglis-Whalen, et al [2105.09277]
- Boer, Mulders, Pijlman [hep-ph/0303034]



SCET

- Modes by method of regions
- Effective action
- Overlap of modes
- Dim.reg.+rap.reg.
- ...

TMD operator expansion

- Modes by parton model
- Background QCD
- Overlap of modes
- Dim.reg.+rap.reg.
- ...

High-energy expansion [Balitsky, et al]

- Modes by parton model
- Background QCD
- No-overlap (?)
- Cut reg.
- ...

Background field method for parton physics
(in a nutshell)

$$\langle h|T\, J^\mu(z)J^\nu(0)|h\rangle = \int [D\bar{q}DqDA]e^{iS_{\text{QCD}}}\Psi^*[\bar{q},q,A]J^\mu(z)J^\nu(0)\Psi[\bar{q},q,A]$$

Cannot be integrated since Ψ is unknown

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Parton model

Ψ contains only collinear particles

$$\Psi[\bar{q},q,A] \to \Psi[\bar{q}_{\bar{n}},q_{\bar{n}},A_{\bar{n}}]$$

$$\{\partial_{+},\partial_{-},\partial_{T}\}q_{\bar{n}} \lesssim \{1,\lambda^{2},\lambda\}q_{\bar{n}}$$

Integral can be partially computed

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$$\{\partial_+,\partial_-,\partial_T\}q_{\bar n}\lesssim\{1,\lambda^2,\lambda\}q_{\bar n}$$

Integral can be partially computed

Background technique

$$\begin{aligned} q &= q_{\bar n} + \psi \\ A &= A_{\bar n} + B \end{aligned}$$

- $q_{\bar n}, A_{\bar n}$: background (external field)
- ψ, B: dynamical (to be integrated)

$$\langle h|T\,J^\mu(z)J^\nu(0)|h\rangle = \int [D\bar q_{\bar n} Dq_{\bar n} DA_{\bar n}] e^{iS_{\text{QCD}}}\Psi^*[\bar q,q,A]\mathcal{J}^{\mu\nu}_{\text{eff}}[\bar q_{\bar n},q_{\bar n},A_{\bar n}](z)\Psi[\bar q,q,A]$$

$$\mathcal{J}^{\mu\nu}_{\text{eff}} = \int [D\bar\psi D\psi DB] e^{iS_{\text{QCD}}+iS_{\text{back}}[\bar q,q,A]}J^\mu[q+\psi](z)J^\nu[q+\psi](0)$$

Generating function for operator product expansion

Background QCD with 2-component background

$$q \to q_n + q_{\bar{n}} + \psi \qquad A^\mu \to A^\mu_n + A^\mu_{\bar{n}} + B^\mu$$

- **Technical note:** S_{QCD} for 2-component background has 1PI vertices!

collinear-fields
(associated with hadron 1)

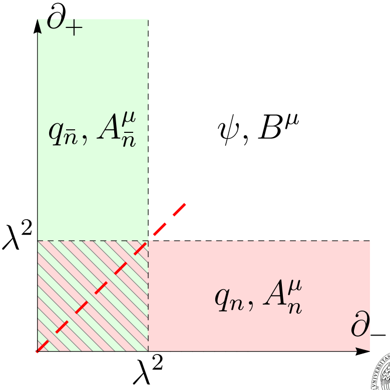
$$\{\partial_+,\,\partial_-,\,\partial_T\}\,q_{\bar{n}} \lesssim Q\{1,\,\lambda^2,\,\lambda\}\,q_{\bar{n}},$$

$$\{\partial_+,\,\partial_-,\,\partial_T\}\,A^\mu_{\bar{n}} \lesssim Q\{1,\,\lambda^2,\,\lambda\}\,A^\mu_{\bar{n}},$$

anti-collinear-fields
(associated with hadron 2)

$$\{\partial_+,\,\partial_-,\,\partial_T\}\,q_n \lesssim Q\{\lambda^2,\,1,\,\lambda\}\,q_n,$$

$$\{\partial_+,\,\partial_-,\,\partial_T\}\,A^\mu_n \lesssim Q\{\lambda^2,\,1,\,\lambda\}\,A^\mu_n.$$



TMD operator expansion
is conceptually similar to ordinary OPE
The only difference is counting rule for y

$$W^{\mu\nu}_{\text{DY}} = \int \frac{d^4y}{(2\pi)^4} e^{-i(yq)} \sum_X \langle p_1, p_2 | J^{\mu\dagger}(y) | X \rangle \langle X | J^\nu(0) | p_1, p_2 \rangle,$$

$$W^{\mu\nu}_{\text{SIDIS}} = \int \frac{d^4y}{(2\pi)^4} e^{i(yq)} \sum_X \langle p_1 | J^{\mu\dagger}(y) | p_2, X \rangle \langle p_2, X | J^\nu(0) | p_1 \rangle,$$

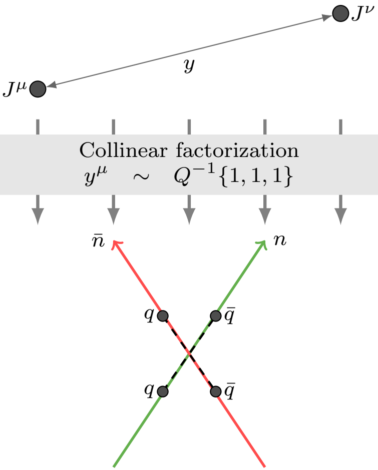
$$W^{\mu\nu}_{\text{SIA}} = \int \frac{d^4y}{(2\pi)^4} e^{i(yq)} \sum_X \langle 0 | J^{\mu\dagger}(y) | p_1, p_2, X \rangle \langle p_1, p_2, X | J^\nu(0) | 0 \rangle.$$

$$(q \cdot y) \sim 1 \qquad \Rightarrow \qquad \{y^+, y^-, y_T\} \sim \{\frac{1}{q^-}, \frac{1}{q^+}, \frac{1}{q_T}\} \sim \frac{1}{Q}\{1, 1, \lambda^{-1}\}$$

To be accounted in operator expansion

$$z_T^\mu \partial_\mu q \sim \text{NLP}, \qquad y_T^\mu \partial_\mu q \sim \text{LP}$$

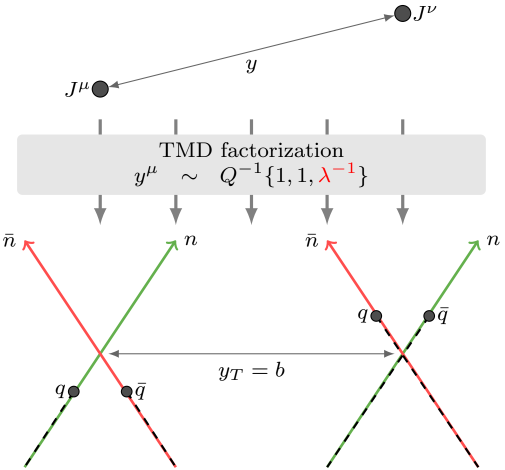
TMD operator expansion
has different geometry



Collinear factorization
$y^\mu \quad \sim \quad Q^{-1}\{1,1,1\}$

$\bar q_i(\lambda n)[\lambda n, 0]q_j(0)$

Two
light-cone operators
⇓
Two
parton distribution function
PDFs & FFs

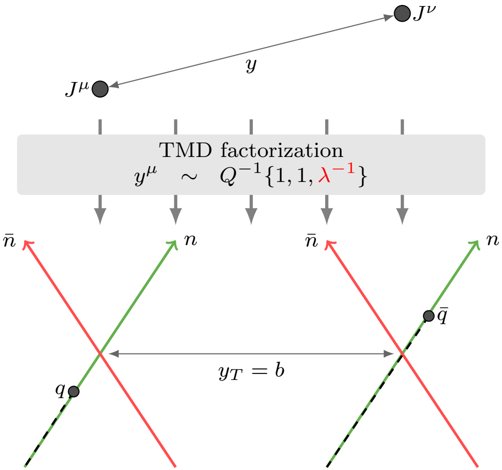


TMD factorization

$$y^\mu \quad \sim \quad Q^{-1}\{1, 1, \lambda^{-1}\}$$

TMD operator expansion has different geometry

Four
light-cone operators
$\Downarrow$
Two
TMD distributions
TMDPDFs & TMDFFs



TMD factorization

$y^\mu \quad \sim \quad Q^{-1}\{1, 1, \lambda^{-1}\}$

$$\bar{q}_i(\lambda n + b)[\lambda n + b, \pm\infty n + b] \quad [\pm\infty n, 0]q_j(0)$$

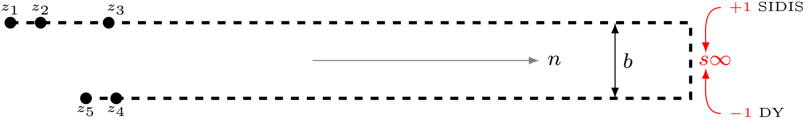
TMD operator expansion has different geometry

Four
light-cone operators
⇓
Two
TMD distributions
TMDPDFs & TMDFFs

TMD operators and their divergences

Any TMD operator is the product of two *semi-compact* operators

$$\mathcal{O}_{NM}(\{z_1,...,z_n\},b) = U_N(\{z_1,...\},b)U_M(\{...,z_n\},b)$$



$$\boxed{\mathcal{O}^{\text{bare}}_{NM}(\{z_1,...,z_n\},b) = R(b^2)Z_{U_N}(\{z_1,...\}) \otimes Z_{U_M}(\{...,z_n\}) \otimes \mathcal{O}_{NM}(\mu,\zeta)}$$

