DIAGRAMMATIC METHODS FOR THRESHOLD RESUMMATION BEYOND LEADING POWER

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OUTLINE

- **Particle scattering near threshold**
- **Factorization beyond Leading Power**
- **Resummation: diagonal and off-diagonal channels**

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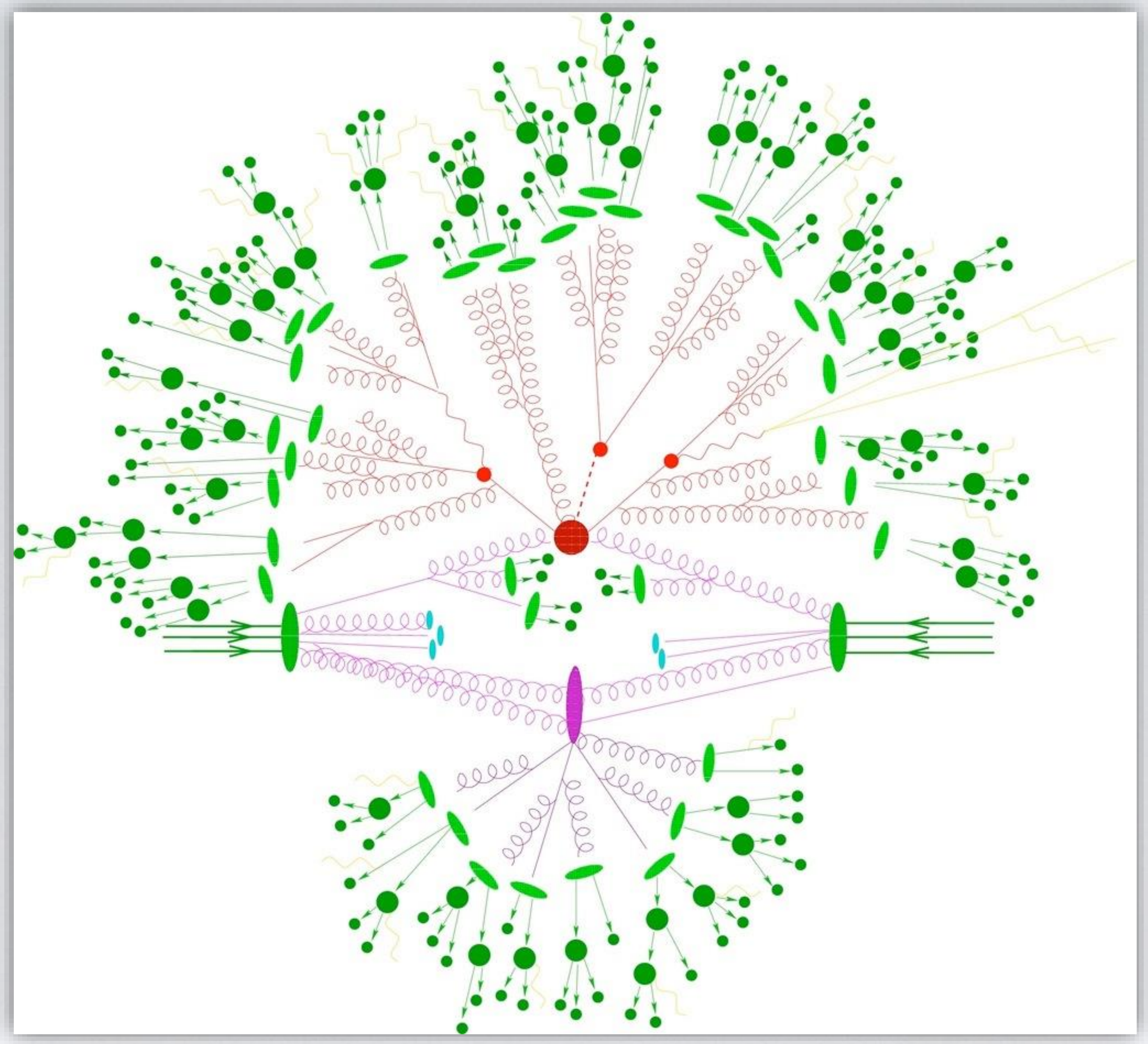
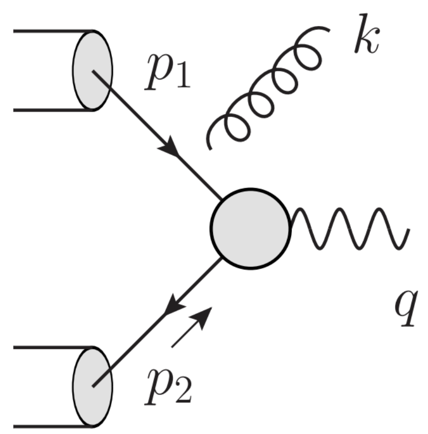
PARTICLE SCATTERING NEAR THRESHOLD

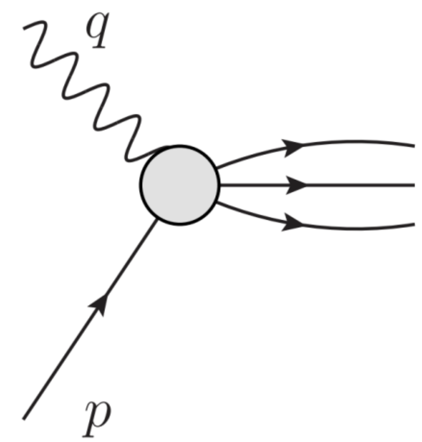


PARTICLE SCATTERING NEAR THRESHOLD

- Consider Drell-Yan and DIS near partonic threshold:



$$(p_1 + p_2)^2 \equiv \hat{s}, \qquad z \equiv \frac{q^2}{\hat{s}} \to 1$$



$$Q^2 = -q^2, \qquad x \equiv \frac{Q^2}{2p\cdot q} \to 1$$

- The partonic cross section has singular expansion

$$\Delta_{ab}(\xi) \sim \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \left[c_n \delta(1-\xi) + \sum_{m=0}^{2n-1} \left(c_{nm} \left[\frac{\ln^m(1-\xi)}{1-\xi}\right]_+ + d_{nm} \ln^m(1-\xi) \right) + \dots \right],$$

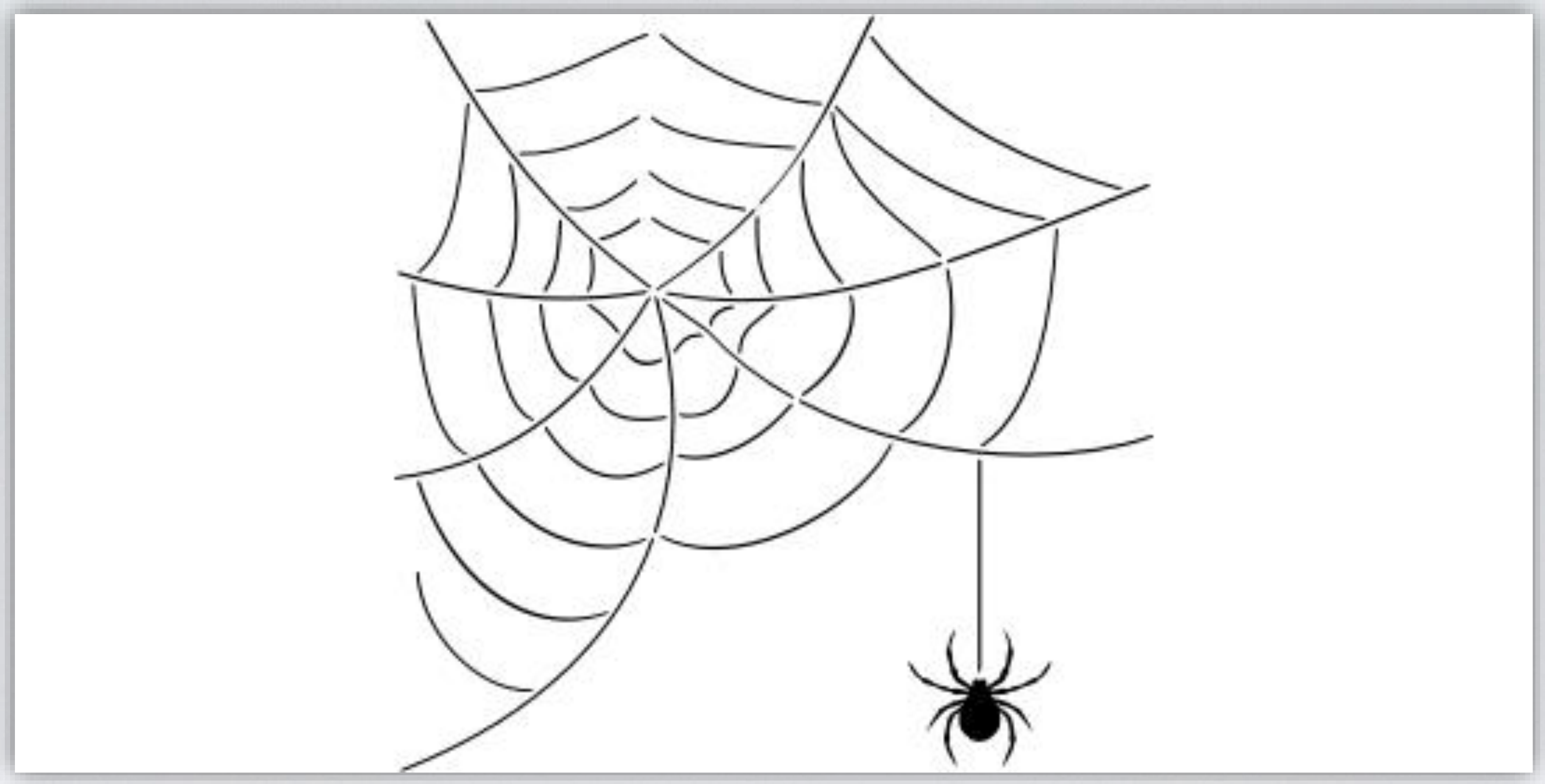
LP **NLP**

with $\xi = z$ for DY or x for DIS.

- Resummation of large logarithms at next-to-leading power (NLP):

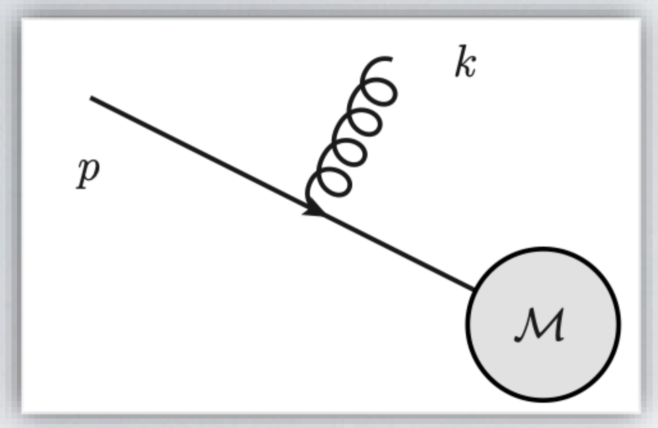
→ interesting theoretical challenge, relevant for precision phenomenology!

FACTORIZATION NEAR THRESHOLD: LP VS NLP



FACTORIZATION OF SOFT GLUONS AT LP

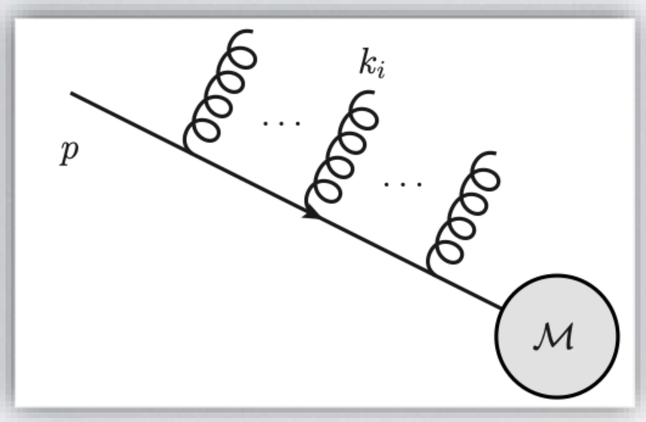
- Emission of a soft gluon from an energetic parton (quark):



"Eikonal" factor

$$= \mathcal{M}\,\frac{\not{p}-\not{k}}{2p\cdot k}\,\gamma^\mu\,T^A\,u(p) \;\sim\; \mathcal{M}\,\frac{p^\mu}{p\cdot k}\,T^A\,u(p).$$

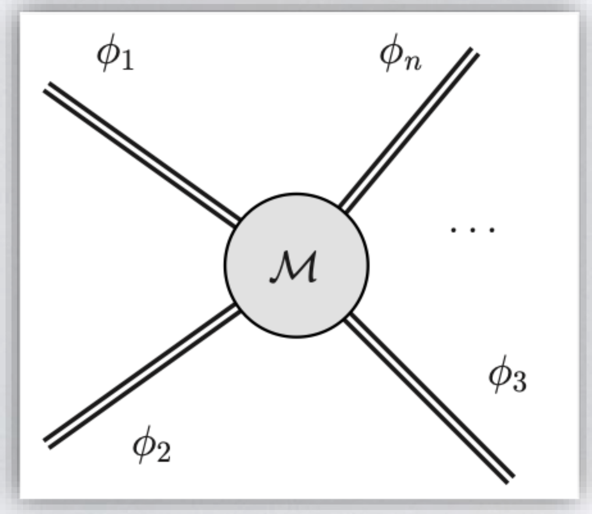
- Emission of multiple soft gluons factorizes:



$$\sim \mathcal{M}\,\mathcal{S}\,u(p), \qquad \mathcal{S} = \langle 0|\Phi_\beta(-\infty,0)|0\rangle,$$

$$\Phi_\beta(\lambda_1,\lambda_2) = \mathcal{P}\exp\left\{i\,g_s\int_{\lambda_1}^{\lambda_2} d\lambda\;\beta\cdot A(\lambda\beta)\right\}.$$

- In general



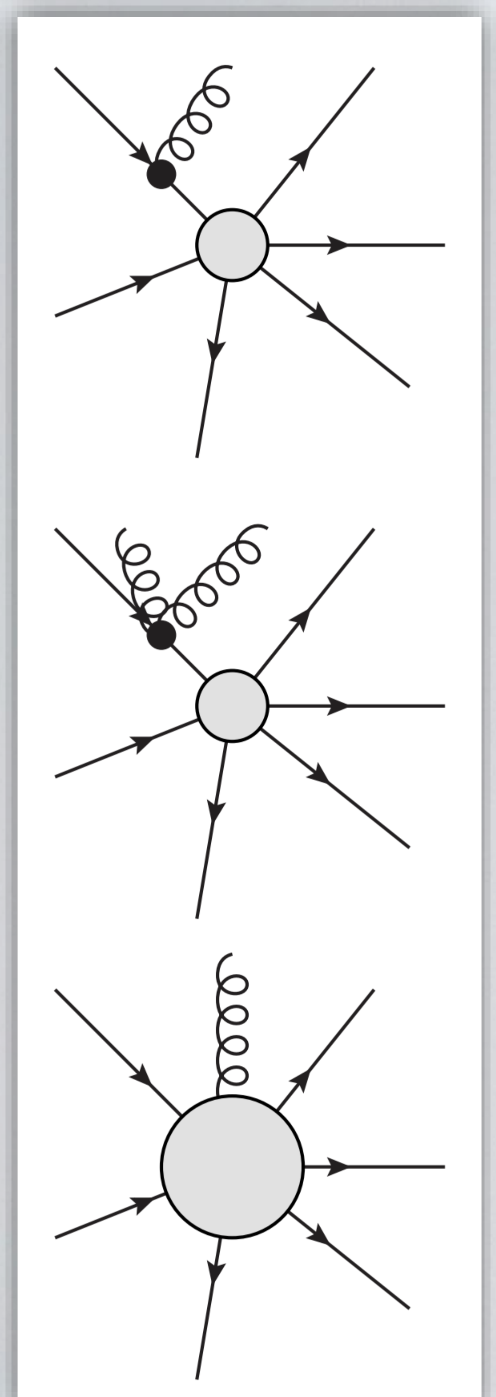
$$\sim \mathcal{M}\,\mathcal{S}\,u(p_1)\bar{v}(p_2)\ldots\bar{u}(p_n),$$

$$\mathcal{S} = \langle 0|\Phi_1\ldots\Phi_n|0\rangle \sim e^{\mathcal{W}_E}.$$

Gatheral, 1983; Frenkel, Taylor, 1984; Sterman, 1987; Catani,Trentadue, 1989; Korchemsky, Marchesini, 1992,1993; ...

