Loop-by-loop Differential Equations for Dual *Elliptic* Feynman Integrals

On work to appear with Andrzej Pokraka

Mathieu Giroux
McGill University

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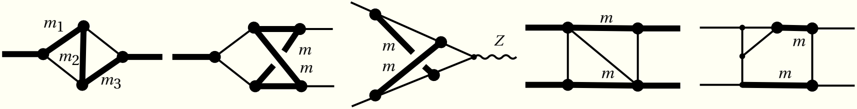
Why bother about elliptic Feynman integrals?

Elliptics integrals in pQFT

Appear at the early stages of pQFT!



Rich literature!

[Sabry ; 61, Broadhurst ; 90, Laporta, Remiddi; 05, Baikov, Chetyrkin, Smirnov, Smirnov, Steinhauser; 09, Adams, Bogner, Schweitzer, Weinzierl; 16, Broedel, Duhr, Dulat, Penante, Tancredi; 19, Duhr, Dulat, Mistlberger; 20, Frellesvig; 21, Duhr, Smirnov, Tancredi; 21, Wilhelm, Zhang; 22, **and many more**!]

$\Downarrow$

An *essential* step in opening a gateway to more precise perturbative calculations of cross-sections

[Talks by Lorenzo and Melih]

Feynman integrals

Within *dim-reg*, any Feynman integral I is a *twisted period*

Multivalued *twist*

$$I = \int_X u(\varepsilon)\, \phi$$

Algebraic differenttial n-form

Space of loop momenta invariants

For each topology, there exists a *finite* set of *spanning* integrals

[Smirnov, Petukhov; 10]

$$\Downarrow$$

Feynman integrals form a *vector space*

[Frellesvig, Gasparotto, Mandal, Mastrolia, Mattiazzi, Mizera; 19]

[Talk by Seva]

Dual of Feynman integrands

The space of *dual forms* $\check{\phi}$ is defined s.t. the *intersection pairing*

$$\langle\check{\phi}|\phi\rangle \sim \int_{\mathbb{C}^n} \check{u}\,\check{\phi} \wedge u\,\phi \overset{\text{s.v.}}{=} \int_{\mathbb{C}^n} \check{\phi} \wedge \phi$$

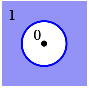
"makes sense" [Caron-Huot, Pokraka; 21] . In particular, we require

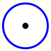
1. Single-valuedness (s.v.) of intersection pairing:

The dual twist is $\check{u} = u^{-1}$

2. Finiteness of intersection pairing:

$\check{\phi}$ must vanish in the neighborhood of poles in ϕ – i.e., $\{\mathsf{D}_i = 0\}$

$\theta_i = \theta(\mathsf{D}_i) \rightsquigarrow$ 

$\mathrm{d}\theta_i = \mathrm{d}\theta(\mathsf{D}_i) \rightsquigarrow$ 

Main *advantage* of using dual forms in this talk:

They do *not* need to look like Feynman integrands!

Why is it *so* hard to integrate?

Algebraic and analytical
complexity

Modelling the LHC with high precision requires *many* massive external and internal states

More variables
⇓
Harder algebra

Multiloop integrals have branch structure dictated by *unitarity cuts*

Multivalued functions
⇓
Algebraic geometry

[Talk by Lorenzo (again!)]

Today:

Focus on a technique to handle better the algebraic side

Loop-by-loop approach to differential equations (DEs)

A *simple* idea

True for DEs too!
Construct DEs one loop at a time

A multiloop problem is a 'bunch' of easier 1-loop problems

Mathematical setup

If our total space X locally looks like $F \times B$:

$(p+q)$-form on X | p-form on B

$$\check{\phi}_j = \check{\phi}_{F,i} \wedge \check{\phi}_{B,ij}$$

q-form on F

Mathematical setup

If our total space X locally looks like $F \times B$:

$(p+q)$-form on X $\qquad$ p-form on B

$$\check{\phi}_j = \check{\phi}_{F,i} \wedge \check{\phi}_{B,ij}$$

q-form on F

We can obtain differential equation "*one loop at a time*":

$$\check{\nabla}\check{\boldsymbol{\phi}} = \overset{(1)}{\check{\nabla}\Big(\check{\boldsymbol{\phi}}_F} \mathbin{\dot\wedge} \underline{\check{\boldsymbol{\phi}}}_B\Big) \simeq \check{\boldsymbol{\phi}}_F \mathbin{\dot\wedge} \overset{(2)}{\underline{\check{\nabla}}\;\underline{\check{\boldsymbol{\phi}}}_B} \simeq \Big(\check{\boldsymbol{\phi}}_F \mathbin{\dot\wedge} \underline{\check{\boldsymbol{\phi}}}_B\Big) \mathbin{\dot\wedge} \check{\boldsymbol{\Omega}}_B \equiv \check{\boldsymbol{\phi}} \mathbin{\dot\wedge} \check{\boldsymbol{\Omega}}_B$$

$$\Big(\check{\nabla} = \mathrm{d} + \check{\omega}\wedge \quad \xrightarrow{(1)} \quad \underline{\check{\nabla}} = \boldsymbol{d} + \check{\omega}(\check{\boldsymbol{\Omega}}_F) \mathbin{\dot\wedge}\Big)$$

Mathematical setup

If our total space X locally looks like $F \times B$:

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$$\check{\phi}_j = \check{\phi}_{F,i} \wedge \check{\phi}_{B,ij}$$

q-form on F

Loop-by-loop constraints:

1. The fibre basis chosen s.t. $\check{\boldsymbol{\omega}}$ is ε-form
2. The base basis chosen s.t. $\check{\phi}_j = \check{\phi}_{F,i} \wedge \check{\phi}_{B,ij}$ is single-valued (algebraic)

We can obtain differential equation "*one loop at a time*":

$$\check{\nabla}\check{\boldsymbol{\phi}} = \underbrace{\check{\nabla}\left(\check{\boldsymbol{\phi}}_F\right.}_{(1)} \dot\wedge \left.\underline{\check{\boldsymbol{\phi}}}_B\right) \simeq \check{\boldsymbol{\phi}}_F \dot\wedge \underbrace{\underline{\check{\nabla}}\; \underline{\check{\boldsymbol{\phi}}}_B}_{(2)} \simeq \left(\check{\boldsymbol{\phi}}_F \dot\wedge \underline{\check{\boldsymbol{\phi}}}_B\right) \dot\wedge \check{\boldsymbol{\Omega}}_B \equiv \check{\boldsymbol{\phi}} \dot\wedge \check{\boldsymbol{\Omega}}_B$$

$$\left(\check{\nabla} = \mathrm{d} + \check{\omega} \wedge \quad \xrightarrow{(1)} \quad \underline{\check{\nabla}} = \boldsymbol{d} + \check{\boldsymbol{\omega}}(\check{\boldsymbol{\Omega}}_F) \dot\wedge \right)$$

X, F and B for the sunrise

Working directly in *momentum space*:

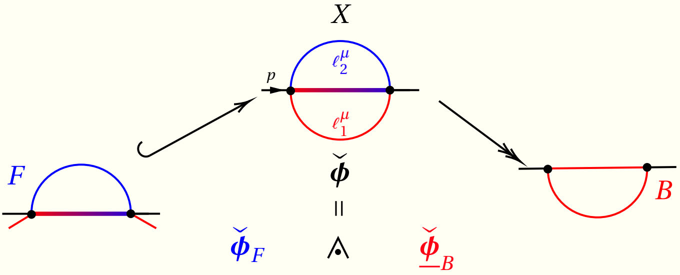
X : Space of Lorentz scalars in ℓ_1 and ℓ_2 for the sunrise

$\shortparallel$

F : Space of Lorentz scalars (say in ℓ_2) for the 1-loop sub-bubble

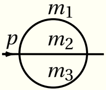
$\times$

B : Space of Lorentz scalars (say in ℓ_1) along which F varies



Example: *The 3-scale sunrise*

The *elliptic* sunrise integral



$$\sim \int \frac{\mathrm{d}^D\ell_1}{\pi^{D/2}}\frac{\mathrm{d}^D\ell_2}{\pi^{D/2}}\frac{\mathrm{d}^D\ell_3}{\pi^{D/2}}\frac{\delta^D(\ell_3-\ell_1+\ell_2-p)}{\left(\ell_1^2+m_1^2\right)\left(\ell_2^2+m_2^2\right)\left(\ell_3^2+m_3^2\right)}$$