- UV divergence for U_N
- UV divergence for U_M
- Rapidity divergence

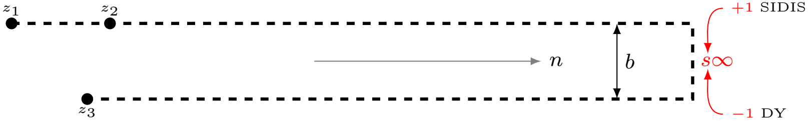
Three independent divergences
Three renormalization constants
Three anomalous dimensions

TMD-twist-(1,1)

Usual TMDs

$U_1 = [..]\xi =$ good-component of quark field (twist-1)

$$\widetilde{\Phi}_{11}^{[\Gamma]}(\{z_1, z_2\}, b) = \langle p, s|\bar{\xi}(z_1 n + b)..\frac{\Gamma}{2}..\xi(z_2 n)|p, s\rangle$$



$$\begin{aligned}
\mu^2 \frac{d}{d\mu^2}\widetilde{\Phi}_{11}(\{z_{1,2}\}, b; \mu, \zeta) &= (\widetilde{\gamma}_1(z_1, \mu, \zeta) + \widetilde{\gamma}_1(z_2, \mu, \zeta))\,\widetilde{\Phi}_{11}(\{z_{1,2}\}, b; \mu, \zeta) \\
\zeta \frac{d}{d\zeta}\widetilde{\Phi}_{11}(\{z_{1,2}\}, b; \mu, \zeta) &= -\mathcal{D}(b, \mu)\widetilde{\Phi}_{11}(\{z_{1,2}\}, b; \mu, \zeta)
\end{aligned}$$

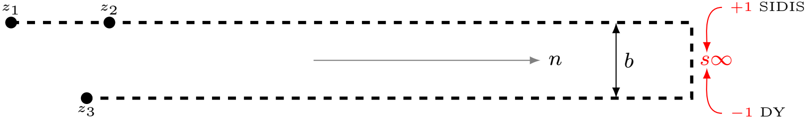
- $\gamma_1 =$ anomalous dimension of U_1 (N^{3}LO)
- $\mathcal{D} =$ CS kernel (NP)

TMD-twist-(2,1)

Appear at NLP

$U_1 = [..]\xi =$ good-component of quark field (twist-1)
$U_2 = [..]F_{\mu+}[..]\xi =$ good-components of gluon and quark fields (twist-2)

$$\widetilde{\Phi}^{[\Gamma]}_{21}(\{z_1, z_2, z_3\}, b) = \langle p, s|\bar{\xi}(z_1 n + b)..F_{\mu+}(z_2 n + b)..\frac{\Gamma}{2}..\xi(z_3 n)|p, s\rangle$$



$$\mu^2 \frac{d}{d\mu^2}\widetilde{\Phi}_{21}(\{z_{1,2,3}\}, b; \mu, \zeta) = (\widetilde{\gamma}_2(z_1, z_2, \mu, \zeta) + \widetilde{\gamma}_1(z_3, \mu, \zeta))\,\widetilde{\Phi}_{21}(\{z_{1,2,3}\}, b; \mu, \zeta)$$
$$\zeta \frac{d}{d\zeta}\widetilde{\Phi}_{21}(\{z_{1,2,3}\}, b; \mu, \zeta) = -\mathcal{D}(b, \mu)\widetilde{\Phi}_{21}(\{z_{1,2,3}\}, b; \mu, \zeta)$$

- $\gamma_1 =$ anomalous dimension of U_1 (N^3LO)
- $\gamma_2 =$ anomalous dimension of U_2 (LO)
- $\mathcal{D} =$ CS kernel (NP)

Similar for TMD-twist-(1,2)

TMD operators of different TMD-twists

(1,1)

$$O_{11}(z,b)=\bar{\xi}(zn+b)[...]\Gamma[...]\xi(0)$$

$\Gamma=\{\gamma^+,\gamma^+\gamma^5,\sigma^{\alpha+}\}$
⇒ well known 8 TMD distributions

(1,2) & (2,1)

$$\begin{aligned} O_{21}(z_{1,2},b) &= \bar{\xi}(z_1n+b)[..]F_{\mu+}(z_2+b)[...]\Gamma[...]\xi(0) \\ O_{12}(z_{1,2},b) &= \bar{\xi}(z_1n+b)[...]\Gamma[...]F_{\mu+}(z_2)[..]\xi(0) \end{aligned}$$

- $\Gamma=\{\gamma^+,\gamma^+\gamma^5,\sigma^{\alpha+}\}$
- 32 TMD distributions
- Related by charge-conjugation ⇔ complex/real

(1,3) & (3,1) & (2,2)

$$\begin{aligned} O_{31;1}(z_{1,2,3},b) &= \bar{\xi}..F_{\mu+}..F_{\nu+}[...]\Gamma[...]\xi(0) \\ O_{22}(z_{1,2,3},b) &= \bar{\xi}..F_{\mu+}[...]\Gamma[...]F_{\nu+}..\xi(0) \\ O_{31;2}(z_{1,2,3},b) &= \bar{\xi}..(\bar{\xi}..\Gamma_2..\xi)[...]\Gamma[...]\xi(0) \\ O_{31;3}(z_{1,2},b) &= \bar{\xi}..F_{-+}[...]\Gamma[...]\xi(0) \\ & ... \end{aligned}$$

- Quasi-partonic and non-quasi-partonic

Operators with different TMD-twists do not mix
renormalization/evolution is independent
independent TMD distributions

Evolution of TMD operator with TMD-twist=(N,M)

$$O_{NM}(\{z_1, ..., z_k\}, b) = \overline{U}_N(\{z_1, ...\}, b) U_M(\{..., z_k\}, 0_T)$$

- Each light-cone operator U renormalizes independently (because there is a finite y_T between them)

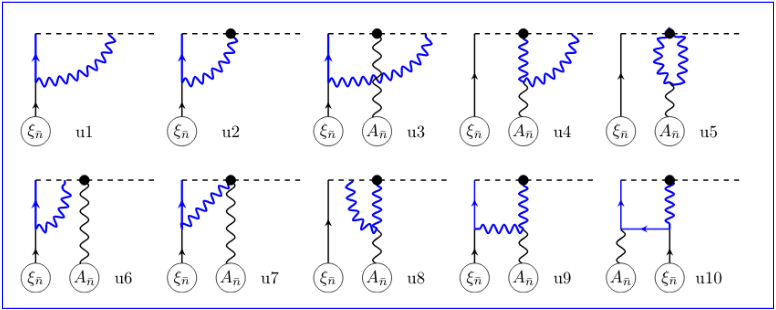
$$\mu \frac{d}{d\mu} U_N(\{z_1, ...\}, b) = \gamma_N \otimes U_N(\{z_1, ...\}, b)$$

- Light-cone operators with different N do not mix (Lorentz invariance!)
- Evolution of TMD operator

$$\mu \frac{d}{d\mu} O_{NM}(\{z_1, ...\}, b) = (\overline{\gamma}_N + \gamma_M) \otimes O_{NM}(\{z_1, ...\}, b)$$

Anatomy of anomalous dimension

$$\begin{aligned}
\widetilde{\gamma}_1(z,\mu,\zeta) &= a_s(\mu)C_F\left(\frac{3}{2}+\ln\left(\frac{\mu^2}{\zeta}\right)+2\ln\left(\frac{q^+}{-s\partial_z^+}\right)\right)+\mathcal{O}(a_s^2),\\
\widetilde{\gamma}_2(z_2,z_3,\mu,\zeta) &= a_s(\mu)\Big\{\mathbb{H}_{z_2z_3}+C_F\left(\frac{3}{2}+\ln\left(\frac{\mu^2}{\zeta}\right)\right)\\
&\qquad +C_A\ln\left(\frac{q^+}{-s\partial_{z_2}^+}\right)+2\left(C_F-\frac{C_A}{2}\right)\ln\left(\frac{q^+}{-s\partial_{z_3}^+}\right)\Big\}+\mathcal{O}(a_s^2),
\end{aligned}$$



Anatomy of anomalous dimension

quark AD + cusp

$$\widetilde{\gamma}_1(z,\mu,\zeta) = a_s(\mu)C_F\boxed{\left(\frac{3}{2}+\ln\left(\frac{\mu^2}{\zeta}\right)\right.}\left.+2\ln\left(\frac{q^+}{-s\partial_z^+}\right)\right)+\mathcal{O}(a_s^2),$$

$$\widetilde{\gamma}_2(z_2,z_3,\mu,\zeta) = a_s(\mu)\Big\{\mathbb{H}_{z_2z_3}+\boxed{C_F\left(\frac{3}{2}+\ln\left(\frac{\mu^2}{\zeta}\right)\right)}$$

$$+C_A\ln\left(\frac{q^+}{-s\partial_{z_2}^+}\right)+2\left(C_F-\frac{C_A}{2}\right)\ln\left(\frac{q^+}{-s\partial_{z_3}^+}\right)\Big\}+\mathcal{O}(a_s^2),$$

Anatomy of anomalous dimension

quark AD + cusp

$$\widetilde{\gamma}_1(z,\mu,\zeta) = a_s(\mu)C_F\left(\boxed{\frac{3}{2}+\ln\left(\frac{\mu^2}{\zeta}\right)}+2\ln\left(\frac{q^+}{-s\partial_z^+}\right)\right)+\mathcal{O}(a_s^2),$$