FACTORIZATION OF SOFT GLUONS BEYOND LP

One needs to consider several effects:

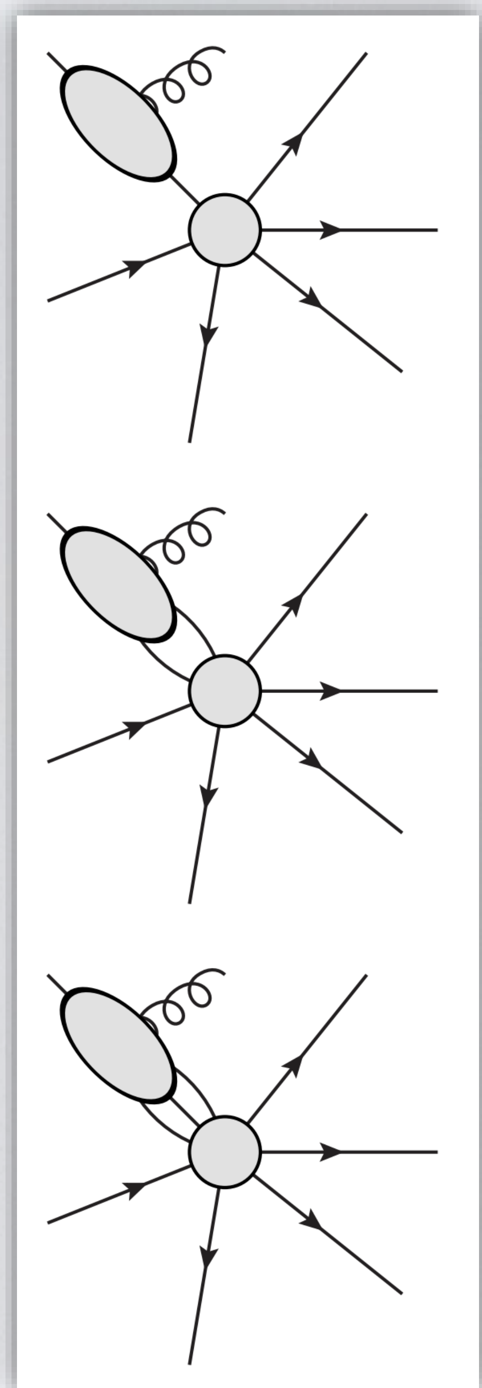


- Emission of soft gluons beyond the eikonal approximation, for instance sensitive to the spin of the emitting particle

Laenen, Magnea, Stavenga, White, 2009, 2010; Bonocore, Laenen, Magnea, LV, White, 2016.

- The soft emission resolve the hard interaction (LBK theorem)

Low 1958, Burnett,Kroll 1968



- Emission of soft gluons from a cluster of collinear particles: one finds several types of "radiative jets".

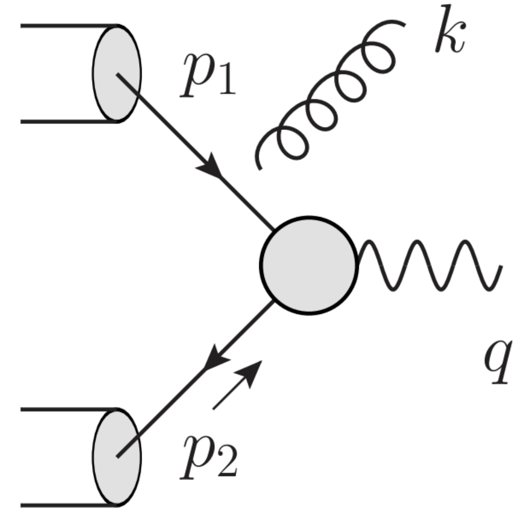
Del Duca 1990;

Bonocore, Laenen, Magnea, Melville, LV, White, 2015,2016;

Gervais 2017;

Laenen, Sinninghe-Damsté, LV, Waalewijn, Zoppi, 2020

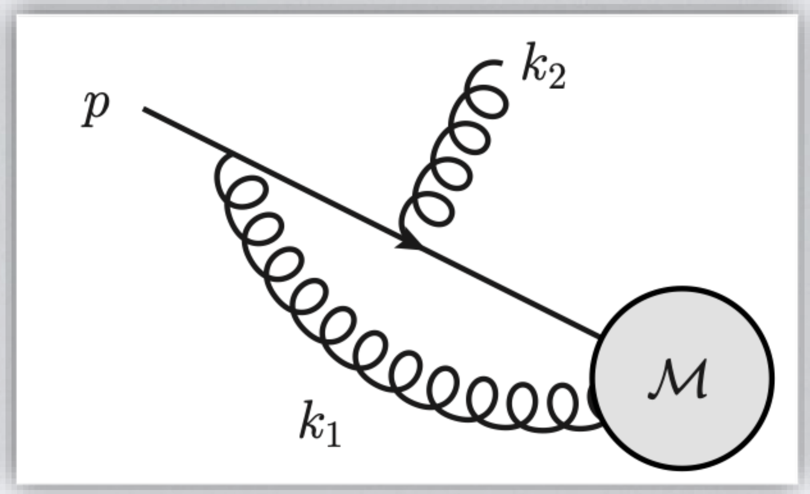
A CASE STUDY: DRELL-YAN



$$(p_1 + p_2)^2 \equiv \hat{s}, \quad z \equiv \frac{q^2}{\hat{s}} \to 1$$

EXPANSION BY REGIONS

- Beyond leading power one has non-trivial effects due to virtual gluons:



- The loop momentum runs over all scales, cannot be treated as soft:

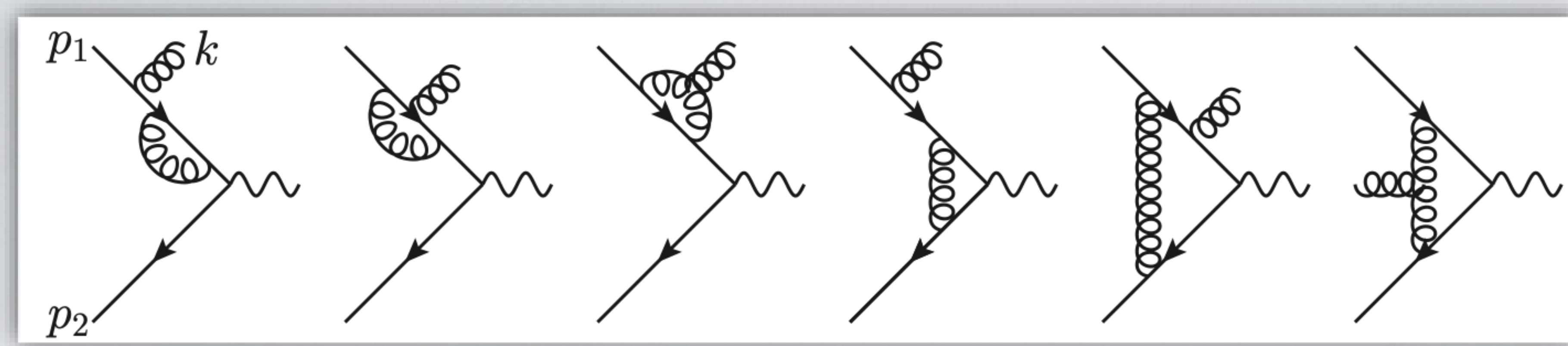
$$k = n_+ \cdot k \frac{n_-}{2} + n_- \cdot k \frac{n_+}{2} + k_\perp,$$

$$n_\pm^2 = 0, \qquad n_- \cdot n_+ = 2, \quad n_- \sim \frac{p}{\hat{s}},$$

$$\begin{aligned} \text{hard:} \quad & k \sim \sqrt{\hat{s}}(1,1,1) \\ \text{collinear:} \quad & k \sim \sqrt{\hat{s}}(1,\lambda,\lambda^2) \\ \text{anti-collinear:} \quad & k \sim \sqrt{\hat{s}}(\lambda^2,\lambda,1) \\ \text{soft:} \quad & k \sim \sqrt{\hat{s}}(\lambda^2,\lambda^2,\lambda^2), \qquad \lambda \ll 1. \end{aligned}$$

Bonocore, Laenen, Magnea, LV, White, 2014

EXPANSION BY REGIONS: LP



$$\hat{s} = (p_1 + p_2)^2,$$
$$t = (p_1 - k)^2,$$
$$u = (p_2 - k)^2.$$

- Virtual gluons gives non-analytical contributions ∝ to the scales of the problem: LP

$$|\mathcal{M}|^2 \propto \overbrace{\frac{\hat{s}^2}{tu}}^{\text{LP}} \left\{ C_F^2 \overbrace{\left(\frac{\mu^2}{-s}\right)^\epsilon}^{\text{hard}} \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} + \dots\right) + C_A C_F \overbrace{\left(\frac{\hat{s}\,\mu^2}{t\,u}\right)^\epsilon}^{\text{soft}} \left(-\frac{1}{\epsilon^2} + \dots\right)\right\} + \dots$$

↓

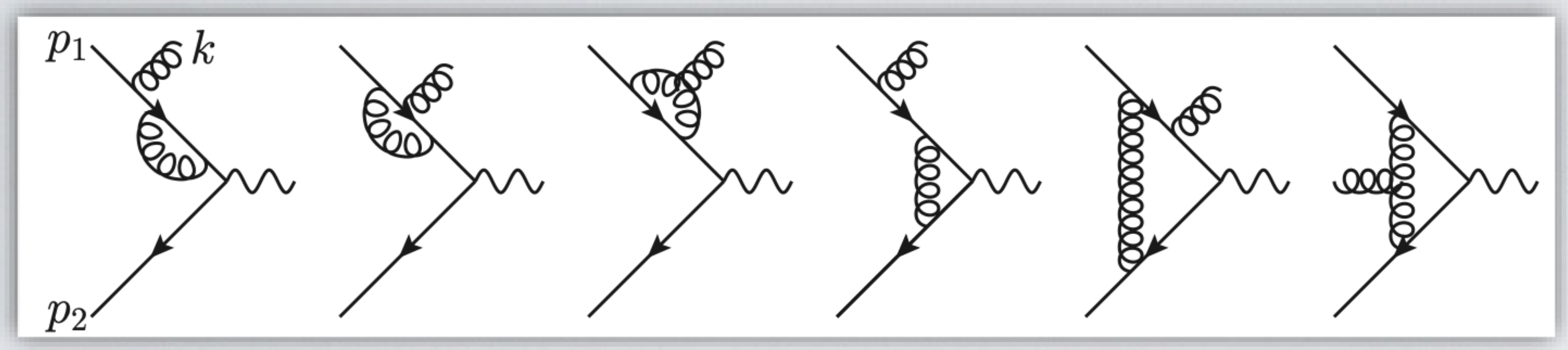
Factorisation

↓

$$S\left[\frac{\hat{s}\,\mu^2}{t\,u}, \epsilon\right] \times H\left[\frac{\mu^2}{-\hat{s}}, \epsilon\right]$$

- Factorization: physics at different scales is uncorrelated.

EXPANSION BY REGIONS: NLP



$$\hat{s} = (p_1 + p_2)^2, \quad t = (p_1 - k)^2, \quad u = (p_2 - k)^2.$$

- Goal: factorize non-analytical contributions ∝ to the scales of the problem:

$$|\mathcal{M}|^2 \propto C_F^2 \Bigg\{ \overbrace{\frac{\hat{s}(t+u)}{tu}}^{\text{NLP}} \overbrace{\left(\frac{\mu^2}{-\hat{s}}\right)^{\epsilon}}^{\text{hard}} \left(-\frac{2}{\epsilon^2} - \frac{1}{\epsilon} + \dots \right) + \Bigg[\overbrace{\frac{s}{t}}^{\text{NLP}} \overbrace{\left(\frac{\mu^2}{-t}\right)^{\epsilon}}^{\text{coll.}} + \overbrace{\frac{\hat{s}}{u}}^{\text{NLP}} \overbrace{\left(\frac{\mu^2}{-u}\right)^{\epsilon}}^{\text{anti-coll.}} \Bigg] \left(-\frac{2}{\epsilon} + \dots \right) \Bigg\}$$

$$+ C_A C_F \overbrace{\frac{\hat{s}(t+u)}{tu}}^{\text{NLP}} \overbrace{\left(\frac{\hat{s}\,\mu^2}{t\,u}\right)^{\epsilon}}^{\text{soft}} \left(-\frac{1}{\epsilon^2} + \dots \right) + \Bigg[\overbrace{\frac{\hat{s}}{t}}^{\text{NLP}} \overbrace{\left(\frac{\mu^2}{-t}\right)^{\epsilon}}^{\text{coll.}} + \overbrace{\frac{\hat{s}}{u}}^{\text{NLP}} \overbrace{\left(\frac{\mu^2}{-u}\right)^{\epsilon}}^{\text{anti-coll.}} \Bigg] \left(-\frac{5}{2} + \dots \right) \Bigg\} + \dots$$

↓

Factorization?

