SCET beyond LP: Basics, Renormalization, Resummation

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Outline

- Introduction
- Elements of SCET beyond leading power
- Renormalization
- Resummation
- Soft fermion processes

Recent work (2017 –) with: Bobeth, Broggio, Garny, Hager, Jaskiewicz, Strohm, Szafron, Vernazza, J. Wang

Power expansion, NLP

Double expansion in α_s and powers.
Two kinds of problems

$$F(Q,\Lambda;\alpha_s) = f_0(\alpha_s) + \frac{\Lambda}{Q} f_1(\alpha_s) + \ldots$$

$Q \gg \Lambda$, "higher-twist", $\lambda = \sqrt{\Lambda/Q}$

$$F(Q,E_s,\Lambda;\alpha_s) = \left[f_0(\alpha_s, \ln\frac{E_s}{Q}) + \frac{E_s}{Q} f_1(\alpha_s, \ln\frac{E_s}{Q}) + \ldots \right] + \mathcal{O}\left(\frac{\Lambda}{Q}\right)$$

$Q \gg E_s \gg \Lambda$, "perturbative resummation problem", $\lambda = \sqrt{E_s/Q}$

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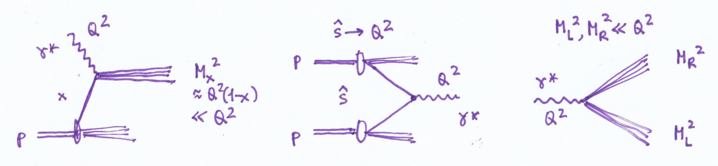
$Q \gg E_s \gg \Lambda$, "perturbative resummation problem", $\lambda = \sqrt{E_s/Q}$

SCET is the natural framework to address the expansion in powers of scale ratios beyond LP.

- Built-in power counting and gauge invariance
- Rules don't change beyond LP

Examples

- Kinematic thresholds of $2 \to 1$ / $1 \to 2$ processes
  $\mathrm{SCET_I}$ processes, no LP rapidity divergences



DIS DY e^+e^- event shapes

- ► Leading power IR logs $\alpha_s^n \left[\dfrac{\ln^m \tau}{\tau}\right]_+$, $m \leq 2m - 1$ by now standard: LL, NLL, NNLL, N3LL, [Moch, Vermaseren, Vogt; 2005; Becher, Neubert, Xu 2006; Becher, Schwartz, 2008], EEC at N4LL [Duhr, Mistlberger, Vita, 2022]

- ► Where there are logs, there are powers, and powers times logs → next-to-leading power (NLP) resummation

- ► Structure of NLP Logs $[\alpha_s^n \ln^m \tau]$, $m \leq 2n - 1$ not well known

- Many other situations at amplitude or cross section level

All-order NLP-LL series $[gg \to H]$ [MB, Garny, Jaskiewicz, Szafron, Vernazza, Wang, 1910.12685]

$$\Delta^{\rm LL}_{\rm NLP}(z,\mu) = \left[\frac{\beta\big(\alpha_s(\mu)\big)}{\alpha_s^2(\mu)}\,\frac{\alpha_s^2(\mu_t)}{\beta\big(\alpha_s(\mu_t)\big)}\right]^2 C_t^2(m_t,\mu_t)$$

$$\times \exp\left[4S^{\rm LL}(\mu_h,\mu) - 4S^{\rm LL}(\mu_s,\mu)\right]\;\frac{-8C_A}{\beta_0}\,\ln\frac{\alpha_s(\mu)}{\alpha_s(\mu_s)}\,\theta(1-z)$$

$\mu_t,\,\mu_h \sim m_H,\,\mu_s \sim m_H(1-z)$

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$\mu_t, \mu_h \sim m_H,\ \mu_s \sim m_H(1-z)$

$$\begin{aligned}
\Delta^{\rm LL}_{\rm NLP}(z,\mu) =&\ -\theta(1-z)\left\{4C_A\frac{\alpha_s}{\pi}\Big[\ln(1-z) - L_\mu\Big]\right.\\
&+ 8C_A^2\left(\frac{\alpha_s}{\pi}\right)^2\Big[\ln^3(1-z) - 3L_\mu\ln^2(1-z) + 2L_\mu^2\ln(1-z)\Big]\\
&+ 8C_A^3\left(\frac{\alpha_s}{\pi}\right)^3\Big[\ln^5(1-z) - 5L_\mu\ln^4(1-z) + 8L_\mu^2\ln^3(1-z) - 4L_\mu^3\ln^2(1-z)\Big]\\
&+ \frac{16}{3}C_A^4\left(\frac{\alpha_s}{\pi}\right)^4\Big[\ln^7(1-z) - 7L_\mu\ln^6(1-z) + 18L_\mu^2\ln^5(1-z) - 20L_\mu^3\ln^4(1-z)\\
&\qquad + 8L_\mu^4\ln^3(1-z)\Big]\\
&+ \frac{8}{3}C_A^5\left(\frac{\alpha_s}{\pi}\right)^5\Big[\ln^9(1-z) - 9L_\mu\ln^8(1-z) + 32L_\mu^2\ln^7(1-z) - 56L_\mu^3\ln^6(1-z)\\
&\qquad \left. + 48L_\mu^4\ln^5(1-z) - 16L_\mu^5\ln^4(1-z)\Big]\right\} + \mathcal{O}(\alpha_s^6\times(\log)^{11})\\
&\qquad \left(L_\mu = \ln\frac{\mu}{m_H}\right)
\end{aligned}$$

$\mathcal{O}(\alpha_s^3)$ agrees with expansion of N3LO FO calculation, $\mathcal{O}(\alpha_s^4)$ with [de Florian et al., 2014] based on "physical kernel conjecture" of Vogt.

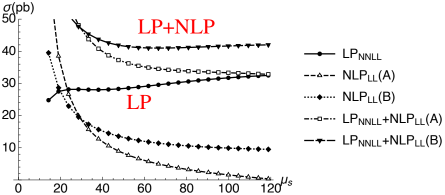
SCET result confirmed by diagrammatic method [Bahjat-Abbas et al., 1905.13710]

NLP numerics for Higgs production

Fixed-order vs threshold resummation at LP-NNLL + NLP-LL

$$\mu_s^{\text{dyn}} = \frac{Q}{\bar{s}_1(\tau)} = 38\,\text{GeV}, \quad \bar{s}_1(\tau) \equiv -e^{\gamma_E} \frac{d \ln \mathcal{L}(y,\mu)}{d \ln y}\bigg|_{y=\tau}$$

[Sterman, Zeng, 2013]



σ (pb)	$\mu_s = \mu_s^{\text{dyn}}$	
	$\mu_h^2 = m_H^2$	$\mu_h^2 = -m_H^2$
$\sigma_{\text{LP}}^{\text{NNLL}}$	24.12	28.04
$\sigma_{\text{LP}}^{\text{NNLO}}$	28.93	
$\sigma_{\text{LP}}^{\text{N}^3\text{LO}}$	29.24	
$\sigma_{\text{NLP}}^{\text{LL}}$ (A)	7.18	12.76
$\sigma_{\text{NLP}}^{\text{LL}}$ (B)	8.82	15.68
$\sigma_{\text{non LP}}^{\text{NNLO}}$	11.90	
$\sigma_{\text{non LP}}^{\text{N}^3\text{LO}}$	16.27	
$\sigma_{\text{LP}}^{\text{NNLL}} + \sigma_{\text{NLP}}^{\text{LL}}$ (A)	31.30	40.80
$\sigma_{\text{LP}}^{\text{NNLL}} + \sigma_{\text{NLP}}^{\text{LL}}$ (B)	32.94	43.72
σ^{NNLO}	40.82	
$\sigma^{\text{N}^3\text{LO}}$	45.52	