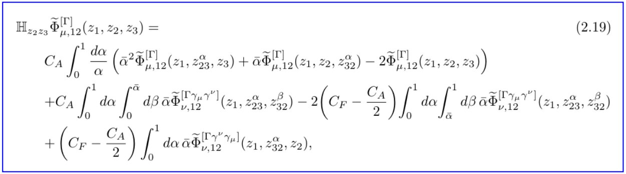
$$\widetilde{\gamma}_2(z_2,z_3,\mu,\zeta) = a_s(\mu)\Big\{\boxed{\mathbb{H}_{z_2z_3}}+\boxed{C_F\left(\frac{3}{2}+\ln\left(\frac{\mu^2}{\zeta}\right)\right)}$$

BFLK

quasi-partonic-kernel

[Bukhvostov, Frolov, Lipatov, Kuraev, 1985]

$$+C_A\ln\left(\frac{q^+}{-s\partial_{z_2}^+}\right)+2\left(C_F-\frac{C_A}{2}\right)\ln\left(\frac{q^+}{-s\partial_{z_3}^+}\right)\Big\}+\mathcal{O}(a_s^2),$$



$$\mathbb{H}_{z_2z_3}\widetilde{\Phi}^{[\Gamma]}_{\mu,12}(z_1,z_2,z_3)= \tag{2.19}$$
$$C_A\int_0^1\frac{d\alpha}{\alpha}\left(\bar{\alpha}^2\widetilde{\Phi}^{[\Gamma]}_{\mu,12}(z_1,z_{23}^{\alpha},z_3)+\bar{\alpha}\widetilde{\Phi}^{[\Gamma]}_{\mu,12}(z_1,z_2,z_{32}^{\alpha})-2\widetilde{\Phi}^{[\Gamma]}_{\mu,12}(z_1,z_2,z_3)\right)$$
$$+C_A\int_0^1 d\alpha\int_0^{\bar{\alpha}}d\beta\,\bar{\alpha}\widetilde{\Phi}^{[\Gamma\gamma_\mu\gamma^\nu]}_{\nu,12}(z_1,z_{23}^{\alpha},z_{32}^{\beta})-2\left(C_F-\frac{C_A}{2}\right)\int_0^1 d\alpha\int_{\bar{\alpha}}^1 d\beta\,\bar{\alpha}\widetilde{\Phi}^{[\Gamma\gamma_\mu\gamma^\nu]}_{\nu,12}(z_1,z_{23}^{\alpha},z_{32}^{\beta})$$
$$+\left(C_F-\frac{C_A}{2}\right)\int_0^1 d\alpha\,\bar{\alpha}\widetilde{\Phi}^{[\Gamma\gamma^\nu\gamma_\mu]}_{\nu,12}(z_1,z_{32}^{\alpha},z_2),$$

Anatomy of anomalous dimension

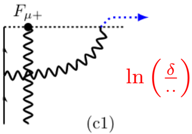
quark AD + cusp

$$\widetilde{\gamma}_1(z,\mu,\zeta) = a_s(\mu)C_F\left(\frac{3}{2}+\ln\left(\frac{\mu^2}{\zeta}\right)+2\ln\left(\frac{q^+}{-s\partial_z^+}\right)\right)+\mathcal{O}(a_s^2),$$

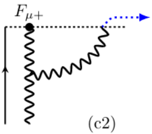
$$\widetilde{\gamma}_2(z_2,z_3,\mu,\zeta) = a_s(\mu)\Big\{\mathbb{H}_{z_2z_3}+C_F\left(\frac{3}{2}+\ln\left(\frac{\mu^2}{\zeta}\right)\right)+C_A\ln\left(\frac{q^+}{-s\partial_{z_2}^+}\right)+2\left(C_F-\frac{C_A}{2}\right)\ln\left(\frac{q^+}{-s\partial_{z_3}^+}\right)\Big\}+\mathcal{O}(a_s^2),$$

BFLK
quasi-partonic-kernel

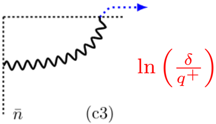
Remnants of collinear divergences
(canceled by SF/reg. by cut)



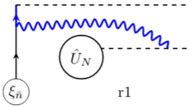
(c1)



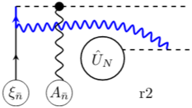
(c2)



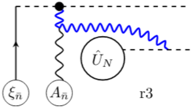
(c3)



r1



r2



r3

$$\widetilde{R}\left(b^2, \frac{\delta^+}{\nu^+}\right) = 1 - 4a_s C_F \Gamma(-\epsilon)\left(-\frac{b^2\mu^2}{4e^{-\gamma_E}}\right)^{\epsilon} \ln\left(\frac{\delta^+}{\nu^+}\right) + O(a_s^2).$$

Rapidity divergence arise from the interaction with the far end of neighbour Wilson line

General facts

- Multiplicatively renormalizable (for QP operators)
- Same for all QP operators (up to overall-color representation)
- Structure for non-QP operator is unknown (in progress)

- Basis of operators ✓
- Anomalous dimensions ✓
- $\Rightarrow$ Wilson lines
- $\Rightarrow$ Hard coefficient (NLO)

Computing hard-coefficient

Keldysh thechnique
to deal with
causality structure

$$J^{(+)\mu}(y)J^{(-)\nu}(0)$$

Details & examples
in [2109.09711]

Computing hard-coefficient

$$\boxed{J^{(+)\mu}(y)J^{(-)\nu}(0)}$$

Details & examples in [2109.09711]

(power) Expand in background fields
sort operators by TMD-twist

$$\begin{aligned}&\bar{q}_{\bar{n}}(y^- n + y_T)\gamma_T^{\mu}q_n(y^+\bar{n} + y_T)\bar{q}_n(0)\gamma_T^{\nu}q_{\bar{n}}(0) + \bar{\psi}_{\bar{n}}(y)\gamma_T^{\mu}q_n(y^+\bar{n} + y_T)\bar{q}_n(0)\gamma_T^{\nu}q_{\bar{n}}(0) + ...\\ &+n^{\mu}\bar{q}_{\bar{n}}(y^- n + y_T)\gamma^- q_n(y^+\bar{n} + y_T)\bar{q}_n(0)\gamma_T^{\nu}q_{\bar{n}}(0) + ...\\ &+y^+\bar{q}_{\bar{n}}(y^- n + y_T)\overleftarrow{\partial}_-\gamma^- q_n(y^+\bar{n} + y_T)\bar{q}_n(0)\gamma_T^{\nu}q_{\bar{n}}(0) + ...\end{aligned}$$

Computing hard-coefficient

Details & examples
in [2109.09711]

$$J^{(+)\mu}(y)J^{(-)\nu}(0)$$

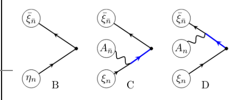
(power) Expand in background fields
sort operators by TMD-twist

$$\bar q_{\bar n}(y^-n+y_T)\gamma_T^\mu q_n(y^+\bar n+y_T)\bar q_n(0)\gamma_T^\nu q_{\bar n}(0)+\bar\psi_{\bar n}(y)\gamma_T^\mu q_n(y^+\bar n+y_T)\bar q_n(0)\gamma_T^\nu q_{\bar n}(0)+\dots$$
$$+n^\mu\bar q_{\bar n}(y^-n+y_T)\gamma^- q_n(y^+\bar n+y_T)\bar q_n(0)\gamma_T^\nu q_{\bar n}(0)+\dots$$
$$+y^+\bar q_{\bar n}(y^-n+y_T)\overleftarrow{\partial}_-\gamma^- q_n(y^+\bar n+y_T)\bar q_n(0)\gamma_T^\nu q_{\bar n}(0)+\dots$$

(loop) Integrate over fast components
with 2-bcg.QCD action

at least NLO is needed
to confirm factorization
(WL direction,
pole-cancelation)

$$\mathcal{J}^{\mu\nu}_{\rm NLP}=-\frac{n^\mu\gamma^\rho_{T,ij}\gamma^\nu_{T,kl}+n^\nu\gamma^\mu_{T,ij}\gamma^\rho_{T,kl}}{N_c}\Big(\frac{\partial_\rho}{\partial_+}\mathcal{O}^{li}_{11,\bar n}\overline{\mathcal{O}}^{jk}_{11,n}+\frac{\partial_\rho}{\partial_+}\overline{\mathcal{O}}^{jk}_{11,\bar n}\mathcal{O}^{li}_{11,n}\Big)$$
$$-\frac{\bar n^\mu\gamma^\rho_{T,ij}\gamma^\nu_{T,kl}+\bar n^\nu\gamma^\mu_{T,ij}\gamma^\rho_{T,kl}}{N_c}\Big(\mathcal{O}^{li}_{11,\bar n}\frac{\partial_\rho}{\partial_-}\overline{\mathcal{O}}^{jk}_{11,n}+\overline{\mathcal{O}}^{jk}_{11,\bar n}\frac{\partial_\rho}{\partial_-}\mathcal{O}^{li}_{11,n}\Big)$$
$$+ig\frac{\delta_{ij}\gamma^\nu_{T,kl}}{N_c}\left\{\mathbb{O}^{li}_{21,\bar n}\left(\frac{\bar n^\mu}{\overrightarrow{\partial}_-}-\frac{n^\mu}{\overleftarrow{\partial}_+}\right)\overline{\mathcal{O}}^{jk}_{11,n}-\overline{\mathbb{O}}^{jk}_{21,\bar n}\left(\frac{\bar n^\mu}{\overrightarrow{\partial}_-}-\frac{n^\mu}{\overleftarrow{\partial}_+}\right)\mathcal{O}^{li}_{11,n}\right.$$



Coincides with [Boer,Mulders,Pijlman,03]

Process dependence

The background can be taken in any gauge (since it is gauge invariant)

- Light-cone gauge kills operators with $A_{+,\bar{n}}$ and $A_{-,n}$ (~ 1 in power counting).
- Convenient choice of gauges
  - Collinear field $A_+ = 0$
  - Anti-Collinear field $A_- = 0$
  - Dynamical field: **Feynman gauge**
- **However** one needs to specify boundary condition. The result depends on it.