↓

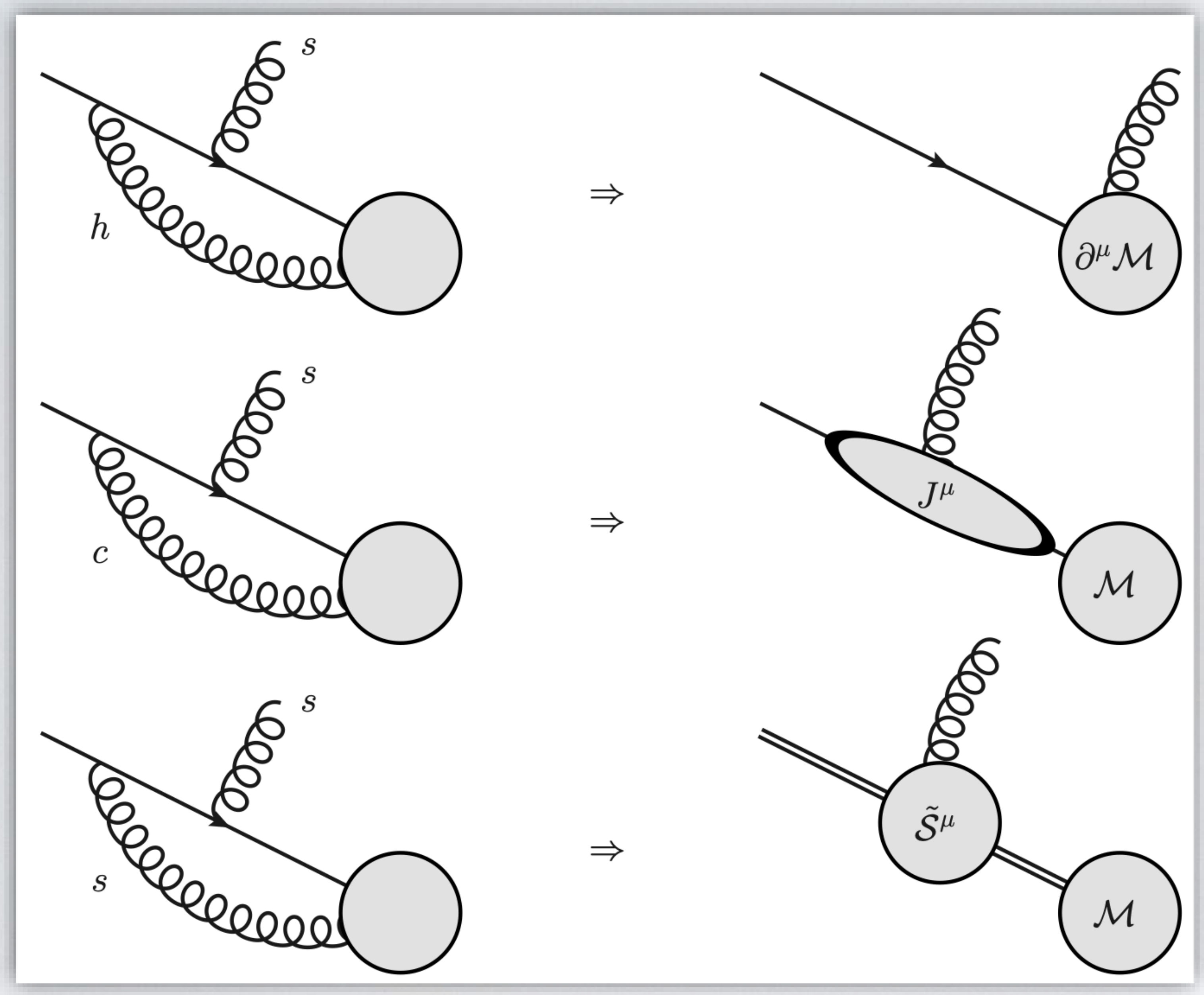
$$S\left[\frac{\hat{s}\,\mu^2}{t\,u}, \epsilon\right] \times J\left[\frac{\mu^2}{-t}, \epsilon\right] \times \bar{J}\left[\frac{\mu^2}{-u}, \epsilon\right] \times H\left[\frac{\mu^2}{-\hat{s}}, \epsilon\right]$$

Bonocore, Laenen, Magnea, LV, White, 2014

- Need an effective approach to take into account hard, collinear and soft modes.

DIAGRAMMATIC FACTORIZATION AT NLP

- Goal: factorize hard, collinear and soft modes;
- describe them in terms of simpler, universal functions in QCD:



Low 1958,

Burnett,Kroll 1968

Del Duca 1990,

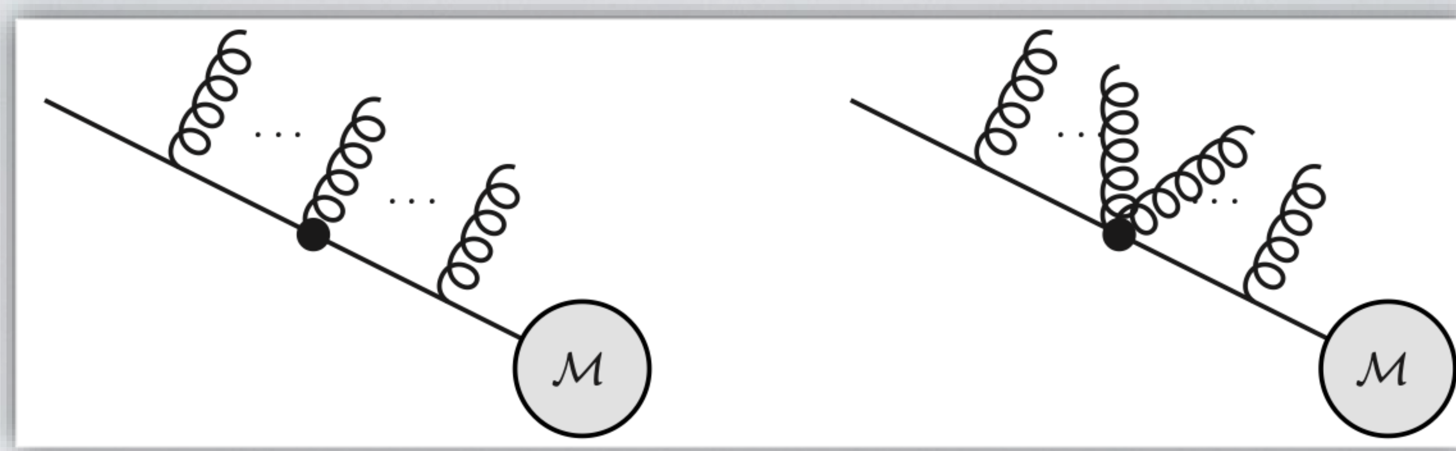
Bonocore, Laenen, Magnea, Melville, LV, White, 2015,2016

Laenen, Magnea, Stavenga, White, 2009, 2010

Bonocore, Laenen, Magnea, LV, White,2016

(NEXT-TO-)SOFT GLUONS

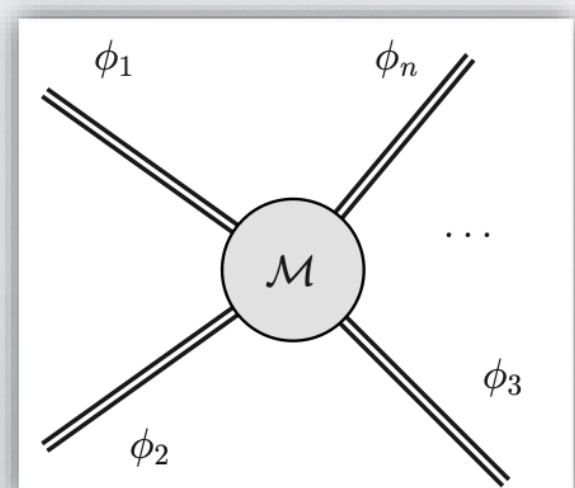
- Emission of soft gluons at NLP described in terms of "NLP" Wilson lines:



Eikonal — NE, spin indep. — NE, spin dep.

$$F_p(-\infty,0)=\mathcal{P}\exp\left[g\int\frac{d^dk}{(2\pi)^d}A_\mu(k)\left(-\frac{p^\mu}{p\cdot k}+\frac{k^\mu}{2p\cdot k}-k^2\frac{p^\mu}{2(p\cdot k)^2}-\frac{\mathrm{i}k_\nu\Sigma^{\nu\mu}}{p\cdot k}\right)\right.$$
$$+\int\frac{d^dk}{(2\pi)^d}\int\frac{d^dl}{(2\pi)^d}A_\mu(k)A_\nu(l)\left(\frac{\eta^{\mu\nu}}{2p\cdot(k+l)}-\frac{p^\nu l^\mu p\cdot k+p^\mu k^\nu p\cdot l}{2(p\cdot l)(p\cdot k)\,[p\cdot(k+l)]}\right.$$
$$\left.\left.+\frac{(k\cdot l)p^\mu p^\nu}{2(p\cdot l)(p\cdot k)\,[p\cdot(k+l)]}-\frac{\mathrm{i}\Sigma^{\mu\nu}}{p\cdot(k+l)}\right)\right].$$

NE, "seagulls"



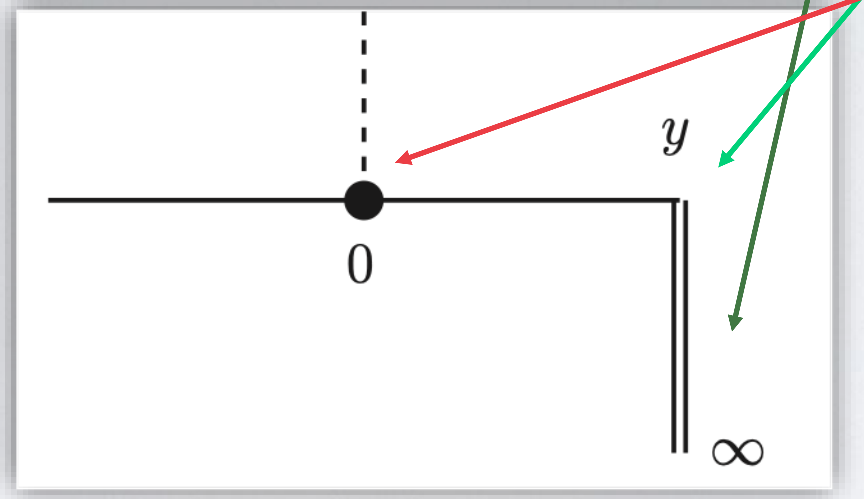
$$\tilde{S}=\langle 0|F_1\dots F_n|0\rangle\sim e^{\sum_{G_e}C_{G_e}\mathcal{W}_e+\sum_{G_{ne}}C_{G_{ne}}\mathcal{W}_{ne}}.$$

Laenen, Magnea, Stavenga, White, 2009, 2010

(VIRTUAL) COLLINEAR MODES

- Collinear modes can be described by introducing a radiative jet function: *Del Duca 1990*

$$J_{\mu,a}\,(p,n,k)\,u(p) \;=\; \int d^d y\;\mathrm{e}^{-\mathrm{i}(p-k)\cdot y}\;\langle 0\,|\,\Phi_n(\infty,y)\,\psi(y)\,j_{\mu,a}(0)\,|\,p\rangle\;.$$



Bonocore, Laenen, Magnea, Melville, LV, White, 2015,2016

- The current $j_a^\mu(x)$ must be conserved: $\partial_\mu j^\mu(x)=0$;
- The radiative jet must satisfy the Ward identity:

$$k_\mu J^{\mu,a}\,(p,n,k) \;=\; g\,\mathbf{T}^a J\,(p,n)\;, \qquad J(p,n)u(p) \;=\; \langle 0\,|\Phi_n(\infty,0)\psi(0)|\,p\rangle\;,$$

- The following current does the job:

$$j_a^\mu(x) \;=\; g\,\Big\{-\overline{\psi}(x)\,\gamma^\mu\,\mathbf{T}_a\,\psi(x) \,+\, f_a{}^{bc}\Big[F_c^{\mu\nu}(x)\,A_{\nu\,b}(x) + \partial_\nu\left(A_b^\mu(x)A_c^\nu(x)\right)\Big]\Big\}\;.$$

Noether current

"Improvement terms"

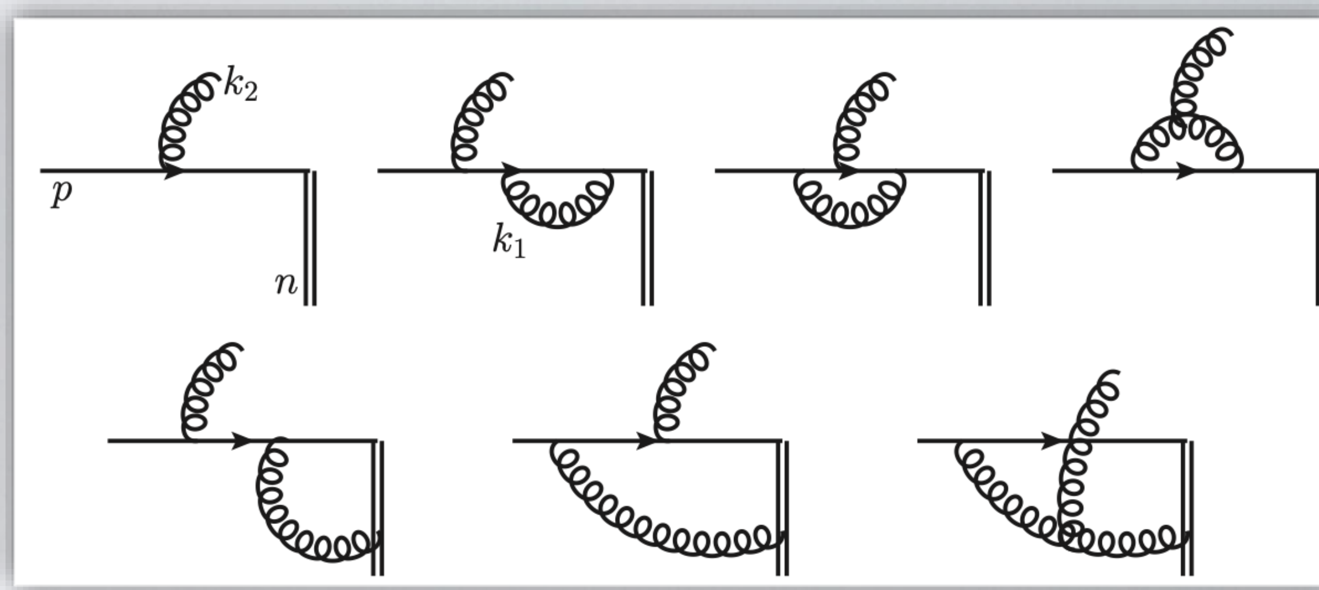
(De Wit, Smith, 1986)

(VIRTUAL) COLLINEAR MODES

- At one loop:

$$J_\mu^{(1)} = (-2p\cdot k)^{-\epsilon}\left[C_F J^{(1)}_{\mu,F} + C_A J^{(1)}_{\mu,A,\mathrm{coll.}}\right]$$

$$+\left(\frac{2p\cdot n}{(-2p\cdot k)(-2n\cdot k)}\right)^{\epsilon} C_A\, J^{(1)}_{\mu,A,\mathrm{soft}}\,,$$



where e.g.

$$J^{(1)}_{\mu,F} = (1+2\epsilon)\,\frac{\not{k}}{t}\,\gamma^\mu + (2+6\epsilon)\,\not{n}_p\gamma^\mu + \left[\frac{2}{\epsilon} - 2 - \epsilon\,(8+\zeta_2)\right]\frac{k^\mu}{t}$$

$$+\left[\frac{4}{\epsilon} + 4 + 2\epsilon\,(2-\zeta_2)\right] n_p^\mu + \left[\frac{4}{\epsilon} + 8 - 2\epsilon\,(-8+\zeta_2)\right]\frac{r}{t}\,p^\mu - 4\,(1+3\epsilon)\,\frac{\not{k}\not{n}_p}{t}\,p^\mu\,,$$

with

$$t = -2p\cdot k\,,\quad n_p^\mu = \frac{n^\mu}{2n\cdot p}\,,\quad n_k^\mu = \frac{n^\mu}{2n\cdot k}\,,\quad r = \frac{n\cdot k}{n\cdot p}\,.$$

- Note: J^μ contains both collinear and soft modes: remove soft modes double counting.

