Symbology for Elliptic Multiple Polylogarithms

Based on the recent work 2206.08378 with M. Wilhelm

and ongoing work with R. Morales, A. Spiering, M. Wilhelm and Q. Yang

Chi Zhang

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Niels Bohr Institute

Introduction and Motivation

MPLs and Beyond in Amplitudes [also see talks by Lorenzo]

Large classes of Feynman integrals are described by Multiple polylogarithms (MPLs).

- One loop Feynman integrals.
- MHV and NMHV amplitudes in planar $\mathcal{N} = 4$ sYM (Conjecture).

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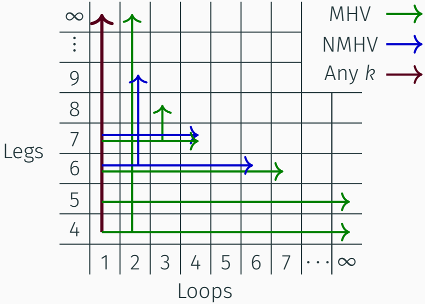

[Bern, Caron-Huot, Del Duca, Dixon, Drummond, Duhr, Foster, Golden, Goncharov, Gürdoğan, He, Henn, von Hippel, Kosower, Li, McLeod, Papathanasiou, Pennington, Roiban, Smirnov, Spradlin, Vergu, Volovich, CZ, ...]

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- Simplify MPLs: 17 pages becomes a few lines for 2-loop hexagon remainder function! [Del Duca, Duhr, Smirnov][Goncharov, Spradlin, Vergu, Volovich]
- Manifest the singularity structures *e.g.* first entry conditions and Steinmann relations $\Rightarrow$ High loop hexagon and heptagon bootstrap programme!
- Bootstrap general Feynman integrals and form factors. [Caron-Huot, Dixon, von Hippel, McLeod, Papathanasiou][Gehrmann, Henn, Lo Presti][Henn, Herrmann, Parra-Martinez][Dixon, McLeod, Wilhelm][He, Li, Yang]. . .

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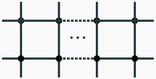
General Amplitudes are much richer than MPLs (even in planar $\mathcal{N} = 4$ sYM):

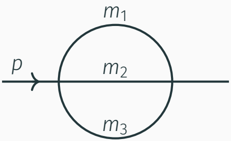
- Calabi-Yau manifolds in multi-loop diagram [Bloch, Kerr, Vanhove][Bourjaily, He, McLeod, von Hippel, Wilhelm][Bönisch, Duhr, Fischbach, Klemm, Nega] . . . [also see the seminars by Lorenzo, Matt, and Christoph]



- CY 1-fold: elliptic curve ($y^2 = P(x)$ with $\deg P(x) = 3$ or 4) @ two-loop.

The emergence of the elliptic curve in the sunrise integral

The sunrise integral in 2-dim . . .[Schnetz][Broedel, Duhr, Dulat, Tancredi]. . .[also see seminars by Lorenzo, Matt, Hjalte and Mathieu]:



$$= \int \frac{d^2\ell_1\, d^2\ell_2}{(\ell_1^2 - m_1^2)(\ell_2^2 - m_2^2)\big((\ell_1 - \ell_2 + p)^2 - m_3^2\big)}$$

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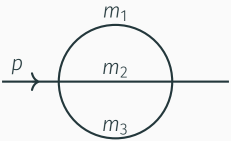

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In terms of Feynman parameter,

$$\begin{aligned} I_{\ominus} &= \int_0^\infty \frac{dx_1\, dx_2}{-p^2\, x_1 x_2 + (m_1^2 x_1 + m_2^2 x_2 + m_3^2)(x_1 x_2 + x_1 + x_2)} \\ &= \frac{1}{m_1^2} \int \frac{dx}{y} \log R_{\ominus}(x, y)\,, \qquad (\text{with } x = x_1) \end{aligned}$$

where

$$y^2 = x^4 + \sum_{i=0}^{3} a_i x^i : \quad \text{elliptic curve!}$$

$$R_{\ominus}(x, y) = \frac{t_3^2 + x(t_1^2 + t_2^2 + t_3^2 - 1) + t_1^2 x^2 + t_1^2 y}{t_3^2 + x(t_1^2 + t_2^2 + t_3^2 - 1) + t_1^2 x^2 - t_1^2 y}\,, \qquad (t_i^2 = m_i^2/p^2).$$

Several elliptic generalizations of MPLs:

- Iterated integrals on elliptic curves: E_4 functions.
- Iterated integrals on tori: $\widetilde{\Gamma}$ functions
- q-deformed polylogarithms: ELi functions. [Adams, Bogner, Weinzier]

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What is "Symbology" for elliptic Polylogarithms?