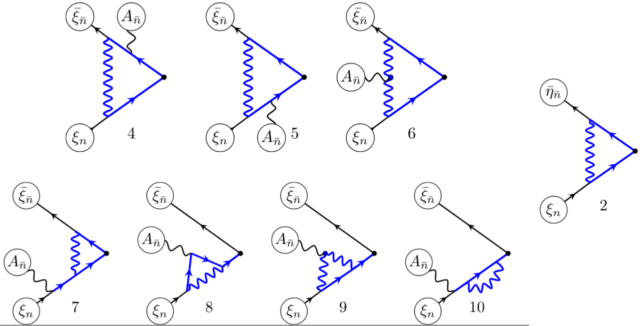
$$A^{\mu}_{\bar{n}}(z) = -g\int_{-\infty}^{0} d\sigma F^{\mu+}_{\bar{n}}(z+n\sigma) \quad \textbf{vs.} \quad A^{\mu}_{\bar{n}}(z) = -g\int_{+\infty}^{0} d\sigma F^{\mu+}_{\bar{n}}(z+n\sigma)$$

$$\bar{q}[z, z-\infty n] \quad \textbf{vs.} \quad \bar{q}[z, z+\infty n]$$

etc.

To specify boundary and WL direction, we should go to NLO

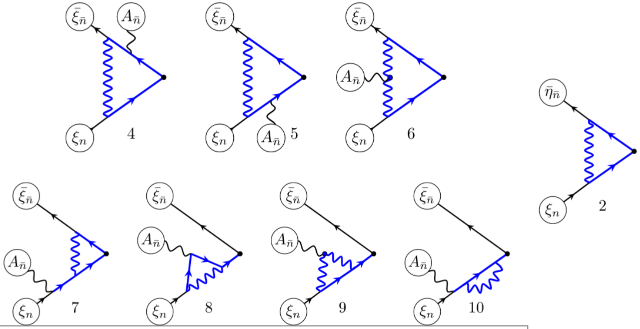
NLO expression in position space



$$\text{diag}_4 + ... + \text{diag}_{10} + \text{diag}_2^{\bar\xi A\xi-\text{part}} = ga_s\frac{\Gamma(-\epsilon)\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)}$$
$$\int\frac{dz^+dz^-}{4^\epsilon\pi}\frac{1}{[-2z^+z^-+i0]^{1-\epsilon}}\int_0^1 ds\Big\{(z^+\bar n^\mu - z^-n^\mu)C_F\frac{2-\epsilon}{\epsilon}\,\mathcal{K}(1,1)$$
$$-\left(C_F\frac{\epsilon(1+\epsilon)}{(1-\epsilon)^2}+C_A\frac{1-\epsilon-\epsilon^2}{(1-\epsilon)^2}\right)\Big[(\epsilon z^-n^\mu+(1-\epsilon)z^+\bar n^\mu)\,\mathcal{K}(s,1)-z^+\bar n^\mu\,\mathcal{K}(0,1)\Big]$$
$$+\left(C_F-\frac{C_A}{2}\right)\frac{2(1-\epsilon-\epsilon^2)}{\epsilon(1-\epsilon)^2}\Big[(\epsilon z^-n^\mu+(1-\epsilon)z^+\bar n^\mu)\mathcal{K}(1,s)-z^+\bar n^\mu\,\mathcal{K}(1,0)\Big]\Big\},$$

$$\mathcal{K}(s,t)=\bar\xi_{\bar n}(sz^-n)\not{A}_{\bar n,T}(tz^-n)\xi_n(z^+\bar n)$$

NLO expression in position space



Depends on boundary conditions

$$\text{diag}_4 + ... + \text{diag}_{10} + \text{diag}_2^{\bar\xi A\xi-\text{part}} = g a_s \frac{\Gamma(-\epsilon)\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)}$$

$$\int \frac{dz^+dz^-}{4^\epsilon \pi} \frac{1}{[-2z^+z^- + i0]^{1-\epsilon}} \int_0^1 ds \Big\{ (z^+\bar n^\mu - z^- n^\mu) C_F \frac{2-\epsilon}{\epsilon} \mathcal{K}(1,1)$$

$$- \left(C_F \frac{\epsilon(1+\epsilon)}{(1-\epsilon)^2} + C_A \frac{1-\epsilon-\epsilon^2}{(1-\epsilon)^2} \right) \Big[(\epsilon z^- n^\mu + (1-\epsilon) z^+ \bar n^\mu)\, \mathcal{K}(s,1) - z^+ \bar n^\mu\, \mathcal{K}(0,1) \Big]$$

$$+ \left(C_F - \frac{C_A}{2} \right) \frac{2(1-\epsilon-\epsilon^2)}{\epsilon(1-\epsilon)^2} \Big[(\epsilon z^- n^\mu + (1-\epsilon) z^+ \bar n^\mu) \mathcal{K}(1,s) - z^+ \bar n^\mu\, \mathcal{K}(1,0) \Big] \Big\},$$

$$\mathcal{K}(s,t) = \bar\xi_{\bar n}(s z^- n) \not{A}_{\bar n,T}(t z^- n) \xi_n(z^+ \bar n)$$

NLO expression in position space

$$I = \int_{-\infty}^{\infty} dz^+ dz^- \, \frac{f_{\bar{n}}(z^-) f_n(z^+)}{[-2z^+ z^- + i0]^\alpha}$$

f's are TMDPDFs or TMDFFs →

	for DY	for SIDIS	for SIA	
$f_{\bar{n}}(z^-)$ is analytical in	lower	lower	upper	half-plane.
$f_n(z^+)$ is analytical in	lower	upper	upper	half-plane.

NLO expression in position space

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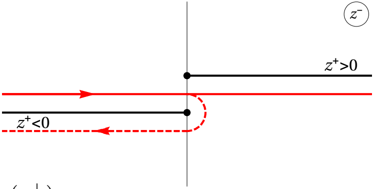

$$I=\int_{-\infty}^{0}dz^+\,\frac{f_n(z^+)}{(-2z^+)^\alpha}\left(I_0+I_1+I_2+I_\infty\right),\qquad I_C=\int_C\frac{f_{\bar n}(z^-)}{(z^-)^\alpha}$$

NLO expression in position space

$$I=\int_{-\infty}^{\infty}dz^+dz^-\,\frac{f_{\bar n}(z^-)f_n(z^+)}{[-2z^+z^-+i0]^\alpha}$$

f's are TMDPDFs or TMDFFs

	for DY	for SIDIS	for SIA	
$f_{\bar n}(z^-)$ is analytical in	lower	lower	upper	half-plane.
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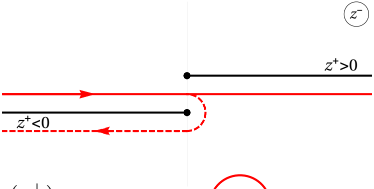


$$I=\int_{-\infty}^{0}dz^+\,\frac{f_n(z^+)}{(-2z^+)^\alpha}\,(I_0+I_1+I_2+I_\infty),\qquad I_C=\int_C\frac{f_{\bar n}(z^-)}{(z^-)^\alpha}$$

0

for DY:	$\lim\limits_{z^-\to-\infty}A_{\bar n}^\mu(z)=0,$	$\lim\limits_{z^+\to-\infty}A_n^\mu(z)=0,$
for SIDIS:	$\lim\limits_{z^-\to+\infty}A_{\bar n}^\mu(z)=0,$	$\lim\limits_{z^+\to-\infty}A_n^\mu(z)=0,$
for SIA:	$\lim\limits_{z^-\to+\infty}A_{\bar n}^\mu(z)=0,$	$\lim\limits_{z^+\to+\infty}A_n^\mu(z)=0.$

Fields at ∞

(= interaction with transverse link)

Reproduce ordinary rules!