DIAGRAMMATIC FACTORIZATION AT NLP

- For Drell Yan we obtain:

$$\frac{k^\nu}{p_l \cdot k}\left[p_{l,\nu}\frac{\partial}{\partial p_l^\mu} - p_{l,\mu}\frac{\partial}{\partial p_l^\nu}\right] = -\frac{\mathrm{i}k^\nu\, \mathrm{L}^{(l)}_{\nu\mu}}{p_i \cdot k}$$

Orbital angular momentum (LBK)

$$\mathcal{A}_{\mu,a}(p_j,k) = \sum_{i=1}^{2}\left(\frac{1}{2}\widetilde{\mathcal{S}}_{\mu,a}(p_j,k) + g\,\mathbf{T}_{i,a}\,G^\nu_{i,\mu}\frac{\partial}{\partial p_i^\nu} + J_{\mu,a}\left(p_i,n_i,k\right)\right)\mathcal{A}(p_j) - \mathcal{A}^{\widetilde{\mathcal{J}}}_{\mu,a}(p_j,k)\,,$$

Removes soft-collinear overlap in the radiative jet

for $n_1 = p_2$, $n_2 = p_1$.

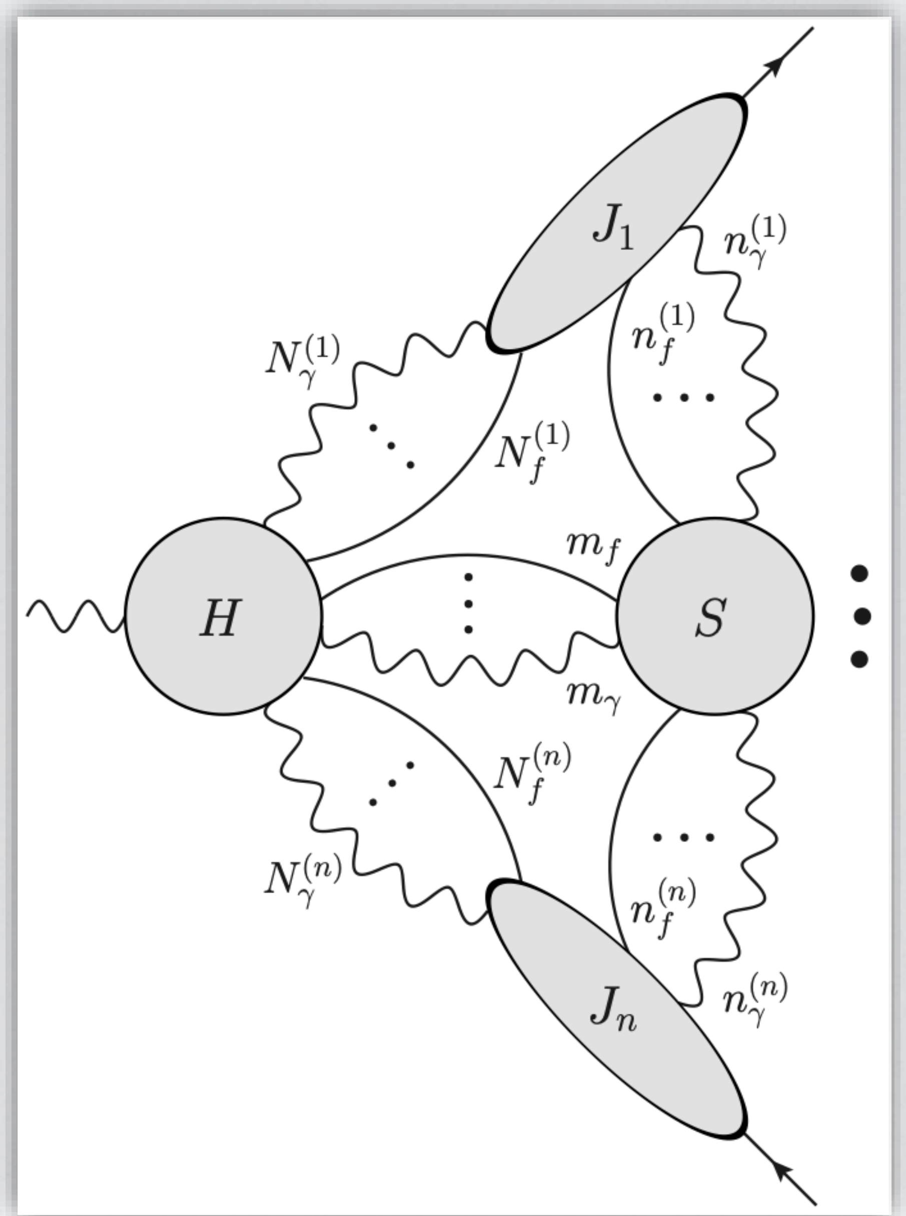
$$g_s\,T^A\left[\frac{p^\mu}{p\cdot k} - \frac{k^\mu}{2p\cdot k} + \frac{i\,k^\alpha\,\Sigma^{\alpha\mu}}{p\cdot k}\right]$$

Spin angular momentum (tree level)

Bonocore, Laenen, Magnea, LV, White, 2015,2016

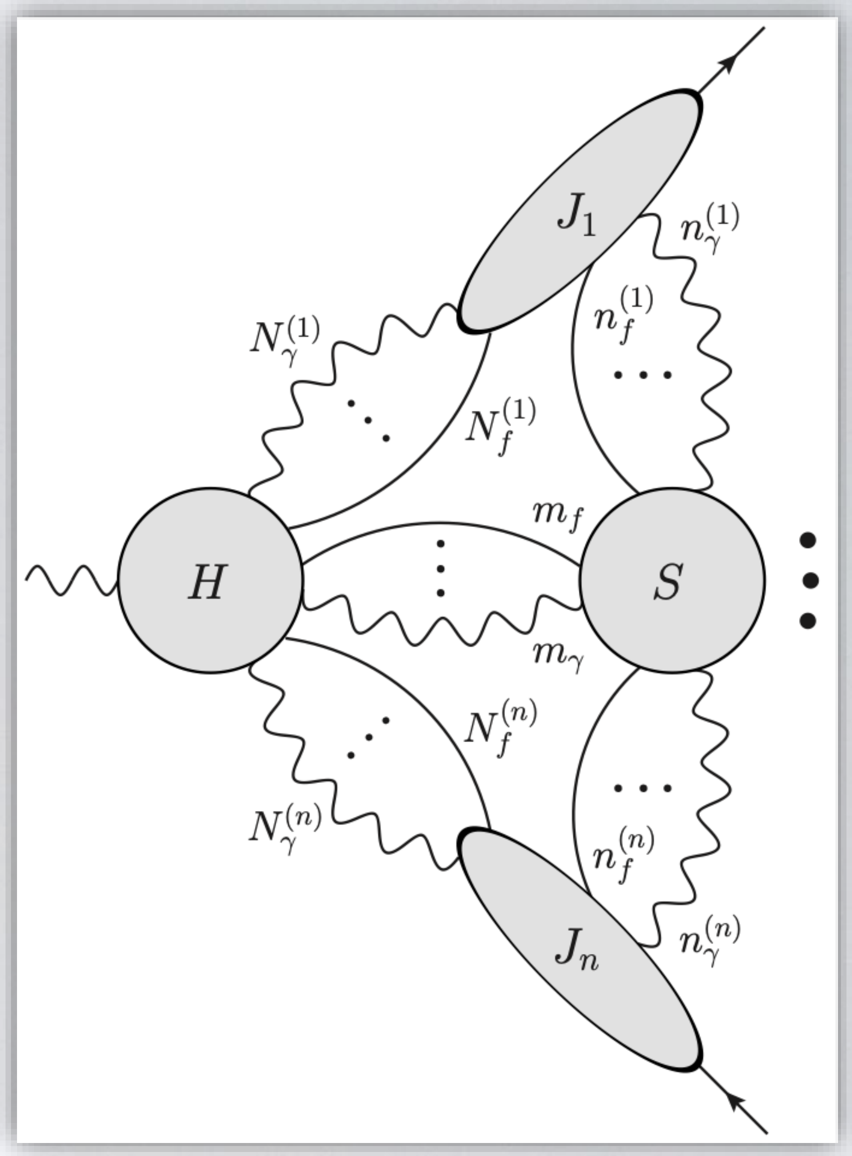
- Reproduces Drell-Yan up to NNLO.
- Hard and soft terms contributes LLs, collinear terms NLLs.

DIAGRAMMATIC FACTORIZATION AT NLP



DIAGRAMMATIC FACTORIZATION AT NLP

- The example discussed so far accurately describes DY up to NNLO. In general, one needs to take into account
  - processes with more than two external directions;
  - factorization beyond one loop;
  - Multiple soft gluon emission.
- Task: obtain a classification of the jet-like structures, consisting of virtual radiation collinear to any of the *n* external hard particles, contributing at subleading power in a parametrically small scale, corresponding to a fermion mass or a soft external momentum.



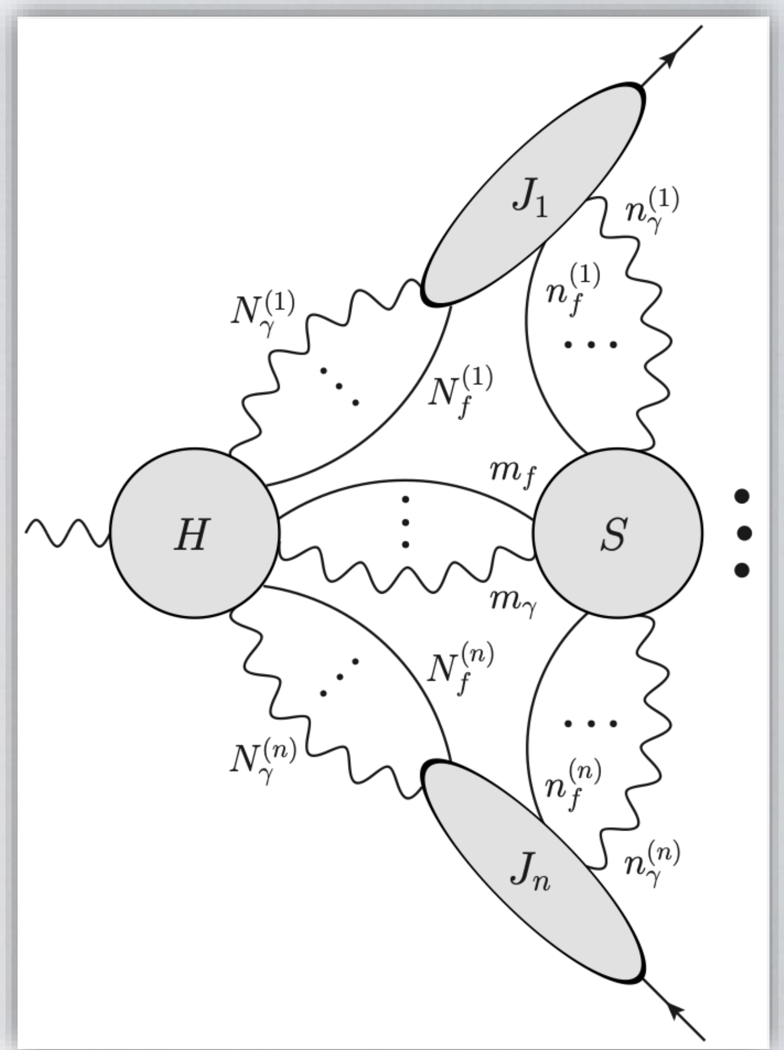
DIAGRAMMATIC FACTORISATION AT NLP

- This can be done systematically:
  - Decompose momenta along light-cone coordinates associated to the directions of the external particles:

$$n_i^\mu = \frac{1}{\sqrt{2}}\left(1, +\frac{\vec{p}_i}{|\vec{p}_i|}\right), \qquad \bar{n}_i^\mu = \frac{1}{\sqrt{2}}\left(1, -\frac{\vec{p}_i}{|\vec{p}_i|}\right), \qquad v^\mu = v^+ n_i^\mu + v^- \bar{n}_i^\mu + v_{\perp i}^\mu.$$

  - Define power counting for momenta and masses, e.g.:

$$\text{Soft:} \qquad k^\mu \sim Q\left(\lambda^2, \lambda^2, \lambda^2\right),$$
$$\text{Collinear:} \qquad k^\mu \sim Q\left(1, \lambda, \lambda^2\right).$$



  - Express an amplitude in terms of "reduced" diagrams: contract off-shell legs to a point, keep on-shell lines;
  - Determine the "superficial degree of divergence", i.e. the λ-scaling for each diagram.
  - It is useful to set up a catalogue of λ-scaling for individual components: for instance a fermion propagator has λ-scaling:

$$\frac{i\left(\not{p} + m\right)}{p^2 - m^2} \sim \frac{\gamma^- + \lambda^2\gamma^+ + \lambda\gamma^\perp + \lambda}{\lambda^2} \sim \frac{1}{\lambda^2}.$$

Sterman, 1978, 1996; Collins, Soper, Sterman, 1989;
Gervais, 2017; Laenen, Sinninghe Damsté, LV, Waalewijn, Zoppi, 2020

DIAGRAMMATIC FACTORISATION AT NLP

QED Vertex	Suppression
$\bar{\psi}^{(c)}\gamma^\mu\psi^{(c)}A_\mu^{(c)}$	λ
$\bar{\psi}^{(c)}\gamma^\mu\psi^{(c)}A_\mu^{(s)}$	1
$\bar{\psi}^{(s)}\gamma^\mu\psi^{(c)}A_\mu^{(c)}$ or $\bar{\psi}^{(c)}\gamma^\mu\psi^{(s)}A_\mu^{(c)}$	1
$\bar{\psi}^{(s)}\gamma^\mu\psi^{(s)}A_\mu^{(s)}$	1

	$m=0$	$m\sim\lambda Q$
Collinear fermion	λ^{-2}	
Soft fermion	λ^{-2}	λ^{-1}
Collinear photon	λ^{-2}	
Soft photon	λ^{-4}	
Collinear loop	λ^{4}	
Soft loop	λ^{8}	

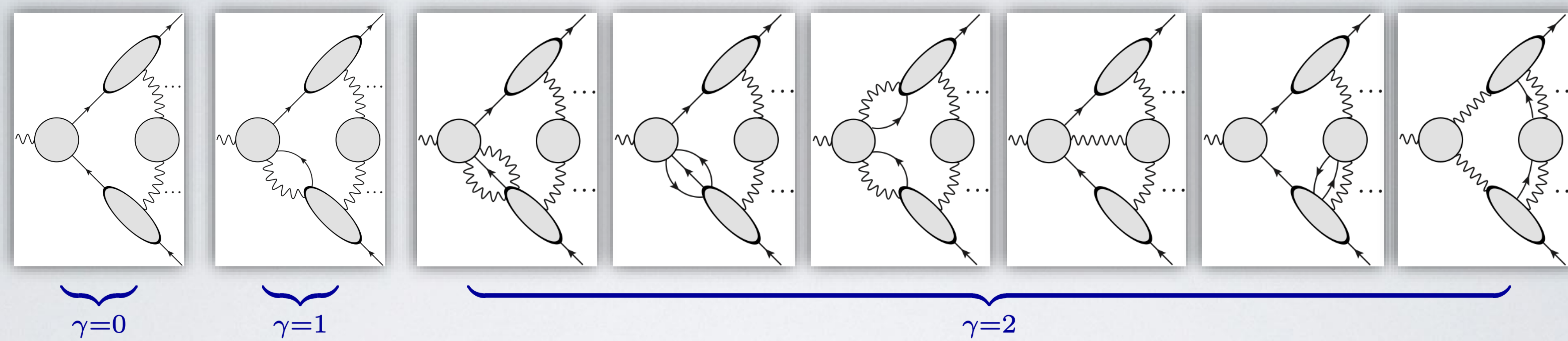
- Express the superficial degree of divergence as a function of the number of fermion and photon connections between the hard, soft and collinear subgraphs:

$$\gamma_{\mathcal{G}} = 2m_\gamma + 3m_f + \sum_{i=1}^{n}\left(N_\gamma^{(i)} + N_f^{(i)} + n_f^{(i)} - 1\right), \qquad (m=0),$$

$$\gamma_{\mathcal{G}} = I_f + 2m_\gamma + 4m_f + \sum_{i=1}^{n}\left(N_\gamma^{(i)} + N_f^{(i)} + 2n_f^{(i)} - 1\right), \qquad (m\sim\lambda Q).$$

- Such formula determines which diagrams (“pinch surfaces”) of an amplitude contributes up to NLP.