- Threshold approximation gives a reasonable approximation – once NLP terms are included.
- Still don't know how to do NLP NLL

Basics of SCET beyond LP

[Simplest case only: $\mathrm{SCET_I}$]

Power counting, modes and fields – SCET$_{\mathrm{I}}$

Mathematically, the EFT constructs the expansion of an observable in powers and logarithms of the ratio of the IR over the UV scales, e.g.

$$\lambda = \sqrt{\Lambda_{\mathrm{QCD}}/Q} \qquad \text{or} \qquad \frac{M}{Q}$$

Every object (fields, derivatives, momenta, ...) in the EFT should have a unique scaling with λ.

collinear mode momentum

$p_c \sim Q(1,\lambda,\lambda^2),\, p_c^2 \sim Q^2\lambda^2$

soft mode momentum

$p_s \sim Q(\lambda^2,\lambda^2,\lambda^2),\, p_s^2 \sim Q^2\lambda^4$

Light-like reference vectors $n_\pm$, $n_\pm^2 = 0$, $n_+ \cdot n_- = 2$

$$\begin{aligned} p^\mu &= (n_+p)\frac{n_-^\mu}{2} + p_\perp^\mu + (n_-p)\frac{n_+^\mu}{2} \\ &= (n_+p, p_\perp, n_-p) \end{aligned}$$

Need separate fields for collinear and soft modes

$$\mathrm{QCD}[A,\psi] \quad \longrightarrow \quad \mathrm{SCET}[A_c, A_s, \xi_c, q_s]$$

Several collinear directions $n_{i\pm} \to$ several copies of collinear fields.

SCET Lagrangian at leading power

[Bauer, Fleming, Pirjol, Stewart (2000); in the given “position-space” form MB, Chapovsky, Diehl, Feldmann (2002)]

$$\mathcal{L}^{(0)}_{\text{SCET}} = \sum_{i=1}^{N} \mathcal{L}^{(0)}_{c_i} + \mathcal{L}_{\text{soft}}$$

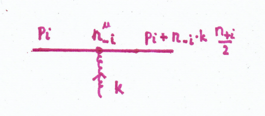
$$\mathcal{L}^{(0)}_c(x) = \bar{\xi}\left(in_-D_c + g_s n_- A_s(x_-) + i\not{D}_{\perp c}\frac{1}{in_+D_c}\, i\not{D}_{\perp c}\right)\frac{\not{n}_+}{2}\,\xi + \mathcal{L}^{(0)}_{c,\text{YM}}$$

$$iD_c = i\partial + g_s A_c, \qquad x^\mu_- = \frac{1}{2} n_+ \cdot x\, n^\mu_-$$

- Non-local, expected since $n_+ p_c \sim Q$.
- Only $n_- A_s$ appears, with eikonal vertex $i g_s n^\mu_-$.
- In soft-collinear interactions, soft fields must be multipole-expanded around the classical trajectory $x^m_- u$ to achieve homogeneous λ-scaling [BCDF, 2002]

$$\phi^2_c(x)\phi_s(x) = \phi^2_c(x)\left[\phi_s(x_-) + \left[x_\perp \partial\phi_{\text{us}}\right](x_-) + \frac{n_- x}{2}\left[n_+\partial\,\phi_{\text{us}}\right](x_-) + \frac{1}{2}\left[x_{\mu\perp}x_{\nu\perp}\partial^\mu\partial^\nu\phi_{\text{us}}\right](x_-) + \ldots\right]$$

Drops small momentum components at vertex and leads to eikonal propagator $1/(p_i+k)^2 \to 1/(p_i^2 + n_+p_i n_- k)$



Soft background field and gauge invariance

- Split field $A^\mu = A_c^\mu + A_s^\mu$ and treat soft field as a background field. Given standard background field gauge invariance of collinear fields.

$$
\begin{array}{lll}
\text{collinear:} & A_c \to U_c\, A_c\, U_c^\dagger + \frac{i}{g}\, U_c \left[D_{\rm us}, U_c^\dagger\right], & \xi \to U_c\, \xi, \\
& A_{\rm us} \to A_{\rm us}, & q \to q, \\
\text{ultrasoft:} & A_c \to U_{\rm us}\, A_c\, U_{\rm us}^\dagger, & \xi \to U_{\rm us}\, \xi, \\
& A_{\rm us} \to U_{\rm us}\, A_{\rm us}\, U_{\rm us}^\dagger + \frac{i}{g}\, U_{\rm us} \left[\partial, U_{\rm us}^\dagger\right], & q \to U_{\rm us}\, q.
\end{array}
$$

- Not homogeneous in λ. Solution [MB, Feldmann, 2002]: pull back soft gauge trafo to the light-cone x_-^μ with a redfinition of collinear fields:

$$
\xi = R\hat{\xi} \quad A_c^\mu = R^\dagger \hat{A}_c R \qquad R(x) = \mathrm{P}\exp\left(ig\int_{x_-}^{x} dy_\mu A_s^\mu(y)\right)
$$

Soft background field and gauge invariance

- The new fields transform under

collinear:

$$n_+\hat{A}_c \to U_c\, n_+\hat{A}_c\, U_c^\dagger + \frac{i}{g}\, U_c \left[n_+\partial, U_c^\dagger\right], \qquad \hat{\xi} \to U_c\, \hat{\xi},$$

$$\hat{A}_{\perp c} \to U_c\, \hat{A}_{\perp c}\, U_c^\dagger + \frac{i}{g}\, U_c \left[\partial_\perp, U_c^\dagger\right],$$

$$n_-\hat{A}_c \to U_c\, n_-\hat{A}_c\, U_c^\dagger + \frac{i}{g}\, U_c \left[n_- D_{\rm us}(x_-), U_c^\dagger\right],$$

$$A_{\rm us} \to A_{\rm us}, \qquad q \to q,$$

ultrasoft:

$$\hat{A}_c \to U_{\rm us}(x_-)\, \hat{A}_c\, U_{\rm us}^\dagger(x_-), \qquad \hat{\xi} \to U_{\rm us}(x_-)\, \hat{\xi},$$

$$A_{\rm us} \to U_{\rm us}\, A_{\rm us}\, U_{\rm us}^\dagger + \frac{i}{g}\, U_{\rm us} \left[\partial, U_{\rm us}^\dagger\right], \qquad q \to U_{\rm us}\, q.$$

- Homogeneous in λ.
  Collinear fields "see" only the background field $n_- A_s(x_-)$.
  Emergent soft gauge symmetry of collinear modes on their classical trajectory. [Key concept for gravity – see Patrick Hager's talk.]

- Gives exact Lagrangian to all orders in λ.