NLO computation

Extra facts

- At LP and NLP one Sudakov form factor is needed (exchange diagrams are NNLP)
- Computation for Sudakov is done for LP and NLP both at NLO
  - Position space
  - LP is well known (up to N^3LO) and coincides
  - Twist-(1,1) part of NLP is the same as LP (**"Reparametrization invariance"**)
    - Required by EM gauge invariance **Non-trivial check**
  - Twist-(1,2) part is totally new
- The UV and rapidity divergences of NLP operators computed independently
  - (position space) BFLK part coincide with [Braun,Manashov,09]
  - (momentum space) "Coincides" with [Beneke, et al, 17] (up to missed channels)
- Checks
  - Pole parts of hard coefficient and operators cancel **very non-trivial check**
  - Some diagrams are computed in momentum space **check**

$$H \otimes \underbrace{[Z_{U_1}(\frac{1}{\epsilon}) \otimes Z_{U_2}(\frac{1}{\epsilon})R(b)]}_{\text{TMD 1}} \otimes \underbrace{[Z_{U_1}(\frac{1}{\epsilon}) \otimes Z_{U_2}(\frac{1}{\epsilon})R(b)]}_{\text{TMD 2}} = \text{finite}$$

TMD factorization at NLP in the terms of operators

Momentum space

Effective operator for any process (DY, SIDIS, SIA)

$$
\begin{aligned}
\mathcal{J}^{\mu\nu}_{\text{eff}}(q) = \int \frac{d^2b}{(2\pi)^2} e^{-i(qb)} \Bigg\{ & \int dx d\tilde{x} \delta\left(x - \frac{q^+}{p_1^+}\right) \delta\left(\tilde{x} - \frac{q^-}{p_2^-}\right) |C_1|^2 \mathcal{J}^{\mu\nu}_{1111}(x,\tilde{x},b) \qquad (6.17) \\
& + \int [dx] d\tilde{x} \delta\left(\tilde{x} - \frac{q^-}{p_2^-}\right) \\
& \quad \times \left(\delta\left(x_1 - \frac{q_1^+}{p_1^+}\right) C_1^* C_2(x_{2,3}) \mathcal{J}^{\mu\nu}_{1211}(x,\tilde{x},b) + \delta\left(x_3 + \frac{q_1^+}{p_1^+}\right) C_2^*(x_{1,2}) C_1 \mathcal{J}^{\mu\nu}_{2111}(x,\tilde{x},b) \right) \\
& + \int dx [d\tilde{x}] \delta\left(x - \frac{q^+}{p_1^+}\right) \\
& \quad \times \left(C_1^* C_2(\tilde{x}_{2,3}) \delta\left(\tilde{x}_1 - \frac{q^-}{p_2^-}\right) \mathcal{J}^{\mu\nu}_{1112}(x,\tilde{x},b) + C_2^*(\tilde{x}_{1,2}) C_1 \delta\left(\tilde{x}_3 + \frac{q^-}{p_2^-}\right) \mathcal{J}^{\mu\nu}_{1121}(x,\tilde{x},b) \right) \\
& + \ldots \Bigg\}
\end{aligned}
$$

TMD factorization at NLP in the terms of operators

Momentum space

Effective operator for any process (DY, SIDIS, SIA)

$$\mathcal{J}^{\mu\nu}_{\text{eff}}(q) = \int \frac{d^2b}{(2\pi)^2} e^{-i(qb)} \Bigg\{ \int dx d\tilde{x} \delta\left(x - \frac{q^+}{p_1^+}\right) \delta\left(\tilde{x} - \frac{q^-}{p_2^-}\right) |C_1|^2 \boxed{\mathcal{J}^{\mu\nu}_{1111}(x,\tilde{x},b)} \tag{6.17}$$

$$+ \int [dx] d\tilde{x} \delta\left(\tilde{x} - \frac{q^-}{p_2^-}\right)$$

$$(x,\tilde{x},b)\Big)$$

$$(x,\tilde{x},b)\Big)$$

$$\begin{aligned}
\mathcal{J}^{\mu\nu}_{1111}(x,\tilde{x},b) &= \frac{\gamma^{\mu}_{T,ij}\gamma^{\nu}_{T,kl}}{N_c}\Big(\mathcal{O}^{li}_{11,\bar{n}}(x,b)\overline{\mathcal{O}}^{jk}_{11,n}(\tilde{x},b) + \overline{\mathcal{O}}^{jk}_{11,\bar{n}}(x,b)\mathcal{O}^{li}_{11,n}(\tilde{x},b)\Big) \\
&+ i\frac{n^{\mu}\gamma^{\rho}_{T,ij}\gamma^{\nu}_{T,kl} + n^{\nu}\gamma^{\mu}_{T,ij}\gamma^{\rho}_{T,kl}}{q^+ N_c}\Big(\partial_\rho\mathcal{O}^{li}_{11,\bar{n}}(x,b)\overline{\mathcal{O}}^{jk}_{11,n}(\tilde{x},b) + \partial_\rho\overline{\mathcal{O}}^{jk}_{11,\bar{n}}(x,b)\mathcal{O}^{li}_{11,n}(\tilde{x},b)\Big) \\
&+ i\frac{\bar{n}^{\mu}\gamma^{\rho}_{T,ij}\gamma^{\nu}_{T,kl} + \bar{n}^{\nu}\gamma^{\mu}_{T,ij}\gamma^{\rho}_{T,kl}}{q^- N_c}\Big(\mathcal{O}^{li}_{11,\bar{n}}(x,b)\partial_\rho\overline{\mathcal{O}}^{jk}_{11,n}(\tilde{x},b) + \overline{\mathcal{O}}^{jk}_{11,\bar{n}}(x,b)\partial_\rho\mathcal{O}^{li}_{11,n}(\tilde{x},b)\Big),
\end{aligned}$$

- Operators of $(1,1)\times(1,1)$ (ordinary TMDs)

$$\mathcal{O}^{ij}_{11}(x,b) = p_+ \int \frac{d\lambda}{2\pi} e^{-ix\lambda p_+} \bar{q}_j[\lambda n + b, \pm\infty n + b][\pm\infty n, 0]q_i$$

- Contains LP and NLP (total derivatives)
- Restores EM gauge invariance up to λ^3

$$q_\mu J^{\mu\nu}_{1111} \quad \sim \quad (p_1^- q_T \; + \; p_2^+ q_T) J_{1111}$$

TMD factorization at NLP in the terms of operators

Momentum space

Effective operator for any process (DY, SIDIS, SIA)

$$\mathcal{J}^{\mu\nu}_{\text{eff}}(q) = \int \frac{d^2b}{(2\pi)^2} e^{-i(qb)} \Bigg\{ \int dx d\tilde{x} \delta\left(x - \frac{q^+}{p_1^+}\right) \delta\left(\tilde{x} - \frac{q^-}{p_2^-}\right) |C_1|^2 \mathcal{J}^{\mu\nu}_{1111}(x,\tilde{x},b) \tag{6.17}$$

$$+ \int [dx] d\tilde{x} \delta\left(\tilde{x} - \frac{q^-}{p_2^-}\right)$$

$$\times \left(\delta\left(x_1 - \frac{q_1^+}{p_1^+}\right) C_1^* C_2(x_{2,3}) \mathcal{J}^{\mu\nu}_{1211}(x,\tilde{x},b) + \delta\left(x_3 + \frac{q_1^+}{p_1^+}\right) C_2^*(x_{1,2}) C_1 \mathcal{J}^{\mu\nu}_{2111}(x,\tilde{x},b) \right)$$

$$+ \int dx [d\tilde{x}] \delta\left(x - \frac{q^+}{p_1^+}\right)$$

$$(x,\tilde{x},b)\Big)$$

$$\mathcal{J}^{\mu\nu}_{1211}(x,\tilde{x},b) =$$
$$\frac{ig}{x_2}\left(\frac{\bar{n}^\nu}{q^-} - \frac{n^\nu}{q^+}\right) \frac{\gamma^\mu_{T,ij}\delta_{kl}}{N_c} \left(\mathcal{O}^{jk}_{12,\bar{n}}(x,b) \overline{\mathcal{O}}^{jk}_{11,n}(\tilde{x},b) - \overline{\mathcal{O}}^{jk}_{12,\bar{n}}(x,b) \mathcal{O}^{li}_{11,n}(\tilde{x},b) \right)$$

- Operators of $(1,2) \times (1,1)$
  $$\mathcal{O}^{ij}_{12}(x_{1,2,3},b) = p_+^2 \int \frac{dz_{1,2,3}}{2\pi} e^{-ix^i z_i p_+} \bar{q}_j [z_1 n + b, \pm\infty n + b][\pm\infty n, z_2 n] \gamma^\mu F_{\mu+} [z_2 n, z_3 n] q_i$$
- EM gauge invarint only up to NNLP
  $$q_\mu J^{\mu\nu}_{1211} \quad \sim \quad (p_1^- \ + \ p_2^+) J_{1211}$$

TMD factorization at NLP in the terms of operators

Momentum space

Effective operator for any process (DY, SIDIS, SIA)

$$\mathcal{J}^{\mu\nu}_{\text{eff}}(q) = \int \frac{d^2b}{(2\pi)^2} e^{-i(qb)} \Bigg\{ \int dx d\tilde{x} \delta \left(x - \frac{q^+}{p_1^+} \right) \delta \left(\tilde{x} - \frac{q^-}{p_2^-} \right) \boxed{|C_1|^2} \mathcal{J}^{\mu\nu}_{1111}(x, \tilde{x}, b) \qquad (6.17)$$

$$+ \int [dx] d\tilde{x} \delta \left(\tilde{x} - \frac{q^-}{p_2^-} \right)$$

$$\times \left(\delta \left(x_1 - \frac{q_1^+}{p_1^+} \right) \boxed{C_1^* C_2(x_{2,3})} \mathcal{J}^{\mu\nu}_{1211}(x, \tilde{x}, b) + \delta \left(x_3 + \frac{q_1^+}{p_1^+} \right) \boxed{C_2^*(x_{1,2}) C_1} \mathcal{J}^{\mu\nu}_{2111}(x, \tilde{x}, b) \right)$$

$$+ \int dx [d\tilde{x}] \delta \left(x - \frac{q^+}{p_1^+} \right)$$

$$(x, \tilde{x}, b) \Big)$$

$$C_1 = 1 + a_s C_F \left(-\mathbf{L}_Q^2 + 3\mathbf{L}_Q - 8 + \frac{\pi^2}{6} \right) + O(a_s),$$

$$C_2(x_{1,2}) = 1 + a_s \Bigg[C_F \left(-\mathbf{L}_Q^2 + \mathbf{L}_Q - 3 + \frac{\pi^2}{6} \right) + C_A \frac{x_1 + x_2}{x_1} \ln \left(\frac{x_1 + x_2}{x_2} \right)$$

$$+ \left(C_F - \frac{C_A}{2} \right) \frac{x_1 + x_2}{x_2} \ln \left(\frac{x_1 + x_2}{x_1} \right) \left(2\mathbf{L}_Q - \ln \left(\frac{x_1 + x_2}{x_1} \right) - 4 \right) \Bigg]$$