***Momentum space* parameterization:**

$$\ell_i^\mu = \ell_{i\parallel}^\mu + \ell_{i\perp}^\mu, \quad \ell_{i\parallel}\cdot\ell_{i\perp} = 0, \quad \ell_{1\parallel}^\mu = x\, p^\mu$$

$$\mathrm{d}^D\ell_i = \left(\ell_{i\perp}^2\right)^{\frac{D-3}{2}}\, \mathrm{d}\Omega_{D-2}\wedge \mathrm{d}\ell_{i\parallel}\wedge \mathrm{d}\ell_{i\perp}^2$$

On the maximal-cut in even D get an integral over:

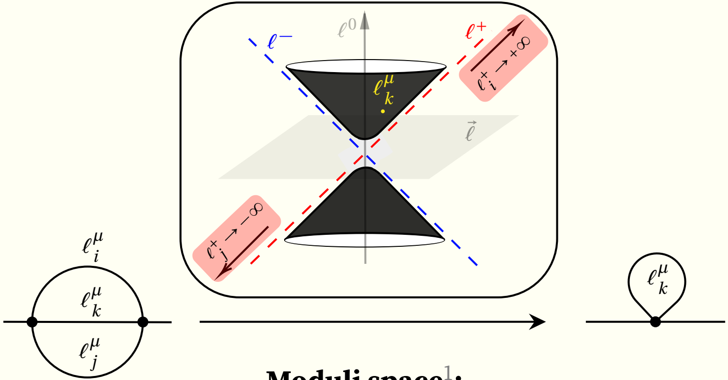
$$E(\mathbb{C}): Y^2-(x-r_1)(x-r_2)(x-r_3)(x-r_4)=0$$

$\exists$ isomorphism with a genus-1 surface:

$$E(\mathbb{C}) \simeq \mathbb{C}/\Lambda_{(1,\tau)}$$

DOFs on the torus

$\mathbb{C}/\Lambda_{(1,\tau)}$ comes with *marked points* inherited from $\binom{3}{2}$ = three special configurations of the sunrise graph

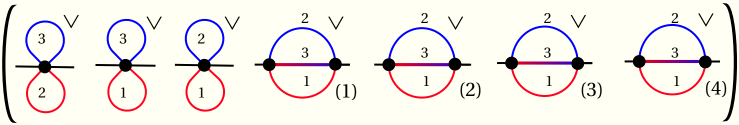


Moduli space[1]**:**

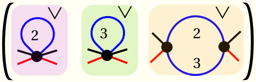
Torus with three marked points: $\{z_i = F(u_i)/K\}_{i=1}^3$

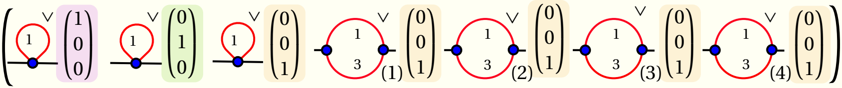
[1]One marked point is fixed by translational symmetry

Schematic splitting of the sunrise

$\{\check{\phi}_j\}_{j=1}^{7} =$ 

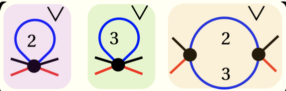
$\check{\phi}_j = \check{\phi}_{F,i} \wedge \check{\phi}_{B,ij}$ — Loop-by-loop splitting

$\{\check{\phi}_{F,i}\}_1^3 =$ 

$\{\check{\boldsymbol{\phi}}_{B,j}\}_1^7 =$ 

Fibre basis & differential equation

The normalized basis in [Caron-Huot, Pokraka; 21]

$$\{\check{\phi}_{F,i}\}_1^3 = \Big($$


$$\Big)$$

$$\check{\phi}_{F,1} = \frac{2\varepsilon}{q\sqrt{\ell_{1\perp}^2}} \frac{\mathrm{d}\theta_2 \wedge \mathrm{d}\ell_{2\parallel}}{\ell_{2\perp}^2\big|_2} \qquad \check{\phi}_{F,2} = \frac{2\varepsilon}{q\sqrt{\ell_{1\perp}^2}} \frac{\mathrm{d}\theta_3 \wedge \mathrm{d}\ell_{2\parallel}}{\ell_{2\perp}^2\big|_3}$$

$$\check{\phi}_{F,3} = \frac{1}{q\sqrt{\ell_{1\perp}^2}} \frac{\mathrm{d}\theta_2 \wedge \mathrm{d}\theta_3}{\sqrt{\ell_{2\perp}^2\big|_{23}}}$$