Wilson lines, gauge invariance and soft decoupling

$$W_c(x) = P\exp\left(ig\int_{-\infty}^{0} ds\, n_+A_c(x+sn_+)\right) \qquad Y(x) = P\exp\left(ig\int_{-\infty}^{0} ds\, n_-A_s(x+sn_-)\right)$$

$$W_c^\dagger\, in_+D_c\, W_c = in_+\partial \qquad Y^\dagger\, in_-D_s\, Y = in_-\partial$$

- Introduce collinear-gauge invariant (background field gauge with $n_-A_s(x_-)$ background field) collinear fields

$$\chi \equiv W_c^\dagger \xi, \qquad \mathcal{A}_{c\perp}^\mu \equiv W_c^\dagger [iD_{c\perp}^\mu W_c], \qquad n_-\mathcal{A}_c \equiv W_c^\dagger [in_-D_{c+s} W_c] - g_s n_- A_s$$

- Soft-decoupling transformation [Bauer, Pirjol, Stewart, 2001]

$$\xi(x) \to Y(x_-)\xi(x) \qquad A_c^\mu(x) \to Y(x_-)A_c^\mu(x)Y^\dagger(x_-)$$

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$$\xi(x) \to Y(x_-)\xi(x) \qquad A^\mu_c(x) \to Y(x_-) A^\mu_c(x) Y^\dagger(x_-)$$

$$\mathcal{L}_c^{(0)}(x) = \bar\chi \left(in_-\partial + n_-\mathcal{A}_c + (i\not\partial_\perp + \not\mathcal{A}_{c\perp}) \frac{1}{in_+\partial} (i\not\partial_\perp + \not\mathcal{A}_{c\perp}) \right) \frac{\not n_+}{2}\chi + \mathcal{L}^{(0)}_{c,\mathrm{YM}}$$

Collinear and soft interactions decoupled at leading power.

SCET Lagrangian, sub-leading power [MB, Feldmann, 2002]

$$\mathcal{L}_c = \mathcal{L}_c^{(0)} + \mathcal{L}_c^{(1)} + \mathcal{L}_c^{(2)} + \ldots$$

$$\mathcal{L}_\xi^{(1)} = \bar{\xi}\left(x_\perp^\mu n_-^\nu\, W_c\, gF^{\rm s}_{\mu\nu}(x_-) W_c^\dagger\right)\frac{\not{n}_+}{2}\,\xi$$

$$\mathcal{L}_{\xi q}^{(1)} = \bar{q}_s(x_-)\, W_c^\dagger i\not{D}_{\perp c}\,\xi - \bar{\xi}\, i\overleftarrow{\not{D}}_{\perp c} W_c\, q_s(x_-)$$

$$\mathcal{L}_\xi^{(2)} = \frac{1}{2}\,\bar{\xi}\left((n_- x)\, n_+^\mu n_-^\nu\, W_c\, gF^{\rm s}_{\mu\nu} W_c^\dagger + x_\perp^\mu x_{\perp\rho} n_-^\nu\, W_c\left[D^\rho_{\rm us}, gF^{\rm us}_{\mu\nu}\right] W_c^\dagger\right)\frac{\not{n}_+}{2}\,\xi$$
$$+\,\frac{1}{2}\,\bar{\xi}\left(i\not{D}_{\perp c}\,\frac{1}{in_+D_c}\, x_\perp^\mu \gamma_\perp^\nu\, W_c\, gF^{\rm us}_{\mu\nu} W_c^\dagger + x_\perp^\mu \gamma_\perp^\nu\, W_c\, gF^{\rm us}_{\mu\nu} W_c^\dagger\,\frac{1}{in_+D_c}\, i\not{D}_{\perp c}\right)\frac{\not{n}_+}{2}\,\xi$$

- Soft gauge field appears only in manifestly covariant field strength.
- No purely collinear subleading interactions. At least one soft field in every vertex.
- Can express in terms of gauge-invariant collinear fields χ_c, $\mathcal{A}_{c\perp}$ and gauge-invariant soft fields after the decoupling transformation, $q = Y^\dagger q_s$, $\mathcal{B}^\mu = Y^\dagger[iD_s^\mu Y]$
- NLP needs $\mathcal{O}(\lambda^2)$, i.e. $\mathcal{L}_\xi^{(1)}$ and $\mathcal{L}_\xi^{(2)}$ but only $\mathcal{L}_{\xi q}^{(1)}$.
- Compendium of Feynman rules in [MB, Garny, Szafron, Wang, 1808.04742]

(No-) Renormalization of the SCET Lagrangian [BCDF, 2002]

Integrating out hard modes (“matching”) usually renormalizes the interactions of the effective Lagrangian:

$$\begin{aligned}\mathcal{L}_c &= \mathcal{L}_c^{(0)} + \mathcal{L}_c^{(1)} + \mathcal{L}_c^{(2)} + \ldots \\ &= \mathcal{L}_{\text{tree}} \equiv \sum_i \mathcal{O}_i \stackrel{\text{matching}}{\longrightarrow} \mathcal{L} = \sum_{i'} C_{i'} \mathcal{O}_{i'}\end{aligned}$$

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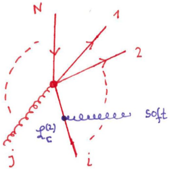
Hard loops

$$\mu^{2\epsilon} \int d^d\ell \, I(\ell; p_{ci}, k_{sj}) = \left(\frac{\rm HI}{\mu^2}\right)^{-\epsilon} \times J(\epsilon) + \ldots \stackrel{\rm scaleless}{=} 0$$

since $\mathrm{HI} = (n_+ p_i)\frac{n_-^\mu}{2} \cdot (n_+ p_{i'})\frac{n_{-\mu}}{2} = 0$ so the integrals are scaleless.

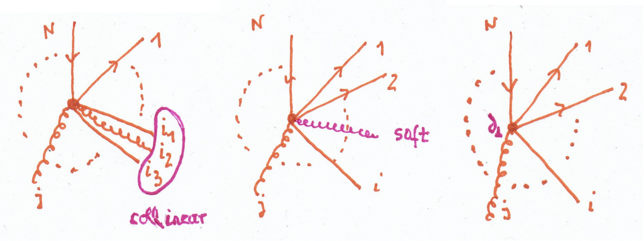
$\hookrightarrow$ **No renormalization, tree Lagrangian is exact [background field + field redefs]**



not renormalized

Hard sources

Hard sub-processes are represented by N-jet light-ray operators of gauge-invariant quark and gluon "jet" fields, which scale $\mathcal{O}(\lambda)$.



- Soft covariant derivatives on collinear fields can be eliminated, e.g. $[in_- D_s, \mathcal{A}_\perp^\mu]$
  No soft fields in NLP operators.

- Collinear building blocks for J_i:

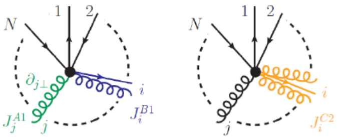
$$\chi_i(t_i n_{i+}) \equiv W_i^\dagger \xi_i, \quad \mathcal{A}^\mu_{\perp i}(t_i n_{i+}) \equiv W_i^\dagger [iD^\mu_{\perp i} W_i]$$

- Operate with $i\partial^\mu_{\perp i}$ on collinear building block or take products of several collinear building blocks in the same collinear sector

General form of the operator:

$$\mathcal{O}(0) = \int \prod_{i=1}^{N} \prod_{k_i=1}^{n_i} dt_{ik_i}\, C(\{t_{ik_i}\})$$

$$\times J_s(0) \times \prod_{i=1}^{N} J_i(t_{i_1}, t_{i_2}, \ldots t_{i_{n_i}})$$



- Every element collinear gauge *invariant* and soft gauge *covariant*.

- Notation: J^{An}, J^{Bn}, J^{Cn}, ...