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In the rest of the talk:

1. Review the framework of elliptic polylogarithms.
2. Manipulation of elliptic letters.
3. Use the sunrise integral as a guideline.
4. Comment on the symbol of the elliptic double box.

Elliptic Polylogarithms

Polylogarithms and Symbols

Polylogarithms of weight n are n-fold iterated integrals with logarithmic singularities on the Riemann sphere: [Poincaré 1884, Lappo-Danilevsky 1927][Chen][Goncharov]

$$G(a_1, a_2, \ldots, a_n; x) = \int_0^x \frac{dx_1}{x_1 - a_1} G(a_2, \ldots, a_n; x_1) \quad a_i \in \mathbb{C} \qquad \text{with } G(;x) := 1.$$

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The singularity structures of MPLs can be characterized by the symbol:
[Goncharov][Goncharov, Spradlin, Vergu, Volovich]

- Derivatives only act on the last entries:

$$dF_n = \sum F_{i,n-1} \times d \log R_i \quad \Rightarrow \quad \mathcal{S}(F_n) = \sum \mathcal{S}(F_{i,n-1}) \otimes \log R_i$$

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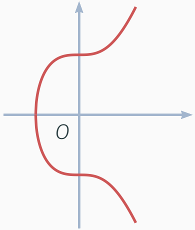
- Symbol letters $\log R_i$ are logarithms of algebraic functions.
- The first entries indicate the loci of branch cuts.
- Easy to manipulate: $\cdots \otimes \log(ab) \otimes \cdots = \cdots \otimes \log(a) \otimes \cdots + \cdots \otimes \log(b) \otimes \cdots$

eMPLs are iterated integrals with logarithmic singularities on elliptic curves $y^2 = P_4(x)$: [Broedel, Duhr, Dulat, Tancredi]



$$\mathrm{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x\right) = \int_0^x dx'\, \psi_{n_1}(c_1, x')\, \mathrm{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; x'\right) \quad \text{with } \mathrm{E}_4(;x) = 1,$$

where

$$\psi_1(c, x) = \frac{1}{x - c},$$

$$\psi_0(0, x) = \frac{1}{y},$$

$$\psi_{-1}(c, x) = \frac{y_c}{y(x - c)}, \qquad \psi_{-1}(\infty, x) = \frac{x}{y},$$

and other $\psi_n(c, x)$ to make a complete basis for the general kernel $R_1(x) + \frac{R_2(x)}{y}$

Elliptic Generalizations I

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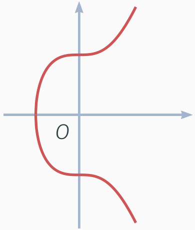

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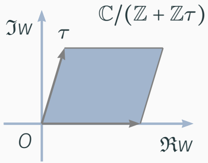
- $\mathrm{E}_4\left(\begin{smallmatrix} 1, & \dots, & 1 \\ c_1, & \dots, & c_n \end{smallmatrix}; x\right) = G(c_1, \dots, c_n, x)$
- Advantage: Directly related to Feynman integrals.
- Disadvantage: Difficulty to compute the differential (not pure).

eMPLs are iterated integrals with logarithmic singularities on torus $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$: [Brown, Levin] [Broedel, Mafra, Matthes, Schlotterer] [Broedel, Duhr, Dulat, Tancredi]



$$\widetilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_k \\ w_1 & \dots & w_k \end{smallmatrix}; w\right) = \int_0^w dw' \, g^{(n_1)}(w'-w_1)\widetilde{\Gamma}\left(\begin{smallmatrix} n_2 & \dots & n_k \\ w_2 & \dots & w_k \end{smallmatrix}; w'\right),$$

with $\widetilde{\Gamma}(;w) = 1$.