$\ell_{1\perp}^2$ = Gram determinant on the 2^{nd} loop
$q = \sqrt{(p+\ell_1)^2}$ (fibre external momentum)

satisfies a dlog-form differential equation $\check{\boldsymbol{\Omega}}_F$ such that

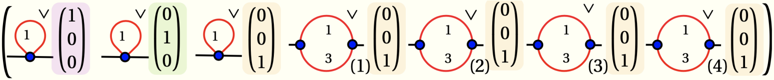
$$\check{\underline{\nabla}} \supset \check{\boldsymbol{\omega}} = \check{\boldsymbol{\omega}}(\check{\boldsymbol{\Omega}}_F) \sim \varepsilon$$

Bubble denominator restricted to last cut is the elliptic curve:

$$Y^2 - \left[\ell_{1\perp}^2(\ell_1+p)^2\ell_{2\perp}^2\right]\Big|_{123} = 0$$

Base basis: Step 1

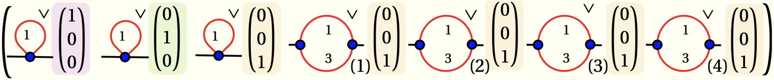
$\{\check{\phi}_{B,j}\}_1^7 =$



1. Uniformly transcendental
2. Satisfies the *second* loop-by-loop constraint ($\check{\phi}_j = \check{\phi}_{F,i} \wedge \check{\phi}_{B,ij}$ is algebraic)
3. Satisfies a lower triangular DE at $\mathcal{O}(\varepsilon^0)$ – i.e., a "linear-form"
4. DEs has a and b independent modular transformation rule

Base basis: Step 1

$\{\check{\phi}_{B,j}\}_1^7 =$



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Tadpoles:

$$\check{\phi}_{B,1} = \mathrm{d}\log\left(\frac{1-\frac{ix}{\sqrt{r_1^2-x^2}}}{1+\frac{ix}{\sqrt{r_1^2-x^2}}}\right) \wedge \mathrm{d}\theta_1 \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\check{\phi}_{B,2} = \mathrm{d}\log\left(\frac{1-\frac{ix}{\sqrt{r_1^2-x^2}}}{1+\frac{ix}{\sqrt{r_1^2-x^2}}}\right) \wedge \mathrm{d}\theta_1 \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\check{\phi}_{B,3} = i\,\varepsilon\,\theta_1\,\mathrm{d}\log\left(\frac{p\,(x+1)+\sqrt{-\ell_{1\perp}^2}}{p\,(x+1)-\sqrt{-\ell_{1\perp}^2}}\right) \wedge \mathrm{d}\log\left(\frac{q_+-q_-}{q_++q_-}\right)\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$\ell_1^\mu = \ell_{1\parallel}^\mu + \ell_{1,\perp}^\mu = x\,p^\mu + \ell_{1\perp}^\mu$, $\psi_1 \sim K$

$q_\pm = \sqrt{(p+\ell_1)^2 + m_\pm^2}$, $m_\pm = m_2 \pm m_3$

c, ∞ = twisted singularities in $D = 4$

Maximal-cut:

$$\check{\phi}_{B,4} = \frac{\psi_1^2}{\pi\,\varepsilon\,W_0}\check{\nabla}_0\check{\phi}_{B,7}$$

$$\check{\phi}_{B,5} = m_1^{-4\varepsilon}\ \mathrm{d}\theta_1 \wedge \frac{(x-r_1)\mathrm{d}x}{Y}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\check{\phi}_{B,6} = m_1^{-4\varepsilon}\ \mathrm{d}\theta_1 \wedge \frac{Y(c)\mathrm{d}x}{(x-c)Y}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\check{\phi}_{B,7} = m_1^{-4\varepsilon}\ \mathrm{d}\theta_1 \wedge \frac{\pi\mathrm{d}x}{\psi_1 Y}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