DIAGRAMMATIC FACTORISATION AT NLP



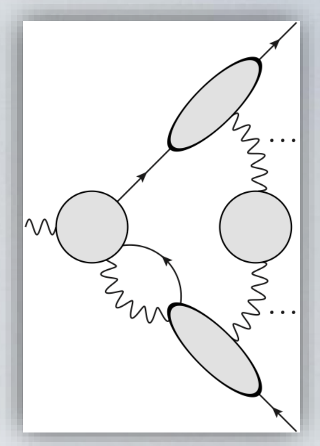
- In QED one obtains the all-orders factorization formula

Laenen, Sinninghe Damsté, LV, Waalewijn, Zoppi, 2020

$$\mathcal{M}^{\rm LP} = \left(\prod_{i=1}^{n} J_{(f)}(\hat{p}_i) \right) \otimes H(\hat{p}_1, \ldots, \hat{p}_n)\, S(n_i \cdot n_j)\,,$$

$$\mathcal{M}^{\rm NLP}_{\rm coll} = \sum_{i=1}^{n} \left(\prod_{j \neq i} J^{j}_{(f)} \right) \left[J^{i}_{(f\gamma)} \otimes H^{i}_{(f\gamma)} + J^{i}_{(f\partial\gamma)} \otimes H^{i}_{(f\partial\gamma)} \right] S \; + \sum_{i=1}^{n} \left(\prod_{j \neq i} J^{j}_{(f)} \right) J^{i}_{(f\gamma\gamma)} \otimes H^{i}_{(f\gamma\gamma)}\, S$$
$$+ \sum_{i=1}^{n} \left(\prod_{j \neq i} J^{j}_{(f)} \right) J^{i}_{(ff\!f)} \otimes H^{i}_{(ff\!f)}\, S \; + \sum_{1 \leq i \leq j \leq n} \left(\prod_{k \neq i,j} J^{k}_{(f)} \right) J^{i}_{(f\gamma)} J^{j}_{(f\gamma)} \otimes H^{ij}_{(f\gamma)(f\gamma)}\, S\,.$$

DIAGRAMMATIC FACTORISATION AT NLP



- More explicitly:

$$\left(\prod_{j\neq i} J^j_{(f)}\right)\left[J^i_{(f\gamma)}\otimes H^i_{(f\gamma)} + J^i_{(f\partial\gamma)}\otimes H^i_{(f\partial\gamma)}\right] S \equiv \; S(\hat{p}_i\cdot\hat{p}_j;\epsilon)\left(\prod_{j\neq i} J_{(f)}(p_j;\epsilon)\right)\underbrace{\int_0^{p_i^+} d\ell_i^+}_{\gamma=1}$$

$$\times\Big[J^{\nu}_{(f\gamma)}(p_i-\hat{\ell}_i,\hat{\ell}_i;\epsilon)\, H_{(f\gamma)\nu}(p_1\ldots;p_i-\hat{\ell}_i,\hat{\ell}_i;\ldots p_n;\epsilon)$$

$$+\, J^{\nu\rho}_{(f\partial\gamma)}(p_i-\hat{\ell}_i,\hat{\ell}_i;\epsilon)\, H_{(f\partial\gamma)\nu\rho}(p_1\ldots;p_i-\hat{\ell}_i,\hat{\ell}_i;\ldots p_n;\epsilon)\Big]\,,$$

where $\hat{p}_i^\mu = p_i^+\, n_i^\mu$.

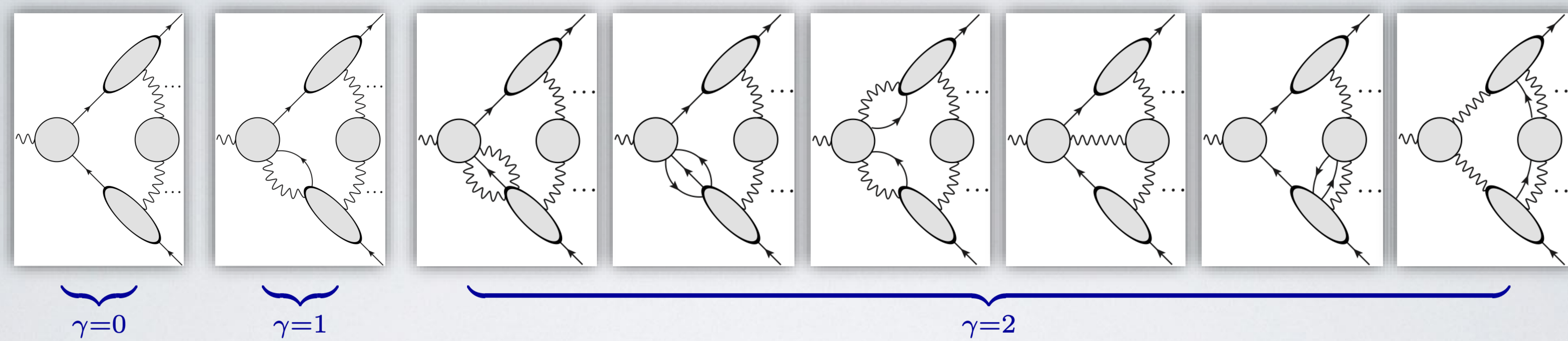
- Multiple particles in the same collinear sector involves a convolution over the momentum fraction along the collinear direction.
- The convolution is well defined in dimensional regularisation; in d=4 it can develop endpoint divergences.
- The leading jet function is defined as

$$J_{(f)}(p_i) = \langle p_i|\overline{\psi}(0)\Phi_{\bar{n}_i}(0,\infty)|0\rangle\,, \qquad \Phi_{\bar{n}_i}(0,\infty) = \mathcal{P}\exp\left[-\,i\,q_i\,e\int_0^\infty ds\,\bar{n}_i\cdot A(s\,\bar{n}_i)\right].$$

- Similar definitions can be derived for the other jet functions. For instance

$$J^{\mu}_{(f\gamma)}(p-\hat{\ell},\hat{\ell}) = \int_{-\infty}^{+\infty} d\xi\, e^{-i\ell\cdot(\xi\bar{n})}\langle p|\left[\overline{\psi}(0)\Phi_{\bar{n}}(0,\infty)\right]\left[\Phi_{\bar{n}}(\infty,0)\,[iD^\mu\,\Phi_{\bar{n}}(0,\infty)]\right](\xi\,\bar{n})|0\rangle\,.$$

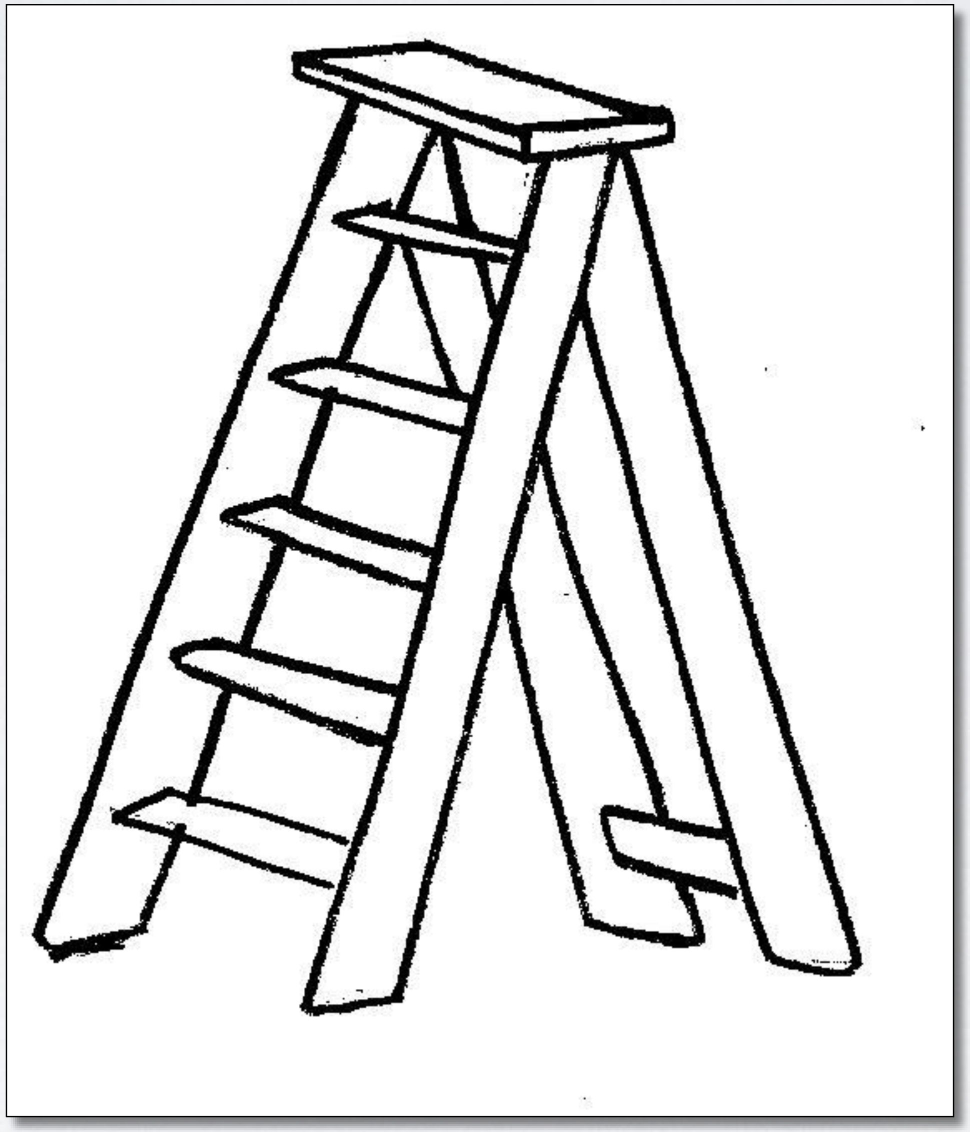
DIAGRAMMATIC FACTORISATION AT NLP



- The main message: in general, more types of jet functions are needed, involving two or more particles along a given collinear direction.
- In SCET, similar dynamical configurations are given in terms of *A0, A1, A2, B1, B2, C2*, etc operators

- There are still open questions:
  - Gauge invariant definitions for the jets in QCD may be non-trivial;
  - Beyond 1 loop there could be a complicated pattern of nested subtractions to remove double counting.
  - Multiple soft gluon emission not yet implemented.

RESUMMATION FOR "DIAGONAL" CHANNELS



DIAGRAMMATIC RESUMMATION: LL@LP

- The hadronic cross section

$$\frac{d\sigma}{d\tau} = \sigma_0(Q^2) \int_0^1 dz\, dx_1\, dx_2\, \delta(\tau - x_1 x_2 z)\, q(x_1)\, \bar{q}(x_2)\, \Delta(z) \,, \qquad \tau = \frac{Q^2}{s},$$

diagonalise in Mellin space:

$$\int_0^1 d\tau\, \tau^{N-1} \frac{d\sigma}{d\tau}\bigg|_{\rm LL} = \sigma_0(Q^2)\, q(N,Q^2,\epsilon)\, \bar{q}(N,Q^2,\epsilon)\, \mathcal{S}(N,Q^2,\epsilon)\,.$$

- The soft function at NLO reads

Sterman, 1987; Catani,Trentadue, 1989

$$\mathcal{S}^{(1)}(N,Q^2,\epsilon)\Big|_{\rm LL} = \left(\frac{\bar{\mu}^2}{Q^2}\right)^{\epsilon} \frac{2\alpha_s}{\pi} C_F \left[\frac{\log N}{\epsilon} + \log^2 N\right]\,.$$

- The singularities cancel with those coming from the PDF. Exponentiation gives

"Replica trick", Gardi, Laenen, Stavenga, White, 2010

$$\int_0^1 d\tau\, \tau^{N-1} \frac{d\sigma_{\rm DY}}{d\tau}\bigg|_{\rm LL} = \sigma_0(Q^2)\, q_{\rm LL}(N,Q^2)\, \bar{q}_{\rm LL}(N,Q^2) \exp\left[\frac{2\alpha_s}{\pi} C_F \log^2 N\right]\,.$$

- In *z* space, this resums LL plus distributions.

$$\int_0^1 dz\, z^{N-1} \frac{\log^{\ell-1}(1-z)}{1-z}\bigg|_+ = \frac{1}{\ell}[-\log(N)]^{\ell} + \ldots\,, \quad \Rightarrow \quad \left(\frac{2\alpha_s C_F}{\pi}\right)^m \frac{2}{(m-1)!} \frac{\log^{2m-1}(1-z)}{1-z}\bigg|_+ .$$

DIAGRAMMATIC RESUMMATION: LL@LP

We have been considering some implicit assumptions:

- The partonic cross section factorizes as

(formal definitions in Kidonakis, Sterman, 1987)

$$\widehat{\Delta}\left(N,Q^2,\epsilon\right) = \left|\mathcal{H}\left(Q^2\right)\right|^2 \frac{\prod_i \psi_i\left(N,Q^2,\epsilon\right)}{\prod_i \psi_{\mathrm{eik},i}\left(N,Q^2,\epsilon\right)} \mathcal{S}\left(N,Q^2,\epsilon\right) .$$

  - *$H(Q^2)$* is finite and contains off-shell virtual contributions;
  - *$S(N,Q^2,\varepsilon)$* collects all soft enhancements associated with (real or virtual) soft radiation;
  - *$\psi_i(N,Q^2,\varepsilon)$* perturbative (anti-)quark distribution function, collecting collinear singularities associated with initial line *i*;
  - double counting is removed by dividing each quark distribution by its own eikonal approximation *$\psi_{eik,i}(N,Q^2,\varepsilon)$*.