The integration kernels $g^{(n)}(w)$ are generated by the *Eisenstein-Kronecker series*

$$\frac{\partial_w \theta_1(0)\theta_1(w+\alpha)}{\theta_1(w)\theta_1(\alpha)} = \sum_{n\geq 0} \alpha^{n-1} g^{(n)}(w) \, .$$

θ_1: odd Jacobi θ-function. $g^{(0)} = 1$, $g^{(1)}(w) = \partial_w \log \theta_1, \dots$

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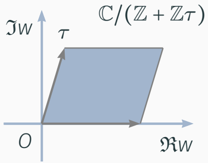

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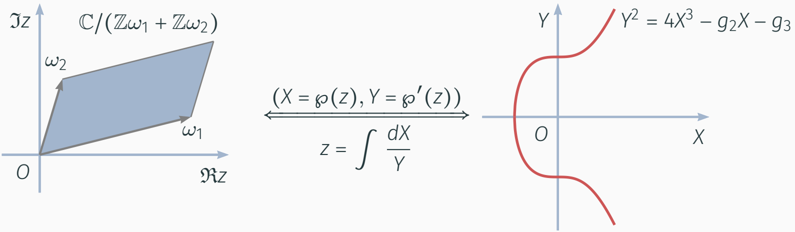
- Length k $\neq$ Weight $n = \sum n_i$
- Easy to compute the differential [Broedel, Duhr, Dulat, Penante, Tancredi]

$$d\widetilde{\Gamma}_k^{(n)} = \sum_i \widetilde{\Gamma}_{k-1}^{(n-j_i)}\omega^{(j_i)}(z_i)$$

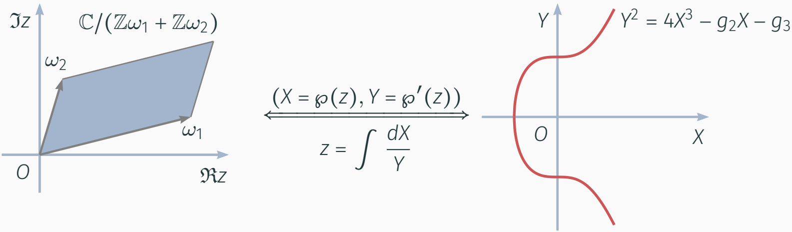
where $\omega^{(j\geq -1)}(w) = g^{(j)}(w,\tau)dw + n(2\pi i)^{-1}g^{(j+1)}(w,\tau)d\tau$

Tori ↔ Elliptic curves



The Weierstrass map and Abel's map for an elliptic curve $y^2 = x^4 + \sum_{i=0}^{3} a_i x^i$:

Torus → Elliptic Curve	$x = \kappa(z) = \dfrac{6a_1 - a_2a_3 + 12a_3\wp(z) - 24\wp'(z)}{3a_3^2 - 8a_2 - 48\wp(z)}, \quad y = \kappa'(z)$
Elliptic Curve → Torus	$z = \displaystyle\int \frac{dx}{y}$



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Remark:

- Each point c in kinematic space → $(c, \pm y_c) \leftrightarrow z_c^{\pm}$.
- $\kappa(m\omega_1 + n\omega_2) = \infty$ → the infinity point $(+\infty, +\infty)$ is mapped to a lattice point.

Transformation of kernels

From now on, we work with the normalized torus $[1 : \tau = \omega_2/\omega_1]$, and denote the images of c as $w_c^{\pm} := z_c^{\pm}/\omega_1$.

Relations between integration kernels for E_4 functions and those for $\widetilde{\Gamma}$ functions:

$$\begin{aligned}
\psi_0 dx &= \omega_1 dw\,, \\
\psi_1(c,x)dx &= \Big(g^{(1)}(w-w_c^+) + g^{(1)}(w-w_c^-) - g^{(1)}(w-w_\infty^+) - g^{(1)}(w-w_\infty^-)\Big)dw\,, \\
\psi_{-1}(c,x)dx &= \Big(g^{(1)}(w-w_c^+) - g^{(1)}(w-w_c^-) + g^{(1)}(w_c^+) - g^{(1)}(w_c^-)\Big)dw\,, \\
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Remark: Another way to package eMPLs: $\mathcal{E}_4(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x)$ [Broedel, Duhr, Dulat, Penante, Tancredi] which are also manifest pure.