  – $A, B, C, \ldots$ refers to 1,2,3, ... fields in a given collinear direction

  – n means $\mathcal{O}(\lambda^n)$ in a given collinear sector relative to A0

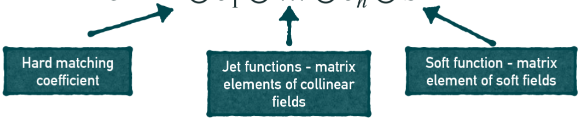
- At $\mathcal{O}(\lambda^2)$ up to two $\partial_\perp$ or up to three fields in one sector. Examples:

$$i\partial_{\perp i} i\partial_{\perp i}\chi_i \ (A2), \quad \chi(t_{i_1})\partial_{\perp i}\mathcal{A}_{\perp i}(t_{i_2}) \ (B2), \quad \chi(t_{i_1})\chi(t_{i_2})\chi(t_{i_3}) \ (C2)$$

- $\mathcal{O}(\lambda^2)$ 3-jet operators could then be $J^{(A0)}J^{(A0)}J^{(B2)}, J^{(B1)}J^{(A0)}J^{(A1)}, \ldots$

Automatic (naive) all-order factorization

$$\sigma \sim H \otimes J_1 \otimes \ldots \otimes J_n \otimes S$$



- Expand process in hard sources + time-ordered products with Lagrangian interactions.
- Strict expansion in $\lambda \sim p_\perp/n_+p \equiv \sqrt{1-x}, \sqrt{1-z}, \sqrt{\tau}$. NLP is $\mathcal{O}(\lambda^2)$
- Factorize into **single scale** ("homogeneous") objects, which have **gauge-invariant operator definitions: hard, jet/collinear and soft functions**
- IR logs in QCD are UV logs in SCET. **Resummation is an operator renormalization / mixing + renormalization group problem**

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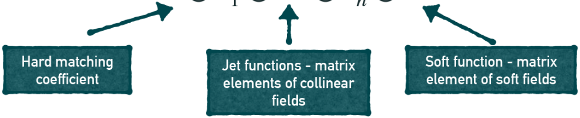

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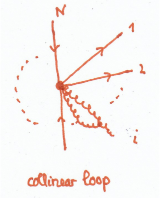
> Too naive – rapidity divergences at leading power, virtuality endpoint divergences at sub-leading power, KSZ theorem violation in renormalization, Glauber modes, . . . ???

Renormalization of $\mathrm{SCET}_{\mathrm{I}}$ operators

[MB, Garny, Szafron, Wang, 1712.04416, 1808.04742, 1907.05463]

Multi-light-cone operators

$$\mathcal{O}(x) = \prod_{i=1}^{N} \left[\prod_{k=1}^{n_i} \int dt_{i_k} \psi_i(x + t_{i_k} n_{+i}) \right]$$



N non-collinear directions defined by momenta
Overall colour-singlet, but not the separate collinear sectors.

Log structure determined by the UV divergences of collinear and soft loops in SCET [Becher, Neubert, 2009]

$$Z_{\mathcal{O}} \prod_{i=1}^{N} \sqrt{Z}_i \langle 0|\mathcal{O}(0)|\mathcal{M}(\{p_i\})\rangle_{|\mathcal{L}_{\rm SCET}} \stackrel{!}{=} \text{finite}$$

SCET_{I} anomalous dimension determined by UV divergences of collinear *and* soft loops.
Use small off-shellness to regulate IR divs.

General structure of the NLP ADM (at one-loop)

$$\Gamma_{PQ}(x,y) = \delta_{PQ}\delta(x-y)\left[-\gamma_{\rm cusp}(\alpha_s)\sum_{i<j}\sum_{l,k}\mathbf{T}_{i_k}\cdot\mathbf{T}_{j_l}\,\ln\left(\frac{-s_{ij}x_{i_k}x_{j_l}}{\mu^2}\right)+\sum_i\sum_k\gamma_{i_k}(\alpha_s)\right]$$
$$+2\sum_i\delta^{[i]}(x-y)\gamma^i_{PQ}(x,y)+2\sum_{i<j}\delta(x-y)\gamma^{ij}_{PQ}$$

- Operators $[\mathcal{O}(\lambda^2)]$

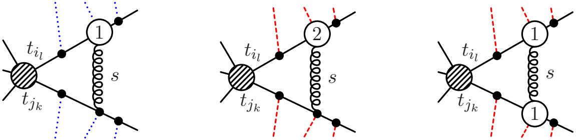
$$P = J^{(A0,A2)},\ J^{(A1,A1)},\ J^{(A1,B1)},\ J^{(A0,B2)},\ J^{(A0,C2)},\ J^{(B1,B1)},$$
$$T(J^{(A0,A0)},\mathcal{L}^{(1)},\mathcal{L}^{(1)}),\ T(J^{(A0,A0)},\mathcal{L}^{(2)}),\ T(J^{(A0,A1)},\mathcal{L}^{(1)}),\ T(J^{(A0,B1)},\mathcal{L}^{(1)})$$

- Collinear contribution depends only on single sectors, but within each sector on x_{i_k}. complicated expressions hidden in collinear term $\gamma^i_{PQ}(x,y)$.

- Soft contributions connect two sectors i,j and have dipole form. Loops with LP soft interaction contribute (only) to the first line. The NLP soft contribution γ^{ij}_{PQ} arises only from mixing of time-ordered products into currents.

$$\gamma^i = \begin{pmatrix}\gamma^i_{PQ} & 0\\ 0 & \gamma^i_{P'Q'}\end{pmatrix},\qquad \gamma^{ij} = \begin{pmatrix}0 & 0\\ \gamma^{ij}_{T(P')Q} & 0\end{pmatrix}.$$

Soft time-ordered product mixing



- Single insertions of $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ vanish, in particular no $\mathcal{O}(\lambda)$ mixing. Double $\mathcal{L}^{(1)}$ insertion is non-zero.

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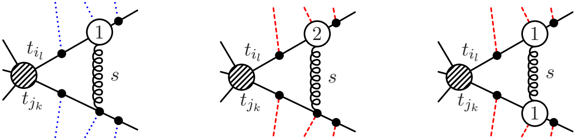


- Single insertions of $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ vanish, in particular no $\mathcal{O}(\lambda)$ mixing. Double $\mathcal{L}^{(1)}$ insertion is non-zero.
- Inconsistent with Lorentz-invariance.

  Take i, j directions non-back-to-back ($p_{\perp i} \neq 0$ but $p_{\perp j} = 0$), QCD cusp logs must be expanded as $s_{ij} = s_{ij}^{(0)} + n_{j+} \cdot p_j n_{j-} \cdot p_{\perp i} + \mathcal{O}(\lambda^2)$

$$\Gamma = -\gamma_{\rm cusp}(\alpha_s) \sum_{i<j} \mathbf{T}_i \cdot \mathbf{T}_j \left(\ln \frac{s_{ij}^{(0)}}{\mu^2} + \frac{n_{j+} \cdot p_j n_{j-} \cdot p_{\perp i}}{s_{ij}^{(0)}} + \ldots \right) \quad \Rightarrow \quad \gamma_{PQ}^{ij} = -\frac{\alpha_s}{\pi} \mathbf{T}_i \cdot \mathbf{T}_j \frac{n_{j-}^\mu}{(n_{i-} n_{j-}) P_i}$$

  There must be $\mathcal{O}(\lambda)$ mixing into (A1,A0).

- The counterterm related to this AD is required to reproduce the IR poles of the on-shell QCD amplitude.