- However, (a) to LL accuracy we can approximate

$$\widehat{\Delta}_{\mathrm{LL}}\left(N,Q^2,\epsilon\right) = \left|\mathcal{H}\left(Q^2\right)\right|^2 \mathcal{S}\left(N,Q^2,\epsilon\right) .$$

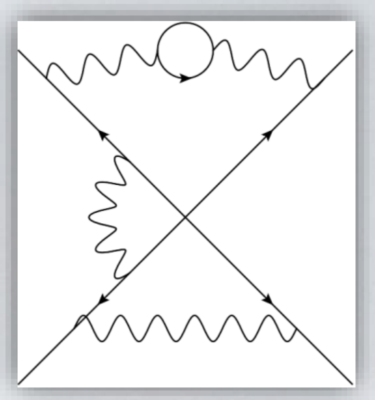
DIAGRAMMATIC RESUMMATION: LL@LP

This can be based on the following observation:

- Threshold singularities, inducing nonanalytic behaviour at $z \to 1$, are directly related to IR singularities of the amplitude.
- IR singularities arise from integrations of the relevant momentum components near singular surfaces in momentum space, (characterised to all orders by means of Landau equations and power counting techniques).
- For massless theories, it can be shown that IR singularities arise only from soft and collinear momentum configurations.
- At LP, at n loops there are precisely $2n$ normal variables that must be integrated with a logarithmic measure: e.g. the n parton energies E_i, with LP integration measure dE_i/E_i and n transverse momenta with respect to the directions defined by external particles, k_{iT}, with LP integration measure dk_{iT}/k_{iT}.
- Threshold logarithms arise when different combinations of normal variables become small at different rates, but LLs arise only when all energies and transverse momenta are strongly ordered, $E_1 \ll \ldots \ll E_n$ and $k_{1T} \ll \ldots \ll k_{nT}$.
- In that limit, at LP, the $2n$ logarithmic integrations yield contributions of the form $\ln^{2n-1}(1 - z)/(1 - z)_+$, since the last logarithmic integration must not performed when computing $d\sigma/dz$.

THE REPLICA TRICK

(b) Exponentiation of the soft function. Consider the soft matrix element



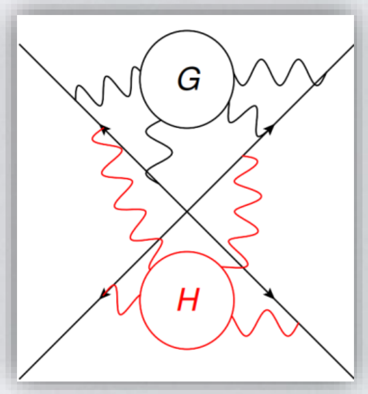
$$\mathcal{S}_n = \langle 0 | \prod_{i=1}^{n} \Phi_i | 0 \rangle \,, \qquad \Phi_i = \exp\left[\mathrm{i}e \int dx_i^\mu A_\mu(x_i) \right] .$$

In path-integral language, this matrix element may be written as

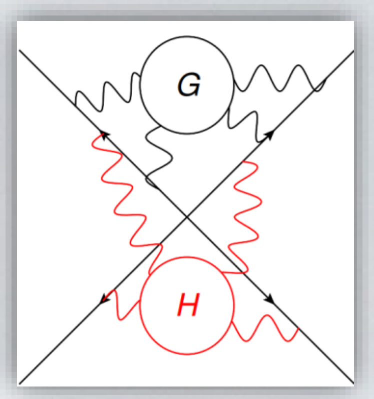
$$\mathcal{S}_n = \int \mathcal{D}A_\mu \left(\prod_{i=1}^{n} \Phi_i \right) \mathrm{e}^{\mathrm{i}S(A_\mu, \bar{\psi}, \psi)} = \int \mathcal{D}A_\mu \, \exp\left[\sum_{i=1}^{n} \mathrm{i}e \int dx_i^\mu A_\mu(x_i) + \mathrm{i}S\left(A_\mu, \bar{\psi}, \psi\right) \right] ,$$

where $S\left(A_\mu, \bar{\psi}, \psi\right)$ represents the QED action. Generate N independent "j" replicas of the gauge and fermion fields, such that particle with different replica number j never interact:

$$\mathcal{S}_{n,R} = \int \mathcal{D}A_\mu^{(1)} \ldots \int \mathcal{D}A_\mu^{(N)} \exp\left[\mathrm{i}e \sum_{j=1}^{N} \sum_{i=1}^{n} \int dx_i^\mu A_\mu^{(j)} + \sum_{j=1}^{N} S\left(A_\mu^{(j)}, \bar{\psi}^{(j)}, \psi^{(j)} \right) \right] .$$



THE REPLICA TRICK



- One has

$$\mathcal{S}_{n,R} = \mathcal{S}_n^N = 1 + N \log\left(\mathcal{S}_n\right) + \mathcal{O}(N^2)\,.$$

  it follows

$$\mathcal{S}_n = \exp\left[\sum_W W\right]\,,$$

  where the sum is over diagrams *W* that are *O(N)* in the replicated theory.

Gardi, Laenen, Stavenga, White, 2010

- Mutual independence of the replicated fields implies that a diagram containing *m* connected subdiagrams must be *O(N^m)*, given that there is a choice of *N* possible replicas for each subdiagram. Thus, the logarithm of the soft function in QED must contain only connected subdiagrams.

- The argument apply in the first place only to the case of virtual contributions to the soft function; However, the replica trick applies also in case of real emissions, provided that real radiations associated with different replica numbers are mutually independent.

- This is fulfilled if the phase space integral for *n* gluon emissions factorizes into *n* decoupled single-gluon phase space integrals, as it happens at LP and at LL NLP.

DIAGRAMMATIC RESUMMATION: LL@NLP

- Partonic cross section at NLP: at LL accuracy only corrections to the matrix element:

$$\hat{\sigma} = \frac{1}{2\hat{s}} \left[\int d\Phi_{\rm LP} \, |\mathcal{M}|^2_{\rm LP} + \int d\Phi_{\rm LP} \underbrace{|\mathcal{M}|^2_{\rm NLP}}_{\Uparrow \, \rm LL} + \int \underbrace{d\Phi_{\rm NLP}}_{\rm NLL \, only} |\mathcal{M}|^2_{\rm LP} + \ldots \right] .$$

Equivalent of the "kinematic corrections"

- Furthermore, the radiative jets contribute only at NLL: at LL only next-to-soft radiation.

In Mellin space

Bonocore, Laenen, Magnea, LV, White, 2014; Bahjat-Abbas, Sinninghe Damsté, LV, White, 2018 for "perturbative" support

$$\int_0^1 d\tau \, \tau^{N-1} \left. \frac{d\sigma}{d\tau} \right|_{\rm NLP,LL} = \sigma_0(Q^2) \, q(N, Q^2, \epsilon) \, \bar{q}(N, Q^2, \epsilon) \, \mathcal{S}_{\rm NLP}(N, Q^2, \epsilon) \, .$$

- where the NLP soft function is made out of generalized Wilson lines:

$$\widetilde{\mathcal{S}} \left(z, Q^2, \epsilon \right) = \frac{1}{N_c} \sum_{n, \, \rm LP} {\rm Tr} \left[\langle 0 | F^\dagger(p_1) F(p_2) | n \rangle \langle n | F^\dagger(p_2) F(p_1) | 0 \rangle \right] \delta \left(z - \frac{Q^2}{\hat{s}} \right) .$$

DIAGRAMMATIC RESUMMATION LL@NLP

- At one loop, in Mellin space, summing LP and NLP gives

$$\mathcal{S}_{\mathrm{LP+NLP}}\left(N, Q^2, \epsilon\right) = \frac{2\alpha_s C_F}{\pi}\left(\frac{\bar{\mu}^2}{Q^2}\right)^{\epsilon}\left[\frac{1}{\epsilon}\left(\log N + \frac{1}{2N}\right) + \log^2 N + \frac{\log N}{N}\right].$$

- The singularities cancel with those coming from the PDF. Exponentiation gives

$$\int_0^1 d\tau\, \tau^{N-1}\, \frac{d\sigma_{\mathrm{DY}}}{d\tau}\bigg|_{\mathrm{LL,\,NLP}} = \sigma_0(Q^2)\, q_{\mathrm{LL,\,NLP}}\left(N, Q^2\right)\; \bar{q}_{\mathrm{LL,\,NLP}}\left(N, Q^2\right) \times \exp\left[\frac{2\alpha_s C_F}{\pi}\left(\log^2 N + \frac{\log N}{N}\right)\right].$$

"Replica trick", Gardi, Laenen, Stavenga, White, 2010

- In z space this gives

Bahjat-Abbas, Bonocore, Sinninghe Damsté, Laenen, Magnea, LV, White, 2019

$$\Delta(z, Q^2)|_{\mathrm{LL,\,NLP}} = \left(\frac{2\alpha_s C_F}{\pi}\right)^m \frac{1}{(m-1)!}\left[\underbrace{2\left(\frac{\log^{2m-1}(1-z)}{1-z}\right)_+}_{\mathrm{LP}} \underbrace{-\, 2\log^{2m-1}(1-z)}_{\mathrm{NLP}}\right].$$

- The coefficient of the NLP LL corresponds to the NLP term in the Altarelli-Parisi splitting kernels.

Kraemer, Laenen, Spira, 1998;
Laenen, Magnea, Stavenga, 2008;
Kidonakis 2009;
Marzani, Bonvini 2014 ....

DIAGRAMMATIC RESUMMATION: UNIVERSALITY

- Production of N colour singlet particles: additional NLP contribution, associated to the orbital angular momentum of incoming particles. Up to NLP:

$$\mathcal{S}_{\rm NLP} = \exp\left[\sum_i W_{\rm LP}^{(i)} + \sum_j W_{\rm NLP}^{(j)}\right] = \exp\left[\sum_i W_{\rm LP}^{(i)}\right]\left(1 + \sum_j W_{\rm NLP}^{(j)}\right),$$

→ Combine orbital angular momentum with the spin angular momentum into W_{NLP}.

- At NLO NLP the partonic cross section take the simple form

$$\widehat{\Delta}_{\rm NLP}^{(q\bar{q})\,\text{or}\,(gg)}(z,\epsilon) = K_{\rm NLP}(z,\epsilon)\,\hat{\sigma}_{\rm LO}^{(q\bar{q})\,\text{or}\,(gg)}(z\hat{s}),$$

Del Duca, Laenen, Magnea, LV, White, 2017

where K_{NLP} is a universal factor

$$K_{\rm NLP}(z,\epsilon) = \frac{\alpha_s}{\pi} C_{F\,\text{or}\,A}\left(\frac{4\pi\mu^2}{\hat{s}}\right)^{\epsilon} z\,(1-z)^{-1-2\epsilon}\frac{\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)\Gamma(1-\epsilon)}.$$

- We can interpret this as

$$\widehat{\Delta}_{\rm NLP}^{(q\bar{q})\,\text{or}\,(gg)}(z,\epsilon) = z\,\mathcal{S}_{\rm LP}(z,\epsilon)\,\hat{\sigma}_{\rm LO}^{(q\bar{q})\,\text{or}\,(gg)}(z\hat{s}),$$

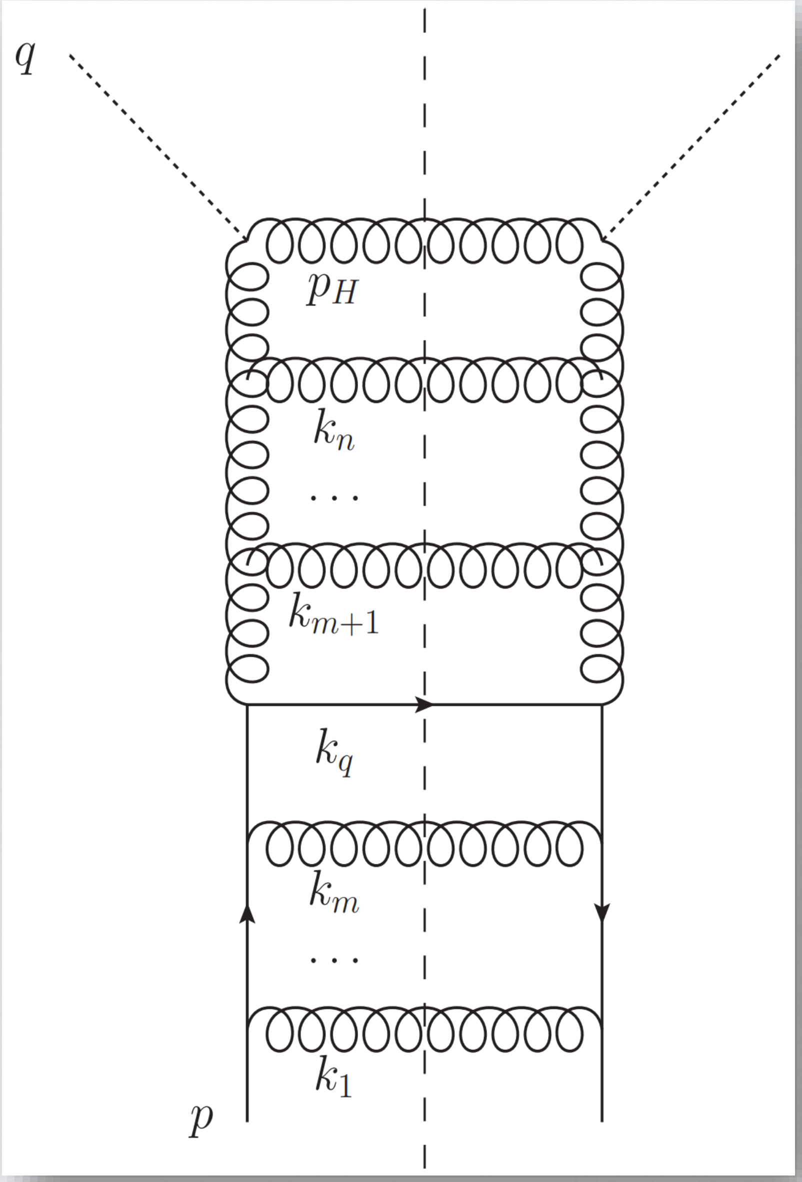
Bahjat-Abbas, Bonocore, Sinninghe Damsté, Laenen, Magnea, LV, White, 2019

- Thus

$$\int_0^1 dz\,z^{N-1}\widehat{\Delta}_{\rm NLP}^{(q\bar{q})\,\text{or}\,(gg)}(z,\epsilon) = \mathcal{S}_{\rm LP}(N+1,\epsilon)\,\hat{\sigma}_{\rm LO}^{(q\bar{q})\,\text{or}\,(gg)}(Q^2)$$

$$= \hat{\sigma}_{\rm LO}^{(q\bar{q})\,\text{or}\,(gg)}(Q^2)\exp\left[\frac{2\alpha_s C_{F\,\text{or}\,A}}{\pi}\log^2(N)\right]\left(1 + \frac{2\alpha_s C_F}{\pi}\frac{\log N}{N}\right).$$