$$\Psi_{\pm n}(c,x)dx = \Big[g^{(n)}(w-w_c^+) \pm g^{(n)}(w-w_c^-)$$
$$- \delta_{\pm n,1}\big(g^{(1)}(w-w_\infty^+) + g^{(1)}(w-w_\infty^-)\big)\Big]dw \,.$$

Recall that,

$$I_{\ominus} = \frac{1}{m_1^2} \int \underbrace{\frac{dx}{y}}_{\Psi_0} \underbrace{\log \frac{t_3^2 + x(t_1^2 + t_2^2 + t_3^2 - 1) + t_1^2 x^2 + t_1^2 y}{t_3^2 + x(t_1^2 + t_2^2 + t_3^2 - 1) + t_1^2 x^2 - t_1^2 y}}_{E_4(\frac{-1}{c};x)+\cdots} .$$

Example: the sunrise integral

Recall that,

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In terms of $\mathcal{E}_4$ functions, the sunrise integral is

$$T^{(1)}_{\ominus} = \frac{m_1^2 I_{\ominus}}{\omega_1} = \mathcal{E}_4(\begin{smallmatrix} 0 & -1 \\ 0 & -1 \end{smallmatrix}; \infty|\tau) - \mathcal{E}_4(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}; \infty|\tau) + \mathcal{E}_4(\begin{smallmatrix} 0 & -1 \\ 0 & r \end{smallmatrix}; \infty|\tau) - \mathcal{E}_4(\begin{smallmatrix} 0 & -1 \\ 0 & \infty \end{smallmatrix}; \infty|\tau)$$
$$+ 4\pi i \mathcal{E}_4(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}; \infty|\tau) - \mathcal{E}_4(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}; \infty|\tau) \log \frac{t_2^2}{t_3^2} ,$$

where

$$\mathcal{E}_4(\begin{smallmatrix} 0 & -1 \\ 0 & c \end{smallmatrix}; x) = \widetilde{\Gamma}(\begin{smallmatrix} 0 & 1 \\ 0 & w_c^+ - w_0^+ \end{smallmatrix}; w_x^+ - w_0^+) - \widetilde{\Gamma}(\begin{smallmatrix} 0 & 1 \\ 0 & w_c^- - w_0^+ \end{smallmatrix}; w_x^+ - w_0^+)$$
$$r = -t_3^2/t_1^2$$
$$\omega_1 : \quad \text{one period of the torus.} .$$

Elliptic Symbology

Symbol for MPLs and eMPLs

Symbol can be recursively defined by taking total differential:

- Multiple Polylogarithms:

$$dF_n = \sum F_{i,n-1} \times d \log R_i \quad \Rightarrow \quad \mathcal{S}(F_n) = \sum \mathcal{S}(F_{i,n-1}) \otimes \log R_i$$

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- elliptic Multiple Polylogarithms [Broedel, Duhr, Dulat, Penante, Tancredi]:

$$d\widetilde{\Gamma}_k^{(n)} = \sum_i \widetilde{\Gamma}_{k-1}^{(n-j_i)} \omega^{(j_i)}(z_i) \Rightarrow \mathcal{S}\Big((2\pi i)^{k-n} \widetilde{\Gamma}_k^{(n)}\Big) = \sum_i \mathcal{S}\Big((2\pi i)^{k-1-n+j_i} \widetilde{\Gamma}_{k-1}^{(n-j_i)}\Big) \otimes \Omega^{(j_i)}$$

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$$\Omega^{(-1)}(w,\tau) = -2\pi i\tau\,, \quad \Omega^{(0)}(w,\tau) = 2\pi i w\,, \quad \Omega^{(1)}(w,\tau) = \log \frac{\theta_1(w|\tau)}{\eta(\tau)}\,,$$