Base basis: Step 2

***Modular properties* of the linear DEs:**

Enough to "bootstrap" the gauge transformation

$$\check{\boldsymbol{\vartheta}}_{B,j} = \mathscr{U}_{ji}\,\check{\boldsymbol{\phi}}_{B,i}$$

integrating out the $\mathcal{O}(\varepsilon^0)$-piece by *pen and paper*

Details in [Giroux, Pokraka; to appear]

$\rightsquigarrow$

ε-form DE spanned by *modular* and *Kronnecker forms*

Base differential equation

$$
\breve{\boldsymbol{\Omega}}_B = \varepsilon \begin{pmatrix}
\mho_2^{(1,2)}\begin{bmatrix}-2\\0\\2\end{bmatrix} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mho_2^{(1,2)}\begin{bmatrix}0\\-2\\2\end{bmatrix} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mho_2^{(1,2)}\begin{bmatrix}-2\\-2\\4\end{bmatrix} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \varpi_2^{(1)}\begin{bmatrix}0\\0\\4\end{bmatrix}+\varpi_2^{(2)}\begin{bmatrix}2\\2\\-6\end{bmatrix} & \varpi_1^{(1)}\begin{bmatrix}\frac{1}{2i}\\\frac{1}{2i}\\0\end{bmatrix} & \varpi_1^{(1)}\begin{bmatrix}-\frac{1}{2i}\\\frac{1}{2i}\\0\end{bmatrix} & \frac{1}{4}\,\omega_0^{\mathrm{Kr}}(\tau) \\
\mho_2^{(1,2)}\begin{bmatrix}4i\\-4i\\-8i\end{bmatrix} & \mho_2^{(1,2)}\begin{bmatrix}-4i\\4i\\-8i\end{bmatrix} & \mho_2^{(1,2)}\begin{bmatrix}-2\\-2\\-4\end{bmatrix} & \varpi_3^{(1)}\begin{bmatrix}-6i\\-6i\\12i\end{bmatrix} & \varpi_2^{(1)}\begin{bmatrix}-1\\-1\\0\end{bmatrix}+\varpi_2^{(2)}\begin{bmatrix}2\\2\\-6\end{bmatrix} & \varpi_2^{(1)}\begin{bmatrix}3\\-3\\0\end{bmatrix} & \varpi_1^{(1)}\begin{bmatrix}\frac{3}{4i}\\\frac{3}{4i}\\0\end{bmatrix} \\
\mho_2^{(1,2)}\begin{bmatrix}-4i\\-4i\\0\end{bmatrix} & \mho_2^{(1,2)}\begin{bmatrix}4i\\4i\\0\end{bmatrix} & \mho_2^{(1,2)}\begin{bmatrix}2\\-2\\0\end{bmatrix} & \varpi_3^{(1)}\begin{bmatrix}-6i\\6i\\0\end{bmatrix} & \varpi_2^{(1)}\begin{bmatrix}1\\-1\\0\end{bmatrix} & \varpi_2^{(1)}\begin{bmatrix}-3\\-3\\4\end{bmatrix}+\varpi_2^{(2)}\begin{bmatrix}2\\2\\-6\end{bmatrix} & \varpi_1^{(1)}\begin{bmatrix}-\frac{1}{4i}\\\frac{1}{4i}\\0\end{bmatrix} \\
\mho_3^{(1,4)}\begin{bmatrix}16i\\-16i\\16i\end{bmatrix} & \mho_3^{(1,4)}\begin{bmatrix}-16i\\16i\\16i\end{bmatrix} & \mho_3^{(1,4)}\begin{bmatrix}-8i\\-8i\\8i\end{bmatrix} & \varpi_4^{(1)}\begin{bmatrix}48\\48\\48\end{bmatrix} & \varpi_3^{(1)}\begin{bmatrix}-4i\\-4i\\8i\end{bmatrix} & \varpi_3^{(1)}\begin{bmatrix}-12i\\12i\\0\end{bmatrix} & \varpi_2^{(1)}\begin{bmatrix}0\\0\\4\end{bmatrix}+\varpi_2^{(2)}\begin{bmatrix}2\\2\\-6\end{bmatrix}
\end{pmatrix}
$$

$$
+\,\varepsilon \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -6\eta_2(\tau) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -6\eta_2(\tau) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -6\eta_2(\tau) & 0 \\
0 & 0 & 0 & -288\eta_4(\tau) & 0 & 0 & -6\eta_2(\tau)
\end{pmatrix}
$$