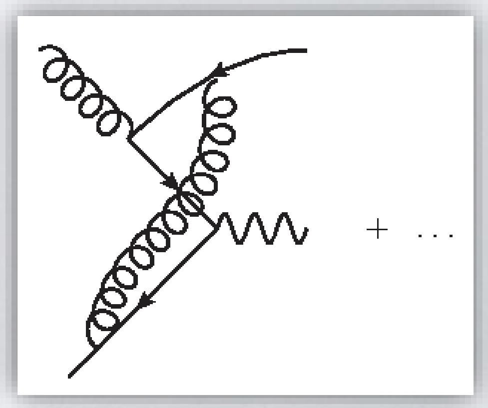
RESUMMATION FOR "OFF-DIAGONAL" CHANNELS



OFF-DIAGONAL CHANNELS

- By off-diagonal channels we refers to the (anti-)quark-gluon channel in DIS and Drell-Yan.
- They are interesting because they start at NLP. Naively one would expect them to be simpler, thus ideal to investigate NLP resummation beyond LL.
- However, this is not the case. They turn out to have a more complicated structure. For instance, collinear modes do contribute to LLs: a region analysis for Drell-Yan gives

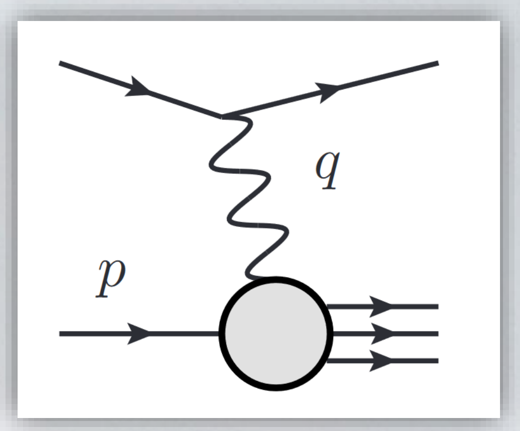
$$
\begin{aligned}
&|\mathcal{M}^{(3)}_{g\bar{q}\to\gamma^*\bar{q}}\mathcal{M}^{*(1)}_{g\bar{q}\to\gamma^*\bar{q}}| \\
&\sim C_F^2\Big[-\frac{s(t+u)}{2\,tu}\Big(\frac{\mu^2}{-s}\Big)^{\epsilon}\Big(-\frac{2}{\epsilon^2}+\ldots\Big)+\Big(\frac{s}{t}\Big(\frac{\mu^2}{-t}\Big)^{\epsilon}+\frac{s}{u}\Big(\frac{\mu^2}{-u}\Big)^{\epsilon}\Big)\Big(-\frac{1}{\epsilon^2}+\ldots\Big)+\frac{s(t+u)}{tu}\Big(\frac{s\,\mu^2}{t\,u}\Big)^{\epsilon}\Big(\frac{1}{\epsilon^2}+\ldots\Big)\Big] \\
&\quad - C_AC_F\Big[\Big(\frac{s}{t}\Big(\frac{\mu^2}{-t}\Big)^{\epsilon}+\frac{s}{u}\Big(\frac{\mu^2}{-u}\Big)^{\epsilon}\Big)\Big(-\frac{1}{\epsilon^2}+\ldots\Big)+\frac{s(t+u)}{2\,tu}\Big(\frac{s\,\mu^2}{t\,u}\Big)^{\epsilon}\Big(\frac{1}{\epsilon^2}+\ldots\Big)\Big].
\end{aligned}
$$



Is there a way to proceed diagrammatically?

DEEP INELASTIC SCATTERING

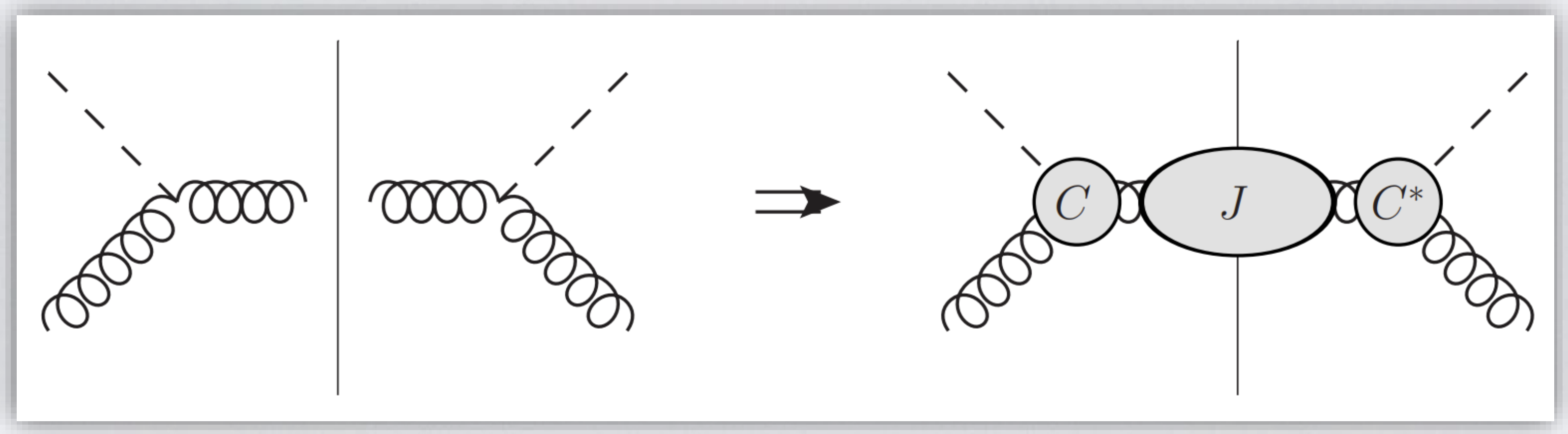
- Consider for instance Deep inelastic scattering (DIS) near threshold:



$$Q^2 \gg P_X^2 \sim Q^2(1-x), \qquad \text{with} \qquad x \equiv \frac{Q^2}{2p\cdot q} \to 1.$$

Sterman 1987; Catani, Trentadue 1989; Korchemsky, Marchesini, 1993; Moch, Vermaseren, Vogt 2005; Becher, Neubert, Pecjak, 2007

- Factorization and resummation well understood at LP. Consider Higgs-induced DIS

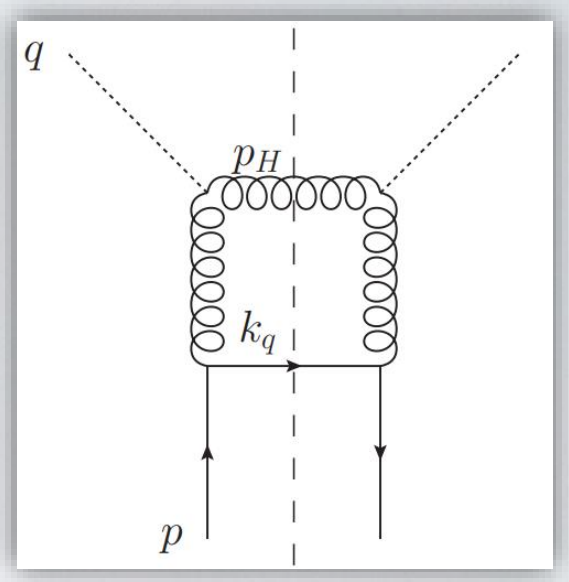


$$W_\phi = \frac{1}{8\pi Q^2}\int d^4x\, e^{iq\cdot x}\,\langle N(P)|\big[G^A_{\mu\nu}G^{\mu\nu A}\big](x)\big[G^B_{\rho\sigma}G^{\rho\sigma B}\big](0)|N(P)\rangle$$

$$= |C(Q^2,\mu)|^2\int_x^1 \frac{d\xi}{\xi}\, J\Big(Q^2\frac{1-\xi}{\xi},\mu\Big)\,\frac{x}{\xi}f_g\Big(\frac{x}{\xi},\mu\Big).$$

DIS: OFF-DIAGONAL CHANNEL

- The off-diagonal channel $q(p) + \phi^*(q) \to X(p_X)$ contributes to DIS at NLP.

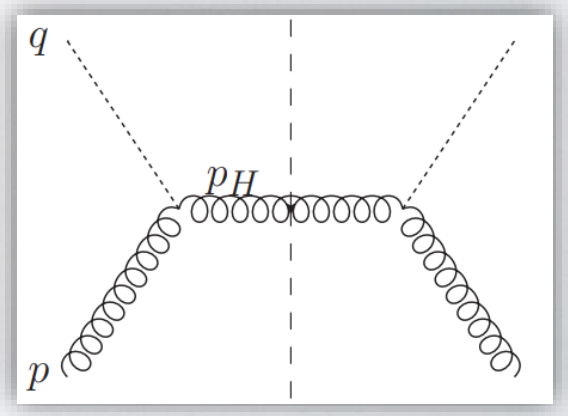


- As for the qg channel in Drell-Yan, the failure of standard resummation methods appears already at the LL order.
  This is manifest for instance in the DGLAP splitting functions, for which Vogt, 2010 found that the all-order quark-gluon splitting function with LL accuracy is given in moment space by

$$P_{gq}^{\mathrm{LL}}(N) = \frac{1}{N}\frac{\alpha_s C_F}{\pi}\mathcal{B}_0(a), \quad \text{where} \quad a = \frac{\alpha_s}{\pi}(C_F - C_A)\ln^2 N \quad \text{and} \quad \mathcal{B}_0(x) = \sum_{n=0}^{\infty}\frac{B_n}{(n!)^2}x^n,$$

- With B_n representing Bernoulli numbers. To understand this result let's consider the structure function

$$F_\phi(x, Q^2) = \int \mathrm{d}\Phi\, \overline{|\mathcal{M}_{\mathrm{DIS}}|^2}\,.$$



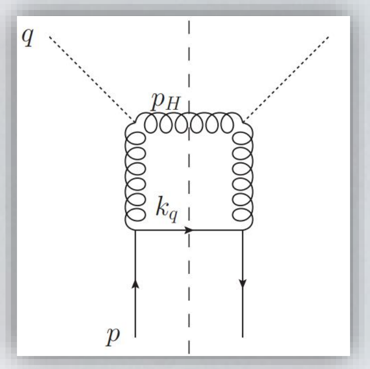
- At tree level

$$F_\phi(x, Q^2)\Big|_{\mathrm{LO}} = \frac{2\pi|\lambda|^2}{2p\cdot q}\,\delta(1-x)\,.$$

DIS: OFF-DIAGONAL CHANNEL

- Normalizing to the tree level, the lowest order in the off-diagonal channel reads

$$W^{(1)}_{\phi,q}(x) = -\frac{2C_F}{\epsilon}(1-x)^{-\epsilon}\,, \qquad \text{with} \qquad W_i(x,Q^2) \equiv \frac{1}{\sigma_0}F_i(x,Q^2),$$



where $W^{(1)}$ is the coefficient of the perturbative expansion according to

$$X = \sum_n a_s^n X^{(n)}\,, \qquad a_s = \frac{\alpha_s}{4\pi}\,.$$

- At higher orders in general one expect the following momentum regions to contribute:

Perturbative modes {

Hard: $p^2 = Q^2,$

Hard-collinear: $p^2 = Q^2\lambda^2 = Q^2/N,$

Non-perturbative modes {

Collinear: $p^2 = \Lambda^2,$

Soft-collinear: $p^2 = \Lambda^2\lambda^2 = \Lambda^2/N.$

$$g(N) \equiv \int_0^1 dx\, x^{N-1}\, g(x),$$

$$x \to 1 \;\Leftrightarrow\; N \to \infty.$$

- We are not able to calculate from scratch, by means of diagrammatic methods, the logarithmic content of each region. However, we can use additional insight, to which we refer to as consistency relations.

Beneke, Garny, Jaskiewicz, Szafron, LV, Wang, 2020

D-DIMENSIONAL CONSISTENCY CONDITIONS

- Hadronic structure function is finite:

$$W = \sum_i W_{\phi,i}\, f_i = \sum_i \tilde{C}_{\phi,k} \tilde{f}_k \,, \qquad \text{with} \qquad \tilde{f}_k = Z_{ki} f_i, \quad W_{\phi,i} = \tilde{C}_{\phi,k} Z_{ki}.$$

- Focus on the bare functions: at LP a single channel contributes:

$$\sum_i (W_{\phi,i} f_i)^{LP} = W_{\phi,g}^{LP} f_g^{LP}.$$

- Work in d-dimensions: the general expansion of the cross section reads

$$W_{\phi,g}\, f_g = f_g(\Lambda) \times \sum_n \left(\frac{\alpha_s}{4\pi}\right)^n \frac{1}{\epsilon^{2n}} \sum_{k=0}^{n} \sum_{j=0}^{n} b_{kj}^{(n)}(\epsilon) \left(\frac{\mu^{2n} N^j}{Q^{2k} \Lambda^{2(n-k)}}\right)^{\epsilon} + \mathcal{O}\left(\frac{1}{N}\right).$$

- In this equation:

Each hard loop gives $\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}$, each hard-collinear loop gives $\left(\frac{\mu^2}{Q^2} N\right)^{\epsilon}$,

Each collinear loop gives $\left(\frac{\mu^2}{\Lambda^2}\right)^{\epsilon}$, each soft-collinear loop gives $\left(\frac{\mu^2}{\Lambda^2} N\right)^{\epsilon}$.

D-DIMENSIONAL CONSISTENCY CONDITIONS

- Invoking pole cancellation, one has the obvious condition:

$$\sum_{k=0}^{n} \sum_{j=0}^{n} b_{kj}^{(n)} = 0 \,,$$

however, also all poles of the type

$$(\ln N)^r \left(\ln \frac{\Lambda}{Q} \right)^s \times \frac{1}{\epsilon^{2n-r-s}} \,.$$

need to cancel. In the end one has the conditions

$$\sum_{k=0}^{n} \sum_{j=0}^{n} j^r k^s \, b_{kj}^{(n)} = 0 \qquad \text{for } s + r < 2n, \; r, s \geq 0 \,.$$

- At order n this gives $n^2 + 2n$ equations, for $(n+1)^2$ coefficients $b_{kj}^{(n)}$: the system can be solved in terms of n unknown coefficient, one per order n.

D-DIMENSIONAL CONSISTENCY CONDITIONS

- Consider now the expansion to NLP:

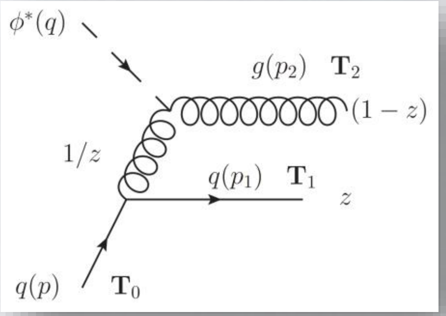
$$\sum_i (W_{\phi,i} f_i)^{NLP} = W_{\phi,q}^{NLP} f_q^{LP} + W_{\phi,\bar{q}}^{NLP} f_{\bar{q}}^{LP} + W_{\phi,g}^{NLP} f_g^{LP} + W_{\phi,g}^{LP} f_g^{NLP} \,.$$

Beneke, Garny, Jaskiewicz, Szafron, LV, Wang, 2020

- The general expansion of the cross section reads

$$\sum_i (W_{\phi,i} f_i)^{NLP} = f_q(\Lambda) \times \frac{1}{N} \sum_{n=1} \left(\frac{\alpha_s}{4\pi}\right)^n \frac{1}{\epsilon^{2n-1}} \sum_{k=0}^{n} \sum_{j=0}^{n} c_{kj}^{(n)}(\epsilon) \left(\frac{\mu^{2n} N^j}{Q^{2k} \Lambda^{2(n-k)}}\right)^{\epsilon} + f_{\bar{q}}(\Lambda),\ f_g(\Lambda) \text{ terms}\,.$$

- Invoking pole cancellations:



$$\sum_{k=0}^{n} \sum_{j=0}^{n} j^r k^s c_{kj}^{(n)} = 0 \qquad \text{for } s + r < 2n-1,\ r,s \geq 0\,,$$

allows $(n+1)^2$ coefficients $c_{kj}^{(n)}$ to be determined from $2n^2-n$ equations.