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$$\Omega^{(\text{odd } j>1)}(w,\tau) = -\frac{2j\zeta_{j+1}\tau}{(2\pi i)^j} + \frac{1}{(j-1)!}\sum_{n=1}^{\infty} n^{j-1}\log\Big(\big(1-\mathrm{e}^{2\pi i(n\tau-w)}\big)\big(1-\mathrm{e}^{2\pi i(n\tau+w)}\big)\Big),$$

$$\Omega^{(\text{even } j)}(w,\tau) = -\frac{2\zeta_j w}{(2\pi i)^{j-1}} + \frac{1}{(j-1)!}\sum_{n=1}^{\infty} n^{j-1}\log\frac{1-\mathrm{e}^{2\pi i(n\tau+w)}}{1-\mathrm{e}^{2\pi i(n\tau-w)}}\,,$$

  where $\eta(\tau)$ is Dedekind eta function and $\zeta_j = \sum_{n\in\mathbb{Z}_+} n^{-j}$ are the Riemann zeta values.

Basic identities for elliptic letters $\Omega^{(j)}$

1. Basic properties:

$$\text{Parity}: \quad \Omega^{(n)}(-w) = (-1)^{n+1}\Omega^{(n)}(w)\,,$$

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2. Recall that

$$\psi_1(c,x)dx = \frac{dx}{(x-c)} = \sum_{\sigma\in\pm}\left(g^{(1)}(w-w_c^\sigma) - g^{(1)}(w-w_\infty^\sigma)\right)dw$$

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Further identities for $\int\psi_0 dx$ and $\int\psi_{-1}dx$?

Abel's addition theorem

$\int \psi_0 dx$ and $\int \psi_{-1} dx$ are Abelian integrals.

- Let $\mathcal{C}$ and $\mathcal{C}'$ be algebraic plane curves given by

$$\begin{aligned} \mathcal{C}: &\quad F(x, y) = 0 \\ \mathcal{C}': &\quad \phi(x, y) = 0 \end{aligned}$$

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- Suppose $\mathcal{C}$ and $\mathcal{C}'$ intersect at n points: $(x_1, y_1), \ldots, (x_n, y_n)$.
- Consider the sum

$$I = \sum_{i=1}^{n} \int_{(x_*,y_*)}^{(x_i,y_i)} R(x,y) dx \,,$$

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- $\partial_{b_\mu} I = \sum_i R(x_i, y_i) \partial_{b_\mu} x_i$ is a *rational* function of $\{b_\mu\}$

Applications of Abel's theorem: Logarithms

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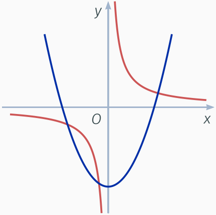

Three intersection points (x_i, y_i) are fixed by

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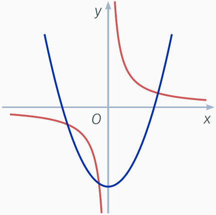

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Consider $I = \sum_i \int_1^{x_i} dx/x$, then

$$\partial_{b_\mu} I = \sum_i \frac{1}{x_i}\frac{\partial x_i}{\partial b_\mu} = \frac{1}{x_1 x_2 x_3}\frac{\partial}{\partial b_\mu}\underbrace{(x_1 x_2 x_3)}_{1} = 0 \quad \Rightarrow \quad I = \text{Const.}$$

Choose $x_1 = x_2 = x_3 = 1 \Rightarrow I = 0$

$$0 = I = \log x_1 + \log x_2 + \log x_3 \quad \Rightarrow \quad \log x_1 + \log x_2 = \log(x_1 x_2)$$

Applications of Abel's theorem: Elliptic integrals

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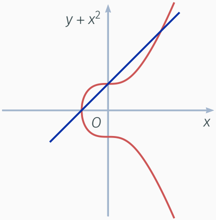
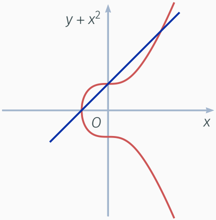


Intersection points: $(x_1, \sqrt{P_4(x_1)})$, $(x_2, \sqrt{P_4(x_2)})$ and $(x_3, -\sqrt{P_4(x_3)})$



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- Recall $\Omega^{(0)}(w_c^+) = \frac{2\pi i}{\omega_1} \int_{-\infty}^{c} \frac{dx}{y}$,

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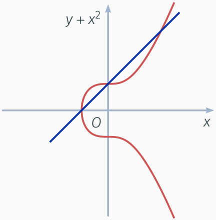