Soft time-ordered product mixing at $\mathcal{O}(\lambda)$ revisited

- Only possibility is the soft loop with one insertion of $\mathcal{L}^{(1)}$.

$$\mathcal{L}_\xi^{(1)} = \bar{\xi}\left(x_\perp^\mu n_-^\nu\, g_s F^{\rm s}_{\mu\nu}\right)\frac{\not{n}_+}{2}\,\xi$$

$$\tilde{\mathcal{L}}_\xi^{(1)} = \bar{\xi}\left(i\not{D}_{\perp c}\frac{1}{in_+D_c}\,g\not{A}_{\perp\rm us} + g\not{A}_{\perp\rm s}\frac{1}{in_+D_c}\,i\not{D}_{\perp c} + [(x_\perp\partial)\,(gn_-A_{\rm s})]\right)\frac{\not{n}_+}{2}\,\xi$$

The 2nd Lagrangian arises in the direct expansion of the quark Lagrangian. The 1st form is obtained from the 2ndby the field redefinition

$$\xi' = (1+g_s x_\perp\cdot A_s)\xi$$

- Alternatively,

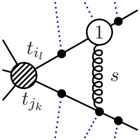
$$\tilde{\mathcal{L}}_\xi^{(1)} = \mathcal{L}_\xi^{(1)} + \Delta\mathcal{L}^{(1)}_{\rm eom}$$

$$\Delta\mathcal{L}^{(1)}_{\rm eom} = \bar{\xi}\,\left[ig_s x_\perp A_{\rm s},\; in_-D + i\not{D}_{\perp c}\frac{1}{in_+D_c}i\not{D}_{\perp c}\right]\frac{\not{n}_+}{2}\,\xi.$$

[To avoid dealing with the YM part of the Lagrangian, we assume abelian gauge fields for simplicity.]

Soft time-ordered product mixing at $\mathcal{O}(\lambda)$ revisited

- Calculate the soft mixing graph with $\tilde{\mathcal{L}}_\xi^{(1)}$



- Relevant integral is (off-shell IR regularization)

$$-i\tilde{\mu}^{2\epsilon}\int\frac{d^d l}{(2\pi)^d}\,\frac{n_{j+}p_j}{l^2(p_i^2-n_{i+}p_i n_{i-}l)^2(p_j^2-n_{j+}p_j n_{j-}l)}$$

$$\times\Big(\underbrace{-\,[n_{j-}p_{\perp i}\,n_{+i}p_i\,n_{-i}l-n_{+i}p_i\,n_{i-}n_{j-}\,p_{\perp i}l]}_{\text{from }\mathcal{L}_\xi^{(1)}\quad\text{[gives 0]}}+\underbrace{n_{j-}p_{\perp i}\,p_i^2}_{\text{from }\Delta\mathcal{L}_{\text{eom}}^{(1)}}\Big)$$

$$=\frac{1}{4\pi^2}\,\frac{n_{j-}\,p_{\perp i}}{n_{i-}n_{j-}\,n_{i+}p_i}\left(\frac{\mu^2 s_{ij}^{(0)}}{p_i^2p_j^2}\right)^{\epsilon}\frac{1}{\epsilon}\neq 0$$

- Off-shell term contributes due to p_i^2/p_i^2.
  Introduce a counterterm for mixing of eom operator into a "physical" transverse derivative operator

$$T(J^{(A0)},\Delta\mathcal{L}_{\text{eom}}^{(1)})\to J^{(A1)}$$

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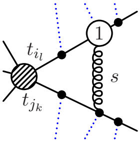
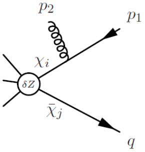
The SCET Lagrangian is not renormalized (Lorentz invariance). Coefficient is uniquely fixed by matching to QCD off-shell.
There is a preferred set of fields for the calculation of the anomalous dimension.

Extra collinear emission and endpoint divergence

- Case-I on-shell amplitude
  - $\Delta\mathcal{L}^{(1)}_{\rm eom}$ does not contribute
  - The counterterm from $T(J^{(A0)}, \Delta\mathcal{L}^{(1)}_{\rm eom}) \to J^{(A1)}$ mixing is needed to renormalize the amplitude.
  - IR divergences of the QCD amplitude are correctly reproduced and include a purely collinear contribution from the matrix element of a B1 operator. This includes the pole part of a divergent convolution.





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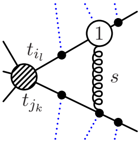
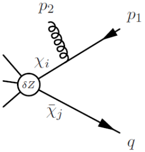




- Case-II off-shell Green function
  - In the sum of all soft contributions a non-local pole term $\frac{1}{\epsilon} \times \frac{1}{(p_1+p_2)^2}$ is left over.
  - This cancels with the endpoint-divergent collinear contribution from B1 operators.
  - Similar cancellations in the presence of an extra collinear emission already occur at leading power, but the dependence on p^2 is logarithmic

Divergent convolutions

- The collinear contribution to the amplitude includes the term

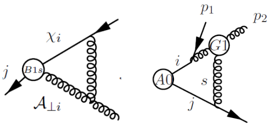
$$\int_0^1 dx\, C^{(A0,B1)}(x)_{|\text{tree}}\, \langle \bar{q}(q) q(p_1) g(p_2) | [\bar{\chi}_j \mathcal{A}_{\perp i} \chi_i](x) | 0 \rangle_{|\text{1-loop}}$$

$$\supset \frac{1}{\epsilon} \int_0^1 dx\, \frac{c}{x}\, \gamma^i_{J^\mu_{B1}(x), J^\nu_{B1}(y)} = \text{ill-defined}$$

- If the convolution of coefficient function and bare matrix element is done in *d* dimensions obtain the non-local pole

$$\frac{1}{\epsilon} \frac{1}{(p_1+p_2)^2}\,,$$

which cancels with the non-local pole in the soft mixing.



  - Divergence arises when collinear gluon becomes soft (momentum fraction $x \to 0$).
  - Violation of soft-collinear factorization already in renormalization
  - Cannot consistently define an anomalous dimension matrix for $J^{A1\mu}$, $J^{B1\mu}$ and J^{T1}

Renormalizing divergent convolutions

$$J^{A1\mu}, \quad J^{B1\mu}, \quad J^{T1} \quad \to \quad J^{A1\mu}, \quad J^{B1\mu}_{\text{reg}}, \quad \check{J} \equiv J^{T1} + \frac{2n^{\mu}_{j-}}{n_{j-}n_{i-}} J^{B1s}_{\mu}$$

- Singular operator includes the $1/x$ factor in the operator definition, its renormaliztion implies the convolution in d dimensions.

$$J^{B1s\,\mu}(t_i,t_j) = \bar{\chi}_j(t_j n_{j+}) \Gamma \left[\frac{1}{in_{i+}\partial} \mathcal{A}^{\mu}_{\perp i}(t_i n_{i+}) \right] \chi_i(t_i n_{i+})\,.$$

Related by reparameterization invariance to J^{T1} and J^{A0}.

- Regular operator defined by $+$-type prescription, which removes the $1/x$ part of the coefficient function in convolutions, making them well-defined.

$$J^{B1\,\mu}_{\text{reg}}(x) = \lim_{\eta \to 0+} \left[\theta(x-\eta) J^{B1\mu}(x) - \eta\delta(x-\eta) \int_{\eta}^{1} dz \frac{J^{B1\mu}(z)}{z} \right].$$

- Consistency requires

$$C^{(A0,B1)}(x) = \frac{c(\alpha_s)}{x} + \text{less singular terms}$$

Checked. ADM can be reconstructed from RPI

[J. Strohm, Master thesis (2020)]

NLP factorization and resummation – DY threshold

[MB, Broggio, Garny, Jaskiewicz, Szafron, Vernazza, Wang, 2018/19]

NLP resummations

Soft gluons

Thrust distribution [Moult, Stewart, Vita, Zhu, 2018]

Back-to-back EEC in $N = 4$ SYM
Moult, Vita, Yan, 2019

Drell-Yan threshold [MB, Broggio, Garny, Jaskiewicz, Szafron, Vernazza, Wang, 2018; Bahjat-Abbas et al., 2019]

All leading log, often only double log

All log order claim for diagonal $q\bar{q}$ channel in large-x DIS and DY [Ajjath, Mukherjee, Ravindran, 2020; Ajjath et al. 2021]

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Soft fermions

Off-diagonal qg channel in large-x DIS and for DY threshold [MB, Garny, Jaskiewicz, Szafron, Vernazza, Wang, 2020; van Beekveld, Vernazza, White, 2021]

"Gluon" thrust [Moult et al., 2019; MB, Garny, Jaskiewicz, Szafron, Strohm, Vernazza, Wang, 2022]

Amplitudes:

$H \to \gamma\gamma$ through light quark loops [Liu, Penin, 2017/18; Liu, Neubert, 2019; Wang, 2019; and NLL Liu, Mecaj, Neubert, Wang, 2020; Anastasiou, Penin, 2020]

μe backscattering [Bell, Böer, Feldmann, 2022]

- Note: soft gluon/quark refers to subleading power interactions. What is resummed to all orders is always gluons.