- C_1 is know up to N^{3}LO
- C_2 **(here is only the real part of it)**

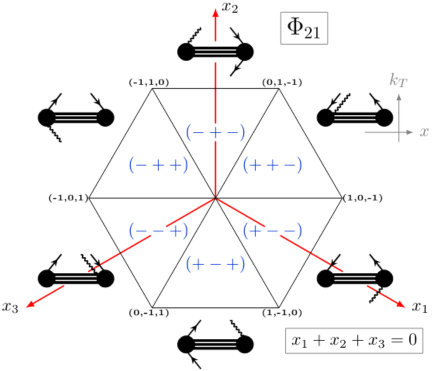
To momentum-fraction space

$$\widetilde{\Phi}_{11}^{[\Gamma]}(z_1,z_2,b) = p^+\int_{-1}^{1} dx e^{ix(z_1-z_2)p^+}\Phi_{11}^{[\Gamma]}(x,b),$$

$$\widetilde{\Phi}_{\mu,21}^{[\Gamma]}(z_1,z_2,z_3,b) = (p^+)^2\int [dx] e^{-i(x_1z_1+x_2z_2+x_3z_3)p^+}\Phi_{\mu,21}^{[\Gamma]}(x_1,x_2,x_3,b),$$

$$\widetilde{\Phi}_{\mu,12}^{[\Gamma]}(z_1,z_2,z_3,b) = (p^+)^2\int [dx] e^{-i(x_1z_1+x_2z_2+x_3z_3)p^+}\Phi_{\mu,12}^{[\Gamma]}(x_1,x_2,x_3,b),$$

$$\int [dx] = \int_{-1}^{1} dx_1 \int_{-1}^{1} dx_2 \int_{-1}^{1} dx_3\, \delta(x_1+x_2+x_3)$$



Support domain $|x_i|<1$
momentum-fractions
could be **positive or negative**

some papers miss this point

- important for divergences-cancelation
- agreement with collinear evolution
- evolution mixture

Evolution equations in the momentum-fraction space

has involved structure

$$\mu^2\frac{d}{d\mu^2}\Phi^{[\Gamma]}_{\mu,21} = \left(\boxed{\frac{\Gamma_{\rm cusp}}{2}\ln\left(\frac{\mu^2}{\zeta}\right)} + \Upsilon_{x_1x_2x_3} + 2\pi i\, s\, \Theta_{x_1x_2x_3}\right)\Phi^{[\Gamma]}_{\mu,21}$$
$$+\mathbb{P}^A_{x_2x_1}\otimes\Phi^{[\gamma^\nu\gamma^\mu\Gamma]}_{\nu,21} + \mathbb{P}^B_{x_2x_1}\otimes\Phi^{[\gamma^\mu\gamma^\nu\Gamma]}_{\nu,21},$$
$$\mu^2\frac{d}{d\mu^2}\Phi^{[\Gamma]}_{\mu,12} = \left(\boxed{\frac{\Gamma_{\rm cusp}}{2}\ln\left(\frac{\mu^2}{\zeta}\right)} + \Upsilon_{x_3x_2x_1} + 2\pi i\, s\, \Theta_{x_3x_2x_1}\right)\Phi^{[\Gamma]}_{\mu,12}$$
$$+\mathbb{P}^A_{x_2x_3}\otimes\Phi^{[\Gamma\gamma^\mu\gamma^\nu]}_{\nu,12} + \mathbb{P}^B_{x_2x_3}\otimes\Phi^{[\Gamma\gamma^\nu\gamma^\mu]}_{\nu,12},$$

Rapidity evolution is the same

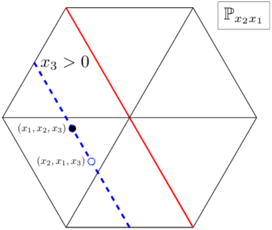
$\Gamma_{\rm cusp}$-part is the same

$$\boxed{\begin{aligned}\zeta\frac{d}{d\zeta}\Phi^{[\Gamma]}_{\mu,12}(x_1,x_2,x_3,b;\mu,\zeta) &= -\mathcal{D}(b,\mu)\Phi^{[\Gamma]}_{\mu,12}(x_1,x_2,x_3,b;\mu,\zeta),\\ \zeta\frac{d}{d\zeta}\Phi^{[\Gamma]}_{\mu,21}(x_1,x_2,x_3,b;\mu,\zeta) &= -\mathcal{D}(b,\mu)\Phi^{[\Gamma]}_{\mu,21}(x_1,x_2,x_3,b;\mu,\zeta).\end{aligned}}$$

Evolution equations in the momentum-fraction space

has involved structure

$$\begin{aligned}
\mu^2\frac{d}{d\mu^2}\Phi^{[\Gamma]}_{\mu,21} &= \left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right)+\Upsilon_{x_1x_2x_3}+2\pi i\,s\,\Theta_{x_1x_2x_3}\right)\Phi^{[\Gamma]}_{\mu,21}\\
&\qquad\boxed{+\mathbb{P}^A_{x_2x_1}\otimes\Phi^{[\gamma^\nu\gamma^\mu\Gamma]}_{\nu,21}+\mathbb{P}^B_{x_2x_1}\otimes\Phi^{[\gamma^\mu\gamma^\nu\Gamma]}_{\nu,21},}\\
\mu^2\frac{d}{d\mu^2}\Phi^{[\Gamma]}_{\mu,12} &= \left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right)+\Upsilon_{x_3x_2x_1}+2\pi i\,s\,\Theta_{x_3x_2x_1}\right)\Phi^{[\Gamma]}_{\mu,12}\\
&\qquad\boxed{+\mathbb{P}^A_{x_2x_3}\otimes\Phi^{[\Gamma\gamma^\mu\gamma^\nu]}_{\nu,12}+\mathbb{P}^B_{x_2x_3}\otimes\Phi^{[\Gamma\gamma^\nu\gamma^\mu]}_{\nu,12},}
\end{aligned}$$



BFLK kernels in momentum space are quite cumbersome

- non-analytic
- continious
- mix-sectors
- longish
- for "$x_i > 0$" region agrees with [Beneke, et al, 17]

$$\begin{aligned}
\mathbb{P}^A_{x_2x_1}\otimes\Phi(x_1,x_2,x_3) = &-\frac{a_s}{2}\Big\{\delta_{x_20}\,C_A\Phi(x_1,0,x_3) \qquad (1)\\
&+C_A\int_{-\infty}^{\infty}dv\Big[\frac{x_2}{v}[(v+x_2)\Phi(x_1,x_2,x_3)-x_2\Phi(x_1-v,x_2+v,x_3)]\frac{\theta(v,x_2)-\theta(-v,-x_2)}{(v+x_2)^2}\\
&\quad+\frac{x_1}{v}(\Phi(x_1,x_2,x_3)-\Phi(x_1-v,x_2+v,x_3))\frac{\theta(v,-x_1)-\theta(-v,x_1)}{v-x_1}\Big]\\
&-C_A\int_{-\infty}^{\infty}dv\Big[\frac{x_2^2(v+2x_2+x_1)}{(x_1+x_2)^2}\frac{\theta(v,x_2)-\theta(-v,-x_2)}{(v+x_2)^2}\\
&\quad+\frac{x_1(2x_2+x_1)}{(x_1+x_2)^2}\frac{\theta(v,-x_1)-\theta(-v,x_1)}{v-x_1}\Big]\Phi(x_1-v,x_2+v,x_3)\\
&+2\left(C_F-\frac{C_A}{2}\right)\int_{-\infty}^{\infty}dv\Big[\frac{-x_2^2}{(x_1+x_2)^2}\frac{\theta(v,x_2)-\theta(-v,-x_2)}{v+x_2}\\
&\quad+\frac{x_1(x_1x_2-2vx_2-vx_1)}{(x_1+x_2)^2}\frac{\theta(v,-x_1)-\theta(-v,x_1)}{(v-x_1)^2}\Big]\Phi(x_2+v,x_1-v,x_3)\Big\}\\
&+\mathcal{O}(a_s^2),
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}^B_{x_2x_1}\otimes\Phi(x_1,x_2,x_3) = &-\frac{a_s}{2}\Big\{\delta_{x_20}\,2(C_A-C_F)\Phi(x_1,0,x_3) \qquad (2)\\
&+C_A\int_{-\infty}^{\infty}dv\Big[\frac{x_2}{v}[(v+x_2)\Phi(x_1,x_2,x_3)-x_2\Phi(x_1-v,x_2+v,x_3)]\frac{\theta(v,x_2)-\theta(-v,-x_2)}{(v+x_2)^2}\\
&\quad+\frac{x_1}{v}(\Phi(x_1,x_2,x_3)-\Phi(x_1-v,x_2+v,x_3))\frac{\theta(v,-x_1)-\theta(-v,x_1)}{v-x_1}\Big]\\
&+2\left(C_F-\frac{C_A}{2}\right)\int_{-\infty}^{\infty}dv\,x_1\Phi(x_2+v,x_1-v,x_3)\frac{\theta(v,-x_1)-\theta(-v,x_1)}{(v-x_1)^2}\Big\}+\mathcal{O}(a_s^2).
\end{aligned}$$