Compact notation:

$$\varpi_n^{(K)}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} := \sum_{i=1}^{3} c_i\ \omega_n^{\mathrm{Kr}}(z_i|K\,\tau), \qquad K\in\mathbb{N}$$

$$\mho_n^{(K,m)}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} := \varpi_n^{(K)}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} + \varpi_n^{(2K)}\begin{bmatrix}-m\ c_1\\-m\ c_2\\-m\ c_3\end{bmatrix}$$

Relation to Feynman integrands

Up to a *constant rescaling* (R) of the ε-form dual basis

$$\mathcal{V} = \mathrm{diag}(1,1,1,1,i/2,i/2,4)$$

our basis is *dual* to the basis of integrands presented in

[Bogner, Müller-Stach, Weinzierl;19]

$$\left\langle \check{\phi}_i^{(\mathrm{R})} \middle| \phi_j \right\rangle = \left\langle \check{\phi}_{F,k} \wedge \check{\vartheta}_{B,ki}^{(\mathrm{R})} \middle| \phi_j \right\rangle = \delta_{ij}$$

Conclusions

Closing Thoughts

✓ Extended dual forms to a multi-scale 2-loop problem

✓ Loop-by-loop dual basis constrained by geometry

$\oplus$

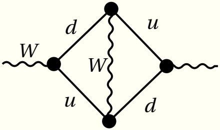
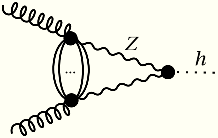
"Modular bootstrap"

$\downarrow$

Ready to integrate ε-form differential equation

✓ Surprisingly simple link to basis of [Weinzierl et al. ;19]

⚠ Go beyond the simplest example!

…  …  …

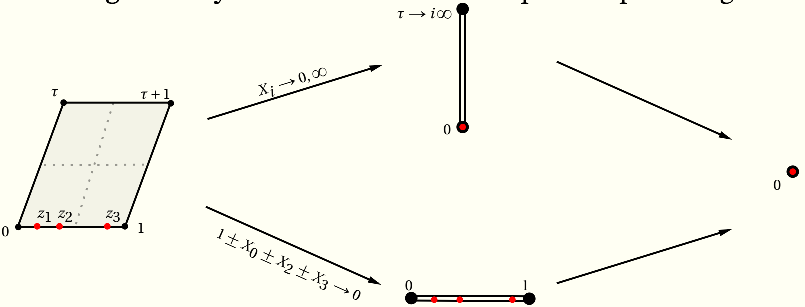
[Talks by Matt, Sebastian, Christoph, Franziska and Chi]

Danke Sehr!

Backup slides

Sunrise alphabet

The geometry *near* the differential equation poles *degenerates*



1. The only *accessible* finite lattice points near these poles:

$$z = 0 \quad \text{and} \quad z = 1$$

2. Target space is known:

$$\text{Kronecker forms:} \quad \omega_n^{\mathrm{Kr}}(z|K\tau)$$

3. ω_n^{Kr} has simple poles only at *lattice points* and is $\mathbb{Z}$*-periodic*

$$\therefore \ \mathscr{A}_{\text{sunrise}} = \{z_1, z_2, z_3\}$$

Function space at 2-loop

- *Modular form* $\omega_k^{\text{modular}}$ of modular weight k:

$$\omega_k^{\text{modular}}(\tau) \xrightarrow{\text{SL}(2,\mathbb{Z})} (c\tau - d)^{k-2}\omega_k^{\text{modular}}(\tau)$$

$$\eta_2(\tau) = [\underbrace{e_2(\tau) - 2e_2(2\tau)}_{\in \mathscr{M}_2(\Gamma_0(2))}]\frac{\mathrm{d}\tau}{2\pi\, i} \text{ and } \eta_4(\tau) = \underbrace{e_4(\tau)}_{\in \mathscr{M}_4(\text{SL}(2,\mathbb{Z}))}\frac{\mathrm{d}\tau}{(2\pi\, i)^3}$$