Drell-Yan production near threshold ($\hat{s} \to Q^2$)

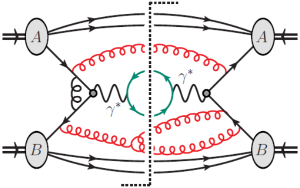
LP resummation [Sterman, 1987; Catani, Trentadue 1989; Korchemsky, Marchesini, 1993; SCET: Idilbi et al. (2005/06); Becher, Neubert, Xi, 2007]

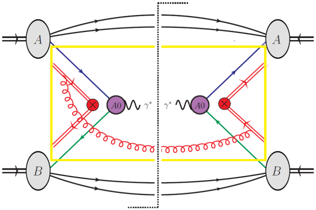
$$\frac{d\sigma_{\rm DY}}{dQ^2} = \frac{4\pi\alpha_{\rm em}^2}{3N_c Q^4} \sum_{a,b} \int_0^1 dx_a dx_b\, f_{a/A}(x_a) f_{b/B}(x_b)\, z\, \frac{\hat{\sigma}_{ab}(z)}{z}$$

$$\hat{\sigma}(z) = H(Q^2)\, Q S_{\rm DY}(Q(1-z))$$

$$S_{\rm DY}(\Omega) = \int \frac{dx^0}{4\pi}\, e^{i\Omega x^0/2}\, \widetilde{S}_0\left(x^0\right)$$

$$\widetilde{S}_0(x) = \frac{1}{N_c} {\rm Tr}\, \langle 0 | \bar{\mathbf{T}} \left[Y_+^\dagger(x) Y_-(x) \right] \mathbf{T} \left[Y_-^\dagger(0) Y_+(0) \right] | 0 \rangle$$





- Hard and soft only. No hard-collinear functions.
- Soft function consists only of Wilson lines – general feature of LP factorization.

  Its RGE sums the large logs of $1 - z = Q^2/\hat{s}$.

Drell-Yan production near threshold – NLP

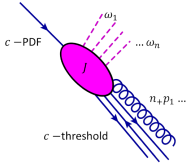
NLP LL resummation [1809.10631] and NLP factorization [1912.01585]

- No subleading power A1, B1, ... SCET currents. Same hard function as at LP.

  Because no energetic radiation into final state.

- New feature: Hard-collinear functions at amplitude level.

  Soft gluons in sub-leading Lagrangian do not decouple. hard-collinear loops no longer scale less, but depend on $n_+ p n_- k$

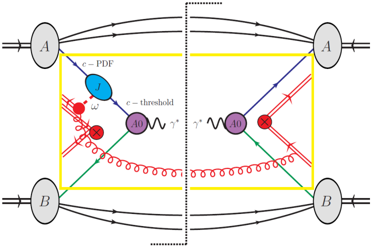


- Matching equation:

$$i \int d^4z \, \mathbf{T}\Big[\chi_{c,\gamma f}(tn_+)\, \mathcal{L}^{(2)}(z)\Big] = 2\pi \sum_i \int \frac{d\omega}{2\pi} \int \frac{dn_+p}{2\pi}\, e^{-i(n_+p)\,t} \int \frac{dn_+p_a}{2\pi}$$

$$\times J_{i;\gamma\beta,\mu,fbd}(n_+p, n_+p_a;\omega)\; \hat{\chi}^{\rm PDF}_{c,\beta b}(n_+p_a) \int dz_-\, e^{-i\omega z_-}\, \mathfrak{s}_{i;\mu,d}(z_-)\,,$$

- Generalized soft functions with soft field strength insertion [MB, Campanario, Mannel, Pecjak, 2004]
- Convolutions $J(\omega) \otimes S(\omega)$



$$\Delta^{dyn}_{\rm NLP}(z) = -\frac{2}{(1-\epsilon)} Q \left[\left(\frac{\not{n}_-}{4}\right) \gamma_{\perp\rho} \left(\frac{\not{n}_+}{4}\right) \gamma^{\rho}_{\perp} \right]_{\beta\gamma}$$
$$\times \int d(n_+p)\, C^{A0,A0}(n_+p, x_b n_- p_B)\, C^{*A0,A0}(x_a n_+ p_A,\, x_b n_- p_B)$$
$$\times \sum_{i=1}^{5} \int \{d\omega_j\}\, J_{i,\gamma\beta}(n_+p, x_a n_+ p_A; \{\omega_j\})\; S_i(\Omega; \{\omega_j\}) + \text{h.c.}\,,$$

Validation: One-loop collinear fns [1912.01585] and two-loop soft [Broggio, Jaskieiwcz, Vernazza, 2021] reproduce diagrammatic NNLO results and beyond [Bonocore et al., 2016]

Generalized soft fn RGE

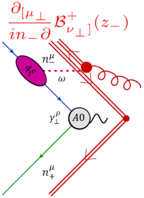
NLP factorization formula (leading-log accurate)

$$\hat{\sigma}_{q\bar{q}}^{\rm NLP}(z) = H^{\rm LP}(Q^2)\, Q\, J(\omega) \otimes_\omega S_{2\xi}(Q(1-z);\omega) + \text{h.c.}$$

Generalized soft function

$$S_{2\xi}(\Omega,\omega) = \mathbf{FT}_{\{x^0,z_-\}} \frac{1}{N_c} \text{Tr}\, \langle 0|\overline{\mathbf{T}}\left[Y_+^\dagger(x^0)Y_-(x)\right] \mathbf{T}\left[Y_-^\dagger(0)Y_+(0)\frac{i\partial_\perp^\nu}{in_-\partial}\mathcal{B}^+_{\perp\nu}(z_-)\right])|0\rangle$$

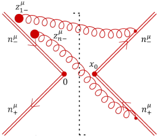
$$= \frac{\alpha_s C_F}{2\pi}\left\{\theta(\Omega)\delta(\omega)\left(-\frac{1}{\epsilon}+\ln\frac{\Omega^2}{\mu^2}\right)+\left[\frac{1}{\omega}\right]_+\theta(\omega)\theta(\Omega-\omega)\right\}$$



Renormalization group equation involves mixing with

$$S_{x_0}(\Omega) = \int \frac{dx^0}{4\pi}\, e^{ix^0\Omega/2}\, \frac{-2i}{x^0-i\varepsilon}\, \frac{1}{N_c} \text{Tr}\, \langle 0|\overline{\mathbf{T}}\left[Y_+^\dagger(x^0)Y_-(x^0)\right] \mathbf{T}\left[Y_-^\dagger(0)Y_+(0)\right]|0\rangle$$

$$\frac{d}{d\ln\mu}\begin{pmatrix} S_{2\xi}(\Omega,\omega) \\ S_{x_0}(\Omega)\end{pmatrix} = \frac{\alpha_s}{\pi}\begin{pmatrix} 4C_F\ln\frac{\mu}{\mu_s} & -C_F\delta(\omega) \\ 0 & 4C_F\ln\frac{\mu}{\mu_s}\end{pmatrix}\begin{pmatrix} S_{2\xi}(\Omega,\omega) \\ S_{x^0}(\Omega)\end{pmatrix}$$



$\hookrightarrow$ leading log resummation

Beyond NLP-LL: divergent convolutions

- LL at NLP in diagonal channel seem to be simple: no double log in collinear function / no colour charge change of collinear particle

  No LL in diagonal DGLAP kernels for $x \to 1$

  Recall: no endpoint divergence in renormalization at for back-to- back particles at $\mathcal{O}(\lambda)$.