Evolution equations in the momentum-fraction space

has involved structure

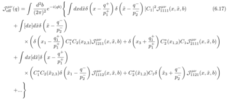
$$\mu^2\frac{d}{d\mu^2}\Phi^{[\Gamma]}_{\mu,21} = \left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right)+\boxed{\Upsilon_{x_1x_2x_3}}+2\pi i\,s\,\Theta_{x_1x_2x_3}\right)\Phi^{[\Gamma]}_{\mu,21}$$
$$+\mathbb{P}^A_{x_2x_1}\otimes\Phi^{[\gamma^\nu\gamma^\mu\Gamma]}_{\nu,21}+\mathbb{P}^B_{x_2x_1}\otimes\Phi^{[\gamma^\mu\gamma^\nu\Gamma]}_{\nu,21},$$
$$\mu^2\frac{d}{d\mu^2}\Phi^{[\Gamma]}_{\mu,12} = \left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right)+\boxed{\Upsilon_{x_3x_2x_1}}+2\pi i\,s\,\Theta_{x_3x_2x_1}\right)\Phi^{[\Gamma]}_{\mu,12}$$
$$+\mathbb{P}^A_{x_2x_3}\otimes\Phi^{[\Gamma\gamma^\mu\gamma^\nu]}_{\nu,12}+\mathbb{P}^B_{x_2x_3}\otimes\Phi^{[\Gamma\gamma^\nu\gamma^\mu]}_{\nu,12},$$

$$\Upsilon_{x_1x_2x_3} = a_s\left[3C_F+C_A\ln\left(\frac{|x_3|}{|x_2|}\right)+2\left(C_F-\frac{C_A}{2}\right)\ln\left(\frac{|x_3|}{|x_1|}\right)\right]+\mathcal{O}(a_s^2).$$

Real-part of collinear logarithms

- Singular at $x_i = 0$
- Integrable
- Checked by NLP coeff.function

q^+ is as in fact. theorem



$$\mathcal{J}^{\mu\nu}_{\text{eff}}(q)=\int\frac{d^2b}{(2\pi)^2}e^{-i(qb)}\Bigg\{\int dx d\tilde{x}\delta\left(x-\frac{q^+}{p_1^+}\right)\delta\left(\tilde{x}-\frac{q^-}{p_2^-}\right)|C_1|^2\mathcal{J}^{\mu\nu}_{1111}(x,\tilde{x},b) \qquad (6.17)$$
$$+\int[dx]d\tilde{x}\delta\left(\tilde{x}-\frac{q^-}{p_2^-}\right)$$
$$\times\left(\delta\left(x_1-\frac{q_1^+}{p_1^+}\right)C_1^*C_2(x_{2,3})\mathcal{J}^{\mu\nu}_{1211}(x,\tilde{x},b)+\delta\left(x_3+\frac{q_1^+}{p_1^+}\right)C_2^*(x_{1,2})C_1\mathcal{J}^{\mu\nu}_{2111}(x,\tilde{x},b)\right)$$
$$+\int dx[d\tilde{x}]\delta\left(x-\frac{q^+}{p_1^+}\right)$$
$$\times\left(C_1^*C_2(\tilde{x}_{2,3})\delta\left(\tilde{x}_1-\frac{q^-}{p_2^-}\right)\mathcal{J}^{\mu\nu}_{1112}(x,\tilde{x},b)+C_2^*(\tilde{x}_{1,2})C_1\delta\left(\tilde{x}_3+\frac{q^-}{p_2^-}\right)\mathcal{J}^{\mu\nu}_{1121}(x,\tilde{x},b)\right)$$
$$+\ldots\Bigg\}$$

Evolution equations in the momentum-fraction space

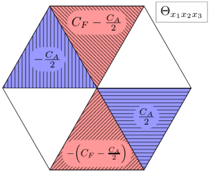
has involved structure

$$\mu^2 \frac{d}{d\mu^2}\Phi^{[\Gamma]}_{\mu,21} = \left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right) + \Upsilon_{x_1x_2x_3} + \boxed{2\pi i\, s\, \Theta_{x_1x_2x_3}}\right)\Phi^{[\Gamma]}_{\mu,21}$$
$$+\mathbb{P}^A_{x_2x_1}\otimes\Phi^{[\gamma^\nu\gamma^\mu\Gamma]}_{\nu,21} + \mathbb{P}^B_{x_2x_1}\otimes\Phi^{[\gamma^\mu\gamma^\nu\Gamma]}_{\nu,21},$$
$$\mu^2 \frac{d}{d\mu^2}\Phi^{[\Gamma]}_{\mu,12} = \left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right) + \Upsilon_{x_3x_2x_1} + \boxed{2\pi i\, s\, \Theta_{x_3x_2x_1}}\right)\Phi^{[\Gamma]}_{\mu,12}$$
$$+\mathbb{P}^A_{x_2x_3}\otimes\Phi^{[\Gamma\gamma^\mu\gamma^\nu]}_{\nu,12} + \mathbb{P}^B_{x_2x_3}\otimes\Phi^{[\Gamma\gamma^\nu\gamma^\mu]}_{\nu,12},$$

Imaginary-part of collinear logarithms

- Discontinious
- Process dependent!
- OMG!

$$\Theta_{x_1x_2x_3} = a_s \times \begin{cases} \frac{C_A}{2} & x_{1,2,3}\in(+,-,-), \\ -\left(C_F-\frac{C_A}{2}\right) & x_{1,2,3}\in(+,-,+), \\ 0 & x_{1,2,3}\in(-,-,+), \\ -\frac{C_A}{2} & x_{1,2,3}\in(-,+,+), \\ C_F-\frac{C_A}{2} & x_{1,2,3}\in(-,+,-), \\ 0 & x_{1,2,3}\in(+,+,-), \end{cases} + \mathcal{O}(a_s^2),$$



Evolution equations in the momentum-fraction space

has involved structure

$$\mu^2\frac{d}{d\mu^2}\Phi^{[\Gamma]}_{\mu,21}=\left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right)+\Upsilon_{x_1x_2x_3}+2\pi i\, s\,\Theta_{x_1x_2x_3}\right)\Phi^{[\Gamma]}_{\mu,21}$$
$$+\mathbb{P}^A_{x_2x_1}\otimes\Phi^{[\gamma^\nu\gamma^\mu\Gamma]}_{\nu,21}+\mathbb{P}^B_{x_2x_1}\otimes\Phi^{[\gamma^\mu\gamma^\nu\Gamma]}_{\nu,21},$$
$$\mu^2\frac{d}{d\mu^2}\Phi^{[\Gamma]}_{\mu,12}=\left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right)+\Upsilon_{x_3x_2x_1}+2\pi i\, s\,\Theta_{x_3x_2x_1}\right)\Phi^{[\Gamma]}_{\mu,12}$$
$$+\mathbb{P}^A_{x_2x_3}\otimes\Phi^{[\Gamma\gamma^\mu\gamma^\nu]}_{\nu,12}+\mathbb{P}^B_{x_2x_3}\otimes\Phi^{[\Gamma\gamma^\nu\gamma^\mu]}_{\nu,12},$$

- Complex
- Discontinious
- Singular

Live is not that bad!

Making story short: we introduce real/T-definite combination of operator and parametrize them

- **32 distributions** ($\bullet \;=\; \oplus$ and $\ominus$)
- **16 T-odd** and **16 T-even**

Example

$$\begin{aligned}\Phi_{\bullet}^{\mu[\gamma^+]}(x_{1,2,3},b) \quad &= \quad \epsilon^{\mu\nu}s_{T\nu}Mf_{\bullet T}(x_{1,2,3},b)+ib^{\mu}M^2f_{\bullet}^{\perp}(x_{1,2,3},b)\\ &\quad +i\lambda\epsilon^{\mu\nu}b_{\nu}M^2f_{\bullet L}^{\perp}(x_{1,2,3},b)+b^2M^3\epsilon_T^{\mu\nu}\left(\frac{g_{T,\nu\rho}}{2}-\frac{b_{\nu}b_{\rho}}{b^2}\right)s_T^{\rho}f_{\bullet T}^{\perp}(x_{1,2,3},b)\end{aligned}$$

$$f_{\oplus,T;DY}=f_{\oplus,T;SIDIS}, \qquad f^{\perp}_{\oplus;DY}=-f^{\perp}_{\oplus;SIDIS}$$

	U	L	T$_{J=0}$	T$_{J=1}$	T$_{J=2}$
U	$f_{\bullet}^{\perp}$	$g_{\bullet}^{\perp}$		$h_{\bullet}$	$h_{\bullet}^{\perp}$
L	$f_{\bullet L}^{\perp}$	$g_{\bullet L}^{\perp}$	$h_{\bullet L}$		$h_{\bullet L}^{\perp}$
T	$f_{\bullet T}, \quad f_{\bullet T}^{\perp}$	$g_{\bullet T}, \quad g_{\bullet T}^{\perp}$	$h_{\bullet T}^{D\perp}$	$h_{\bullet T}^{A\perp}$	$h_{\bullet T}^{S\perp}, \quad h_{\bullet T}^{T\perp}$

Evolution equations split into two cases:
Evolution with kernels $\mathbb{P}^A$ or $\mathbb{P}^B$