- Three unknowns: these can be reduced using information from the region expansion:
  - $c_{n0}^{(n)} = 0$ (final state cannot be purely hard)
  - $c_{00}^{(n)} = 0$ (without any hard or anti-hardcollinear loops there must be at least one soft-collinear loop).
  - The system can be solved again in terms of *n* unknown coefficient, one per order *n*.

OFF-DIAGONAL DIS: THE DIAGRAMMATIC WAY

- This information can be used in two ways:

- On the one hand, one can use SCET methods to determine the towers of coefficients from the hard region, i.e. $c_{n1}^{(n)}$.

  See: Beneke, Garny, Jaskiewicz, Szafron, LV, Wang, 2020

- On the other hand, the tower of all-real soft emissions $c_{nn}^{(n)}$ is particularly suitable for a diagrammatic approach.

  See: van Beekveld, LV, White 2021

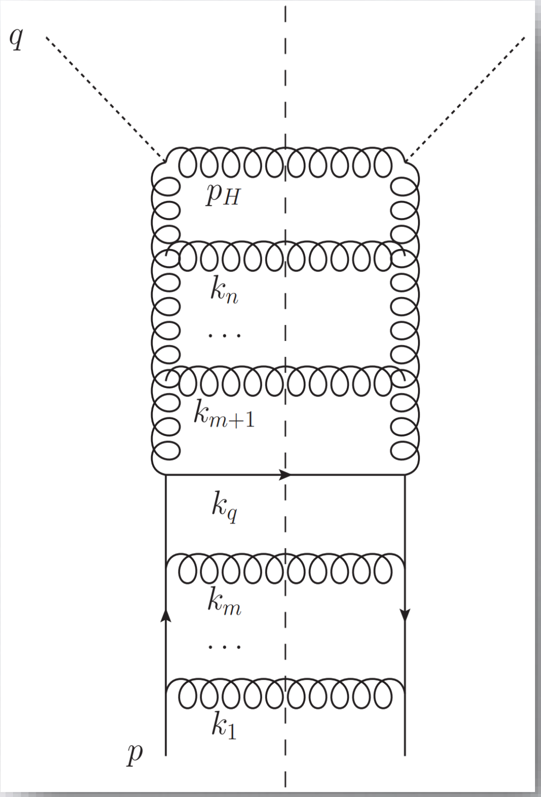
  It can be determined based on the following considerations:

- In a physical polarization gauge in which

  Gribov, Lipatov, 1972; Dokshitzer, Diakonov, Troian, 1980; Dokshitzer, Khoze, Mueller, Troian, 1991.

$$\sum_{\text{pols.}} \epsilon^\dagger_\mu(k)\epsilon_\nu(k) = -\eta_{\mu\nu} + \frac{k_\mu c_\nu + k_\nu c_\mu}{c \cdot k}, \qquad c = q + xp.$$

  only ladder diagrams contribute to the LLs.



- The power suppression is given by the soft quark polarization sum:

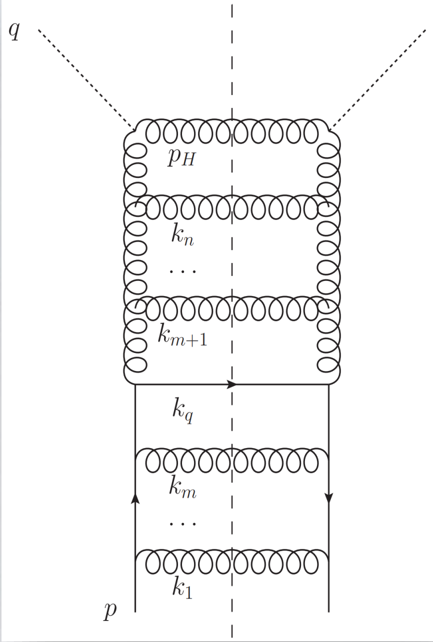
$$\sum_{\text{spins}} u(k_q)\bar{u}(k_q) = \not{k}_q,$$

  and gluon emissions are eikonal (LP): $V^\mu = \pm g_s \mu^{\frac{4-d}{2}} \mathbf{T}_i \frac{p_i^\mu}{p_i \cdot k}$.

OFF-DIAGONAL DIS: THE DIAGRAMMATIC WAY

- One has

$$\overline{|\mathcal{M}_{qh\to qg_1\dots g_n}|^2}$$
$$= \frac{|\lambda|^2 C_F^{m+1} C_A^{n-m} g_s^{2(n+1)}}{8\mu^{(d-4)(n+1)}} \left(\prod_{i=1}^{n} \frac{2q\cdot p\, p\cdot k_i}{q'\cdot k_i}\right) \mathrm{Tr}[\not{p}\gamma^\beta \ \not{k}_q\gamma^\alpha] \left(-\eta_{\alpha\beta} + \frac{q'_\alpha p_{H,\beta} + q'_\beta p_{H,\alpha}}{q'\cdot p_H}\right)$$
$$\times \frac{1}{(p\cdot k_1)^2 [p\cdot(k_1+k_2)]^2 \dots [p\cdot(k_1+\dots+k_m+k_q)]^2 \dots [p\cdot(k_1+\dots+k_n+k_q)]^2}.$$

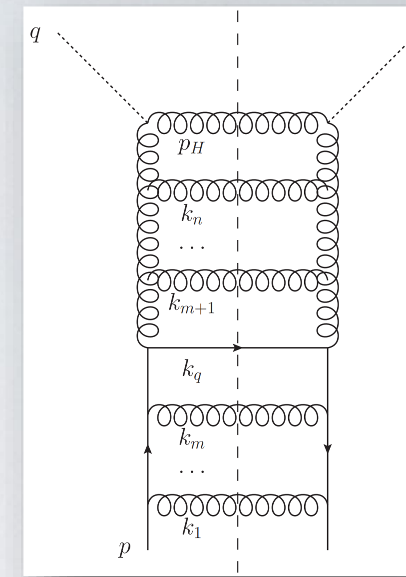


- Phase space can be also approximated to LP and factorizes in Laplace space.

$$\int \mathrm{d}\Phi^{(n+2)} = \frac{2\pi}{(4\pi)^{\frac{(n+1)d}{2}}} (Q^2)^{n+(n+1)\frac{(d-4)}{2}} \frac{x^{-(n+1)\frac{(d-4)}{2}-n}(1-x)^{-(n+1)\frac{d-2}{2}}}{\Gamma\left(\frac{d-2}{2}\right)^{n+1}} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}T}{2\pi i}\, e^{T(1-x)}$$
$$\times \int \mathrm{d}\alpha_q\, \mathrm{d}\beta_q\, (\alpha_q\,\beta_q)^{\frac{d-4}{2}}\, e^{-T(\alpha_q+\beta_q)} \left[\prod_{i=1}^{n} \int \mathrm{d}\bar{\alpha}_i\, \mathrm{d}\bar{\beta}_i\, (\bar{\alpha}_i\,\bar{\beta}_i)^{\frac{d-4}{2}}\, e^{-T(\bar{\alpha}_i+\bar{\beta}_i)}\right],$$

$$\bar{\alpha}_i \simeq \frac{q'\cdot k}{p\cdot q}, \quad \bar{\beta}_i \simeq \frac{p\cdot k}{p\cdot q}\,.$$

OFF-DIAGONAL DIS: THE DIAGRAMMATIC WAY



- In Mellin space one obtains

$$W_{\phi,q}^{(n+1)} = -\left(\sum_{m=0}^{n} C_F^{m+1} C_A^{n-m}\right) \frac{2}{\epsilon}\frac{N^\epsilon}{N}\left(\frac{4N^\epsilon}{\epsilon^2}\right)^n \frac{1}{(n+1)!}\,,$$

which sums to

$$W_{\phi,q}\Big|_{\rm LL} = \sum_{n=1}^{\infty} a_s^n W_{\phi,q}^{(n)} = -\frac{2a_s C_F}{\epsilon}\frac{N^\epsilon}{N}\frac{1}{C_F - C_A}\left(\frac{4a_s N^\epsilon}{\epsilon^2}\right)^{-1}\left\{\exp\left[\frac{4a_s C_F N^\epsilon}{\epsilon^2}\right] - \exp\left[\frac{4a_s C_A N^\epsilon}{\epsilon^2}\right]\right\}.$$

- The full result is found requiring that virtual corrections modify the real emission contributions at each order, removing singularities which are simultaneously soft and collinear: this leads to the parametrization

$$W_{\phi,q}\Big|_{\rm LL} = -\frac{2a_s C_F}{\epsilon}\frac{N^\epsilon}{N}\frac{1}{C_F - C_A}\left(\frac{4a_s (N^\epsilon + \lambda_1)}{\epsilon^2}\right)^{-1}\exp\left[\frac{4a_s(\lambda_2 C_F + \lambda_3 C_A)}{\epsilon^2}\right]$$
$$\times\left\{\exp\left[\frac{4a_s C_F (N^\epsilon + \lambda_4)}{\epsilon^2}\right] - \exp\left[\frac{4a_s C_A (N^\epsilon + \lambda_5)}{\epsilon^2}\right]\right\}.$$

which can be easily fixed, given that the system is over constrained. We find

$$W_{\phi,q}\Big|_{\rm LL} = -\frac{2a_s C_F}{\epsilon}\frac{N^\epsilon}{N}\frac{1}{C_F - C_A}\left(\frac{4a_s (N^\epsilon - 1)}{\epsilon^2}\right)^{-1}\left\{\exp\left[\frac{4a_s C_F (N^\epsilon - 1)}{\epsilon^2}\right] - \exp\left[\frac{4a_s C_A (N^\epsilon - 1)}{\epsilon^2}\right]\right\},$$

van Beekveld, LV, White 2021

OFF-DIAGONAL DIS: FINITE STRUCTURE FUNCTION

- Furthermore, recall

$$W = \sum_i W_{\phi,i} f_i = \sum_k \tilde{C}_{\phi,k} \tilde{f}_k, \quad \Rightarrow \quad W^{NLP}_{\phi,q} = \tilde{C}^{NLP}_{\phi,q} Z^{LP}_{qq} + \tilde{C}^{LP}_{\phi,g} Z^{NLP}_{gq} \,.$$

- We obtain a solution for $\tilde{C}$:

$$\tilde{C}^{NLP,LL}_{\phi,q}\Big|_{\epsilon\to 0} = \frac{1}{2N\ln N}\frac{C_F}{C_F - C_A}\Bigg(\mathcal{B}_0(a)\exp\left[C_A\frac{\alpha_s}{\pi}\left(\frac{1}{2}\ln^2 N + \ln N \ln\frac{\mu^2}{Q^2}\right)\right] - \exp\left[\frac{\alpha_s C_F}{\pi}\left(\frac{1}{2}\ln^2 N + \ln N\ln\frac{\mu^2}{Q^2}\right)\right]\Bigg), \qquad \text{with} \qquad a = \frac{\alpha_s}{\pi}(C_F - C_A)\ln^2 N,$$

and

$$P_{ij} = -\gamma_{ij} = \frac{dZ_{ik}}{d\ln\mu}(Z^{-1})_{kj}\,, \qquad \gamma^{NLP,LL}_{gq}(N) = -\frac{1}{N}\frac{\alpha_s C_F}{\pi}\mathcal{B}_0(a)\,, \qquad \mathcal{B}_0(x) = \sum_{n=0}^{\infty}\frac{B_n}{(n!)^2}x^n,$$

with Bernoulli numbers $B_0 = 1$, $B_1 = -1/2$, …

- Reproduces earlier conjecture by *Vogt, 2010*.

van Beekveld, LV, White 2021; see also Beneke, Garny, Jaskiewicz, Szafron, LV, Wang, 2020

OFF-DIAGONAL DIS: THE DIAGRAMMATIC WAY

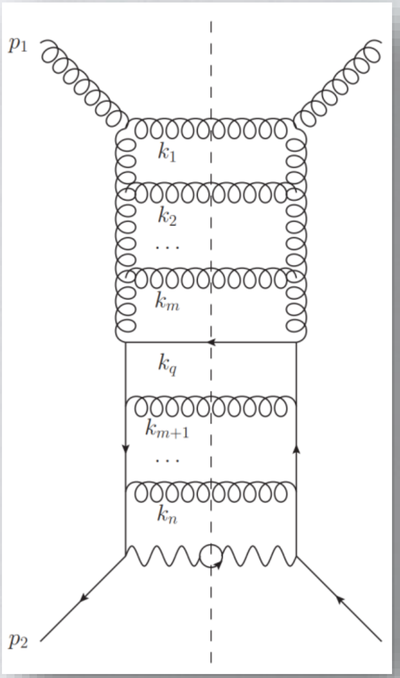
- The same procedures can be easily adapted to the subleading qg channel in Drell-Yan (and Higgs production).
- Consistency conditions can be studied to determine the smallest set of parameters necessary to determine the whole partonic cross section;
- The set of parameters can be determined
  - by assuming exponentiation of a given region, justified within a refactorization approach.
  - by direct calculation of the ladder diagrams contributing to the real emission.
- Either way, one reproduces an earlier conjecture in Lo Presti, Almasy, Vogt 2014:

Beneke, Garny, Jaskiewicz, Strohm, Szafron, LV, Wang, 2020 (unpublished)

van Beekveld, LV, White 2021

$$W_{\mathrm{DY},g\bar{q}}\Big|_{\mathrm{LL}} = -\frac{T_R}{2(C_F - C_A)}\frac{1}{N}\frac{\epsilon(N^{\epsilon-1})}{N^\epsilon - 1}\exp\left[\frac{4a_s C_F(N^\epsilon - 1)}{\epsilon^2}\right] \times \left\{\exp\left[\frac{4a_s C_F N^\epsilon(N^\epsilon - 1)}{\epsilon^2}\right] - \exp\left[\frac{4a_s C_A N^\epsilon(N^\epsilon - 1)}{\epsilon^2}\right]\right\},$$

$$\widetilde{C}_{\mathrm{DY},g\bar{q}}\Big|_{\mathrm{LL}} = \frac{T_R}{C_A - C_F}\frac{1}{2N\ln N}\left[e^{8C_F a_s \ln^2 N}\mathcal{B}_0[4a_s(C_A - C_F)\ln^2 N] - e^{(2C_F + 6C_A)a_s \ln^2 N}\right].$$



CONCLUSION

- Historically, diagrammatic methods have been used as a first tool to investigate the factorization of cross sections near threshold, and the resummation of the corresponding large logarithms.

- Diagrammatic methods have now been applied to study factorization at NLP, and have been used to derive resummation of LLs at NLP, in both diagonal and off-diagonal channels in Drell-Yan and DIS.

- It remains an open question whether diagrammatic methods will be able to provide the tools to address the resummation of logarithms at NLP beyond LL accuracy.

- On the phenomenological side, various analyses shows that LLs at NLP are competitive to NNLLs at LP.