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  No LL in diagonal DGLAP kernels for $x \to 1$

  Recall: no endpoint divergence in renormalization at for back-to- back particles at $\mathcal{O}(\lambda)$.

- At NLL a convolution in $J \otimes S$ appears, that exists in d dimensions, but does not for $\epsilon \to 0$.

$$\int_0^\Omega d\omega \, \underbrace{\left(n_+ p \, \omega\right)^{-\epsilon}}_{\text{collinear piece}} \underbrace{\frac{1}{\omega^{1+\epsilon}} \frac{1}{(\Omega - \omega)^\epsilon}}_{\text{soft piece}}$$

- Do not have a <u>renormalized</u> factorization theorem for the partonic cross section. Have to refactorize the parton distributions for $x \to 1$ as well from NLP.

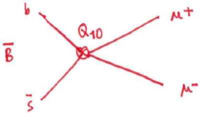
For the off-diagonal parton channels [$\to$ soft fermions] the problem appears already at NLP LL, while LP is zero $\to$ study soft fermion emission

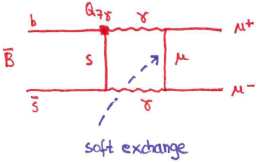
Next-to-leading power – soft fermions

[MB, Garny, Jaskiewicz, Szafron, Vernazza, Wang, 2020;
MB, Garny, Jaskiewicz, Strohm, Szafron, Vernazza, Wang, 2022]

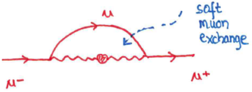
Soft fermion exchange - a curious flavour physics example

$B_s \to \mu^+\mu^-$ [MB, Bobeth, Szafron, 1708.09152]





$$C_{10} \quad \text{vs.} \quad C_7^{\rm eff}\,\frac{\alpha_{\rm em}}{\pi}\,\frac{m_B}{\Lambda_{\rm QCD}}\,\ln^2\frac{m_B\Lambda_{\rm QCD}}{m_\mu^2}$$



$\hookrightarrow$ soft fermion exchange dominates in a hypothetical world with larger $\alpha_{\rm em}$ and m_B.

Soft fermion exhange leads to a transfer of charge (lepton flavour, electric, colour, ...) and particle identity in the collinear directions.

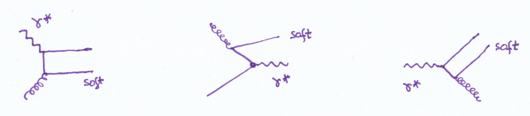
Causes a new type of Sudakov double logs proportional to the transfer of charge.

“Off-diagonal” channels and soft quarks

- Soft fermion coupling to collinear modes is power-suppressed. Leading interaction

$$\mathcal{L}^{(1)}_{\xi q} = \bar{q}_s\, W_c^\dagger i\not{D}_{\perp c}\, \xi_c - \bar{\xi}_c\, i\overleftarrow{\not{D}}_{\perp c} W_c\, q_s + \mathcal{O}(\lambda^2)$$

- $1 \to 2\,/\,2 \to 1$ off-diagonal high-energy scattering (threshold)



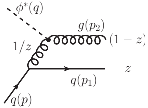
- Two intriguing observations:
  - Off-diagonal parton splitting kernel [Vogt et al., 2010] is a two-scale object

$$P^{\rm LL}_{gq}(N) = \frac{1}{N}\frac{\alpha_s C_F}{\pi}\,\mathcal{B}_0(a), \qquad a = \frac{\alpha_s}{\pi}(C_F - C_A)\ln^2 N\,,$$

  - Quark Sudakov exponentiation conjecture for the $\gamma^* \to g + (q\bar{q})$ amplitude for $s_{q\bar{q}} \ll Q^2$ [Moult et al. 2019] contains

$$\frac{\alpha_s}{4\pi}\ln N \exp\left(-\alpha_s C_F/\pi \ln^2 N\right) \times \frac{e^{-a}-1}{a}$$

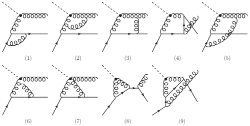
$q + \phi^* \to X$ DIS for $M_X^2 \ll Q^2$



Virtual correction to the structure function

$$W_{\phi,q}\big|_{q\phi^*\to qg} = \int_0^1 dz \left(\frac{\mu^2}{s_{qg} z\bar{z}}\right)^{\epsilon} \mathcal{P}_{qg}(s_{qg}, z)\Big|_{s_{qg}=Q^2\frac{1-x}{x}}$$

$$\mathcal{P}_{qg}(s_{qg}, z)\big|_{\text{tree}} = \frac{\alpha_s C_F}{2\pi}\frac{\bar{z}^2}{z} + \mathcal{O}(\epsilon, \lambda^2)$$



Endpoint divergent, gives NLP LL at $\mathcal{O}(\alpha_s)$.

One-loop correction, double pole part ($\leftrightarrow$ NLP LL)

$$\mathscr{P}_{qg}(z)|_{1-\text{loop}} = \mathscr{P}_{qg}(z)|_{\text{tree}} \frac{\alpha_s}{\pi}\frac{1}{\epsilon^2}\left\{ T_1\cdot T_0 \left(\frac{\mu^2}{zQ^2}\right)^{\epsilon} + T_2\cdot T_0 \left(\frac{\mu^2}{\bar{z}Q^2}\right)^{\epsilon} + T_1\cdot T_2 \left[\left(\frac{\mu^2}{Q^2}\right)^{\epsilon} - \left(\frac{\mu^2}{zQ^2}\right)^{\epsilon}\right]\right\}$$

Formally single pole

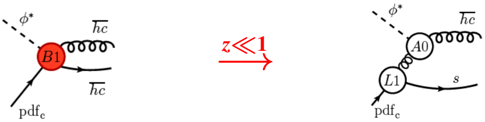
Promoted to leading pole after integration

$$\frac{1}{\epsilon^2}\int_0^1 dz \, \frac{1}{z^{1+\epsilon}}(1 - z^{-\epsilon}) = -\frac{1}{2\epsilon^3}$$

- Expansion in ϵ before integration gives wrong $-1/\epsilon^3$. No leading pole for soft gluon emission.
- Resummation of terms singular as $z \to 0$ is required to all orders, z^{ϵ} counts as $\mathcal{O}(1)$
- **zQ^2 as an emergent new intermediate scale**

z-SCET interpretation and refactorization

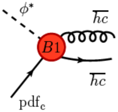
$$\int_0^1 dz C^{B1}(Q,z) \quad \times$$



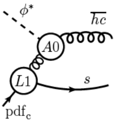
For $z \ll 1$ the C^{B1} matching coefficient is a two-scale object and must be resummed. Construct an “auxiliary” z-SCET containing z-soft and z-anti-softcollinear modes.

z-SCET interpretation and refactorization

$$\int_0^1 dz C^{B1}(Q,z) \quad \times$$



$\xrightarrow{z \ll 1}$



For $z \ll 1$ the C^{B1} matching coefficient is a two-scale object and must be resummed. Construct an “auxiliary” z-SCET containing z-soft and z-anti-softcollinear modes.