- *Eisenstein-Kronecker series*:

$$F(z,\alpha,\tau) = \vartheta_1'(0,\bar{q})\frac{\vartheta_1(z+\alpha,\bar{q})}{\vartheta_1(z,\bar{q})\vartheta_1(\alpha,\bar{q})} = \sum_{k=0}^{\infty} g^{(k)}(z|\tau)\alpha^{k-1}$$

- *Kronecker forms*:

$$\omega_k^{\text{Kr}}(z,\tau) = (2\pi\, i)^{2-k}\left(g^{(k-1)}(z,\tau)\mathrm{d}z + (k-1)g^{(k)}(z,\tau)\frac{\mathrm{d}\tau}{2\pi\, i}\right)$$

Period matrix of the ε-form basis

The *period matrix* for the ε-form basis takes the form

$$\check{\boldsymbol{P}}_{ij} := \oint_{\gamma_i} \check{\boldsymbol{\vartheta}}_{B,j} = \check{\boldsymbol{P}} = \varepsilon^{-1}\,\check{\boldsymbol{P}}^{(-1)} + \check{\boldsymbol{P}}^{(0)}$$

The fact that

$$\check{\boldsymbol{\Omega}}_B^{(0)} = 0$$

for a basis with non-uniform ε-depth requires

$$\boldsymbol{d}\check{\boldsymbol{P}}^{(0)} + \varepsilon^{-1}\,\check{\boldsymbol{\omega}} \cdot \check{\boldsymbol{P}}^{(-1)} = 0$$

In particular, our basis is *not* of constant leading singularity

$$\boldsymbol{d}\check{\boldsymbol{P}} \neq 0$$

[Talk Hjalte]

3-loop: Elliptically fibered K3

$$\mathscr{S}_{\nu_1\nu_2\nu_3\nu_4} \sim \int \frac{\mathrm{d}^D\ell_1}{\pi^{D/2}}\frac{\mathrm{d}^D\ell_2}{\pi^{D/2}}\frac{\mathrm{d}^D\ell_3}{\pi^{D/2}}\frac{\mathrm{d}^D\ell_4}{\pi^{D/2}}\frac{\delta^D(p-\ell_1-\ell_2-\ell_3-\ell_4)}{\mathsf{D}_1^{\nu_1}\mathsf{D}_2^{\nu_2}\mathsf{D}_3^{\nu_3}\mathsf{D}_4^{\nu_4}}, \quad \mathsf{D}_i = \ell_i^2 + m_i^2$$

maximal cut in $D = 4$

$$\mathscr{S}_{1111}|_{\text{m.c.}} \sim \int \mathrm{d}x_2 \wedge \mathrm{d}x_3 \; f_6\left(\sqrt{\prod_{i=1}^{4}(x_2 - \mathfrak{r}_i(x_3))}\right)$$

$$\mathfrak{r}_i \in \mathbb{Q}[x_3]$$

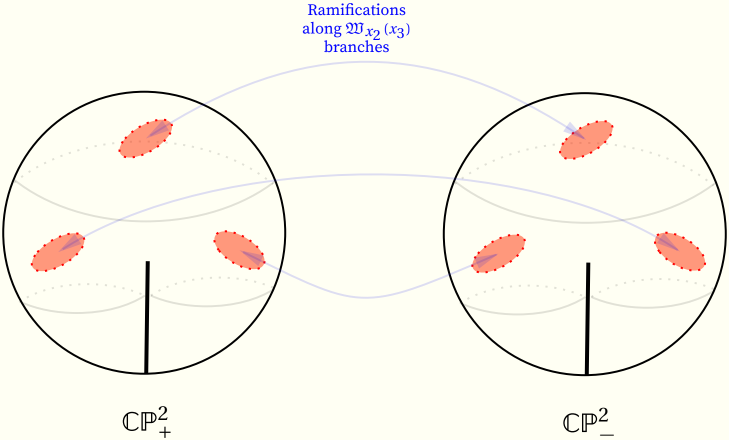
Geometry G admits the structure of an *elliptic fibrations* over the Riemann sphere

$$E(\mathbb{C}) \hookrightarrow G \twoheadrightarrow \mathbb{CP}^1$$

Geometers call such space G a *complex algebraic K3-surface*

The K3-type

The 3-loop banana K3-surface is a double cover of $\mathbb{CP}^2$ branched over the locus of a sextic



Right (co)homology bases? Right moduli space?