Example $\mathbb{P}^A$

$$\mu^2\frac{d}{d\mu^2}\begin{bmatrix} H^A_\oplus \\ H^A_\ominus \end{bmatrix} = \left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right)+\Upsilon_{x_1x_2x_3}\right)\begin{bmatrix} H^A_\oplus \\ H^A_\ominus \end{bmatrix}$$
$$+\begin{bmatrix} 2\mathbb{P}^A_{x_2x_1} & 2\pi s\Theta_{x_1x_2x_3} \\ -2\pi s\Theta_{x_1x_2x_3} & 2\mathbb{P}^A_{x_2x_1} \end{bmatrix}\begin{bmatrix} H^A_\oplus \\ H^A_\ominus \end{bmatrix},$$

$$\begin{pmatrix} f^\perp_\oplus+g^\perp_\ominus \\ f^\perp_\ominus-g^\perp_\oplus, \end{pmatrix}, \quad \begin{pmatrix} f^\perp_{\oplus,L}+g^\perp_{\ominus,L} \\ f^\perp_{\ominus,L}-g^\perp_{\oplus,L} \end{pmatrix}, \qquad \begin{pmatrix} f_{\oplus,T}+g_{\ominus,T} \\ f_{\ominus,T}-g_{\oplus,T} \end{pmatrix}, \quad \begin{pmatrix} f^\perp_{\oplus,T}+g^\perp_{\ominus,T} \\ f^\perp_{\ominus,T}-g^\perp_{\oplus,T} \end{pmatrix},$$
$$\begin{pmatrix} h_\oplus \\ h_\ominus \end{pmatrix}, \quad \begin{pmatrix} h_{\oplus,L} \\ h_{\ominus,L} \end{pmatrix}, \qquad \begin{pmatrix} h^{A\perp}_{\oplus,T} \\ h^{A\perp}_{\ominus,T} \end{pmatrix}, \quad \begin{pmatrix} h^{D\perp}_{\oplus,T} \\ h^{D\perp}_{\ominus,T} \end{pmatrix}.$$

- Real functions = real evolution
- Mixes T-odd and T-even distributions
- Mixing is proportional to s, so T-parity is preserved, and distributions are universal

TMD distributions of twist-three are *generalized functions*
No definite value at $x_i = 0$, but definite integrals

A typical term in the cross-section

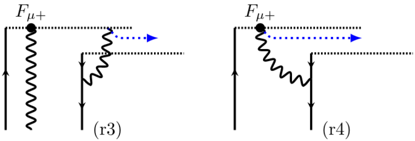
$$\int [dx]\delta(x-x_3)\frac{\Phi_{12}(x_{1,2,3},b)}{x_2-is0}\Phi_{11}(-\tilde{x},-b)+\int [dx]\delta(\tilde{x}-\tilde{x}_1)\Phi_{11}(x,b)\frac{\Phi_{21}(\tilde{x}_{1,2,3},-b)}{\tilde{x}_2-is0}$$

- The integral is divergent since $\Phi_\bullet$ is discontinuous at $x_2 = 0$
  - Important: integral from [-1,1], otherwise it would be just singular
- In fact, divergences cancel
- Let us redefine TMDs subtracting terms such that the cross-section is finite term-by-term

$$\int [dx]\delta(x-x_3)\frac{\Phi_{12}(x_{1,2,3},b)}{x_2 - is0} \quad \longleftrightarrow \quad \int_{s\infty}^{y} d\sigma\ \Phi_{12}(\{y,\sigma,0\},b)$$

- It is the rapidity divergence
- It can be computed

$$\int \frac{\Phi_{12}}{x_2} \sim \ln(\delta^+)\,\partial_\mu \mathcal{D}\,\Phi_{11}$$



Physical TMD distributions of twist-three

$$\mathbf{\Phi}^{[\Gamma]}_{21,\mu}(x_{1,2,3},b) = \Phi^{[\Gamma]}_{21,\mu}(x_{1,2,3},b) - [\mathcal{R}_{21}\otimes\Phi_{11}]^{[\Gamma]}_{\mu}(x_{1,2,3},b)$$

similar for Φ_{12}

- Obey same evolution equations
- $\mathcal{R}$ is know at $\mathcal{O}(\alpha_s)$

$$[\mathcal{R}_{21}\otimes\Phi_{11}]^{[\Gamma]}_{\mu}(x_1,x_2,x_3,b) = i\partial_\mu\mathcal{D}(b)\,\Phi^{[\Gamma]}_{11}(-x_1,b)(\theta(x_2,x_3)-\theta(-x_2,-x_3))+\mathcal{O}(a_s^2),$$

- Produce term-by-term finite cross-section
- Leaves remnant

$$\int[dx]\delta(x-x_3)\frac{\Phi_{12}(x_{1,2,3},b)}{x_2-is0}\Phi_{11}(-\tilde{x},-b)+\int[d\tilde{x}]\delta(\tilde{x}-\tilde{x}_1)\Phi_{11}(x,b)\frac{\Phi_{21}(\tilde{x}_{1,2,3},-b)}{\tilde{x}_2-is0}$$

$$\to\int[dx]\delta(x-x_3)\frac{\mathbf{\Phi}_{12}(x_{1,2,3},b)}{x_2-is0}\Phi_{11}(-\tilde{x},-b)+\int[d\tilde{x}]\delta(\tilde{x}-\tilde{x}_1)\Phi_{11}(x,b)\frac{\mathbf{\Phi}_{21}(\tilde{x}_{1,2,3},-}{\tilde{x}_2-is0}$$

$$+\partial_\mu\mathcal{D}(b)\Phi_{11}(x)\Phi_{11}(-\tilde{x})\ln\left(\frac{\zeta}{\bar{\zeta}}\right)$$

Not important for DY, SIDIS, SIA ($\zeta=\bar{\zeta}$)
Important for qTMD at lattice ($\zeta\neq\bar{\zeta}$)

TMD hadron tensor at NLP/NLO – too long
(all polarizations, angles, etc)
unpolarized-part only

$$
\begin{aligned}
W_{\text{NLP}}^{(0)\mu\nu} \;=\; \frac{-1}{N_c}\Bigg\{ & -\frac{i(n^\mu b^\nu + b^\mu n^\nu)}{q^+}M^2\left(\Sigma_+[\mathring{f}_1, f_1] + \frac{\mathring{\mathcal{D}}}{4}\Sigma_-[f_1,f_1]\ln\left(\frac{\bar\zeta}{\zeta}\right)\right)\\
& -\frac{i(\bar n^\mu b^\nu + b^\mu \bar n^\nu)}{q^-}M^2\left(\Sigma_+[f_1, \mathring{f}_1] + \frac{\mathring{\mathcal{D}}}{4}\Sigma_-[f_1,f_1]\ln\left(\frac{\bar\zeta}{\zeta}\right)\right)\\
& -i\frac{n^\mu b^\nu + b^\mu n^\nu}{q^+}2M^2\left(\Sigma_+[h_1^\perp, h_1^\perp] + \frac{b^2M^2}{2}\Sigma_+[\mathring{h}_1^\perp, h_1^\perp] + \frac{b^2M^2}{2}\frac{\mathring{\mathcal{D}}}{4}\Sigma_-[h_1^\perp,h_1^\perp]\ln\left(\frac{\bar\zeta}{\zeta}\right)\right)\\
& -i\frac{n^\mu b^\nu + b^\mu n^\nu}{q^-}2M^2\left(\Sigma_+[h_1^\perp, h_1^\perp] + \frac{b^2M^2}{2}\Sigma_+[h_1^\perp, \mathring{h}_1^\perp] + \frac{b^2M^2}{2}\frac{\mathring{\mathcal{D}}}{4}\Sigma_-[h_1^\perp,h_1^\perp]\ln\left(\frac{\bar\zeta}{\zeta}\right)\right)\Bigg\}.
\end{aligned}
$$

$$
\begin{aligned}
& W_{\text{NLP}}^{(1)\mu\nu} = \frac{-1}{N_c}\Bigg\{\\
& +\left(\frac{n^\mu b^\nu + b^\mu n^\nu}{q^+} - \frac{\bar n^\mu b^\nu + b^\mu \bar n^\nu}{q^-}\right)M^2\Big(\Sigma_-[f_\oplus^\perp, f_1] + \Sigma_-[f_1, f_\oplus^\perp]\Big)\\
& -i\left(\frac{n^\mu b^\nu - b^\mu n^\nu}{q^+} - \frac{\bar n^\mu b^\nu - b^\mu \bar n^\nu}{q^-}\right)M^2\Big(\Sigma_+[f_\ominus^\perp, f_1] - \Sigma_+[f_1, f_\ominus^\perp]\Big)\\
& +\left(\frac{n^\mu b^\nu + b^\mu n^\nu}{q^+} - \frac{\bar n^\mu b^\nu + b^\mu \bar n^\nu}{q^-}\right)2M^2\Big(\Sigma_-[h_\oplus, h_1^\perp] + \Sigma_-[h_1^\perp, h_\oplus]\Big)\\
& -i\left(\frac{n^\mu b^\nu - b^\mu n^\nu}{q^+} - \frac{\bar n^\mu b^\nu - b^\mu \bar n^\nu}{q^-}\right)2M^2\Big(\Sigma_+[h_\ominus, h_1^\perp] - \Sigma_+[h_1^\perp, h_\ominus]\Big)\\
& +i\left(\frac{n^\mu b^\nu - b^\mu n^\nu}{q^+} - \frac{\bar n^\mu b^\nu - b^\mu \bar n^\nu}{q^-}\right)M^2\Big(\Sigma_+[f_1, g_\oplus^\perp] + ?\Sigma_+[g_\oplus^\perp, f_1]\Big)\\
& +\left(\frac{n^\mu b^\nu + b^\mu n^\nu}{q^+} - \frac{\bar n^\mu b^\nu + b^\mu \bar n^\nu}{q^-}\right)M^2\Big(\Sigma_-[f_1, g_\ominus^\perp] - ?\Sigma_-[g_\ominus^\perp, f_1]\Big)\Bigg\}
\end{aligned}
$$

Conclusion

TMD factorization at NLP

- Operator expression at NLP/NLO is known
- Full classification is done
- Restoration of EM-conservation
- Also for qTMDs

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Thank you for attention!