Multi-scale object

$$C^{B1}(Q,z)J^{B1}(z) \xrightarrow{z\to 0} C^{A0}(Q^2)\int d^4x\, T\left\{J^{A0}, \mathscr{L}^{(1)}_{\xi q_{z-\overline{sc}}}(x)\right\} = C^{A0}(Q^2)\, D^{B1}(zQ^2,\mu^2)\, J^{B1}_{z-\overline{sc}}$$

Single-scale objects

- D^{B1} appears as a universal coefficient that renormalizes soft quark emission. Its double logarithms are proportional to the change of colour charge of the collinear particles.
- The same coefficient appears in the endpoint factorization theorem for $H \to gg$ through light-quark loops [Liu, Neubert, Schnubel, Wang, 2022]

Renormalization of the soft-quark emission coefficient

$$[\![C_1^{\mathrm{B1}}(Q^2,r)]\!] = C^{\mathrm{A0}}(Q^2) \times \frac{D^{\mathrm{B1}}(rQ^2)}{r}$$

The soft-quark limit can be obtained from the limit $r \to 0$ of the full NLP B1 operator and its RGE:

$$D^{\mathrm{B1}}(p^2) = 1 + \frac{\alpha_s}{4\pi}(C_F - C_A)\left(\frac{2}{\epsilon^2} - 1 - \frac{\pi^2}{6}\right)\left(\frac{\mu^2}{-p^2 - i\varepsilon}\right)^{\epsilon} + \mathcal{O}(\alpha_s^2).$$

$$\frac{d}{d\ln\mu}D^{B1}(p^2) = \int_0^\infty d\hat{p}^2\,\gamma_D(\hat{p}^2,p^2)D^{B1}(\hat{p}^2)\,,$$

with the asymptotic anomalous dimension

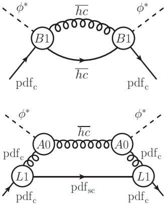
$$\gamma_D(\hat{p}^2,p^2) = \frac{\alpha_s(C_F - C_A)}{\pi}\,\delta(\hat{p}^2 - p^2)\ln\left(\frac{\mu^2}{-p^2 - i\varepsilon}\right)$$

$$+ \underbrace{\frac{\alpha_s}{\pi}\left(\frac{C_A}{2} - C_F\right)p^2\left[\frac{\theta(\hat{p}^2 - p^2)}{\hat{p}^2(\hat{p}^2 - p^2)} + \frac{\theta(p^2 - \hat{p}^2)}{p^2(p^2 - \hat{p}^2)}\right]_+}_{\text{relevant to NLL only}}$$

Off-diagonal DIS for $x \to 1$

TH
- *Hard*, $p^2 = Q^2$
- *Anti-hardcollinear*, $p^2 = Q^2\lambda^2 = Q^2/N$

PDF
- *Collinear*, $p^2 = \Lambda^2$
- *Soft-Collinear*, $p^2 = \Lambda^2\lambda^2 = \Lambda^2/N$



Consistency relations: Each region contribution different dependence on N,Q and ϵ. Impose pole cancellation on the general Ansatz.

$$W^{NLP}_{\phi,q} U^{LP}_{qq} + W^{LP}_{\phi,g} U^{NLP}_{gq} = W^{NLP}_{\phi,q} \exp\left[-\frac{\alpha_s C_F}{\pi \epsilon^2}\left(\frac{\mu^2}{\Lambda^2}\right)^{\epsilon}(N^{\epsilon}-1)\right] + \exp\left[\frac{\alpha_s C_A}{\pi\epsilon^2}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}(N^{\epsilon}-1)\right] U^{NLP}_{gq}$$

$$\stackrel{!}{=} \frac{1}{N}\sum_{n=1}\left(\frac{\alpha_s}{4\pi}\right)^n \frac{1}{\epsilon^{2n-1}} \sum_{k=0}^{n}\sum_{j=0}^{n} c^{(n)}_{kj} \left(\frac{\mu^{2n} N^j}{Q^{2k}\Lambda^{2(n-k)}}\right)^{\epsilon}$$

$$c^{(n)}_{n1} = \frac{1}{2}(-4)^n \frac{C_F}{C_F - C_A}\frac{C_F^n - C_A^n}{n!} = \frac{(-4)^n}{2n!} C_F \left(C_F^{n-1} + C_F^{n-2} C_A + \cdots + C_A^{n-1}\right) .$$

All $c^{(n)}_{kj}$ can be obtained from the all-hard loop region, $c^{(n)}_{n1}$, or the $D^{B1}(p^2)$ coefficient from consistency. Booststrap the full solution algebraically from the soft quark Sudakov factor.

Off-diagonal DIS – LL DGLAP kernel and coefficient function

$$Z_{gq}^{NLP,LL} = \frac{1}{2N\ln N}\frac{C_F}{C_F-C_A}\exp\left[\frac{\alpha_s C_F}{\pi}\frac{\ln N}{\epsilon}\right]$$

$$F_{\rm pole}(w,a)\gamma_{gq}^{NLP,LL}(N) = \frac{1}{N}\frac{\alpha_s C_F}{\pi}\left[F_{\rm pole}(w,a) - w\frac{d}{da}F_{\rm pole}(w,a)\right] = -\frac{1}{N}\frac{\alpha_s C_F}{\pi}\mathcal{B}_0(a)$$

$$F_{\rm pole}(w,a) = \sum_{k\geq 1}\frac{1}{w^k}\sum_{n\geq 0}\frac{B_n}{n!(n+k)!}a^{n+k}\,, \qquad \mathcal{B}_0(x) = \sum_{n=0}^{\infty}\frac{B_n}{(n!)^2}x^n \quad \text{and} \quad a = \frac{\alpha_s}{\pi}(C_F-C_A)$$

$$\tilde{C}_{\phi,q}^{NLP,LL}\Big|_{\epsilon\to 0} = \frac{1}{2N\ln N}\frac{C_F}{C_F-C_A}\left(\mathcal{B}_0(a)\exp\left[C_A\frac{\alpha_s}{\pi}\left(\frac{1}{2}\ln^2 N + \ln N\ln\frac{\mu^2}{Q^2}\right)\right]\right.$$
$$\left. - \exp\left[\frac{\alpha_s C_F}{\pi}\left(\frac{1}{2}\ln^2 N + \ln N\ln\frac{\mu^2}{Q^2}\right)\right]\right)$$

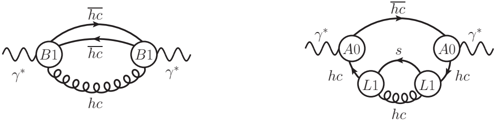
The Bernoulli function arises as a consequence of $\overline{\rm MS}$ factorization. Proves Vogt's conjecture for the all-order series.

Extensions and remarks

- Used factorization in d dimensions. To apply standard 4D RGE techniques, must perform subtractions and rearrangements between the B1 and $T(\text{A0}, \mathcal{L}^{(1)})$ term.

  $\longrightarrow$ J. Strohm's talk on "gluon thrust"

- Same framework applies to thrust in the 2-jet region.
  No Bernoulli numbers, since no PDF factorization but final state jets.



- By crossing and reinterpetation of modes, one obtains a result for the NLP LLs in the off-diagonal DY cross section near threshold. Here the soft-quark Sudakov factor applies to the collinear function in the refactorized convolution $J \otimes S$ of the collinear and soft function.

- Provides insight into why for the off-diagonal channel even the splitting function contains double logs. Soft gluon emission does not cause this effect, because the collinear particle before and after emission has the same colour charge.