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- Recall $\psi_{-1}(c,x) = \frac{y_c}{y(x-c)}$

$$\text{Abel's theorem}: \quad \left(\int_{-\infty}^{x_1} + \int_{-\infty}^{x_2} - \int_{-\infty}^{x_3} \right) \psi_{-1}(c,x) dx = \log \frac{c^2 - b_1 c - b_2 + y_c}{c^2 - b_1 c - b_2 - y_c} .$$

$$\psi_{-1}(c,x)dx = \left(g^{(1)}(w - w_c^+) - g^{(1)}(w - w_c^-) + g^{(1)}(w_c^+) - g^{(1)}(w_c^-) \right) dw$$

$\Rightarrow$ identities for $\Omega^{(1)}$

Elliptic Bloch Relations: Symbol prime for symbol letters

The elliptic generalization of the five-term identity

$$D(x) + D(y) + D\left(\frac{1-x}{1-xy}\right) + D(1-xy) + D\left(\frac{1-y}{1-xy}\right) = 0$$

where $D(z) = \Im(\mathrm{Li}_2(z)) + \arg(1-z)\log|z|$ [Bloch][Zagier, Gangl].

$\Rightarrow$ identities for $\Omega^{(2)}$'s [Brödel, Kaderli] $\Rightarrow$ $\Omega^{(2)}$'s are as complicated as dilogarithms.

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Proof: $\mathcal{S}(2\pi i \widetilde{\Gamma}(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}; w)) = \Omega^{(0)}(w) \otimes \Omega^{(1)}(w) - \Omega^{(2)}(w) \otimes (2\pi i \tau)$,

$\Rightarrow \mathcal{S}\left(2\pi i \sum_j c_j \widetilde{\Gamma}(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}; w_j)\right) = -\left(\sum_j c_j \Omega^{(2)}(w)\right) \otimes (2\pi i \tau)$

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$$f(\tau) = 0 \quad \supseteq \quad \text{Elliptic Bloch Relations.}$$

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- Quasi-periodicity gives

$$\mathcal{S}'(\Omega^{(1)}(w)) = \Omega^{(0)}(w) \otimes' \Omega^{(0)}(w) + \Omega^{(-1)} \otimes' \Omega^{(1)}(w), \quad \mathcal{S}'(\log c) = \Omega^{(-1)} \otimes' \log c.$$

Example: the sunrise integral

Recall that

$$T^{(1)}_{\ominus} = \frac{m_1^2}{\omega_1}$$

$$= \mathcal{E}_4\left(\begin{smallmatrix}0 & -1\\ 0 & -1\end{smallmatrix}; \infty\right) - \mathcal{E}_4\left(\begin{smallmatrix}0 & -1\\ 0 & 0\end{smallmatrix}; \infty\right) + \mathcal{E}_4\left(\begin{smallmatrix}0 & -1\\ 0 & r\end{smallmatrix}; \infty\right) - \mathcal{E}_4\left(\begin{smallmatrix}0 & -1\\ 0 & \infty\end{smallmatrix}; \infty\right)$$

$$+ 4\pi i \mathcal{E}_4\left(\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}; \infty\right) - \mathcal{E}_4\left(\begin{smallmatrix}0\\ 0\end{smallmatrix}; \infty\right) \log \frac{t_2^2}{t_3^2}, \qquad (\text{with } r = -t_3^2/t_1^2)$$

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Apply the above identities gives

$$\mathcal{S}\left(2\pi i T^{(1)}_{\ominus}\right) = \log\frac{t_2^2}{t_1^2}\otimes\Omega^{(0)}(w_0^+) + \log\frac{t_1^2}{t_3^2}\otimes\Omega^{(0)}(w_{-1}^+)$$

$$+\left[\frac{1}{2\pi i}\left(2\mathcal{E}_4\left(\begin{smallmatrix}-2\\ -1\end{smallmatrix};\infty\right) - \mathcal{E}_4\left(\begin{smallmatrix}-2\\ 0\end{smallmatrix};\infty\right) - \mathcal{E}_4\left(\begin{smallmatrix}-2\\ \infty\end{smallmatrix};\infty\right)\right) + \log\frac{t_3^2}{t_2^2}\right]\otimes(2\pi i\tau)\,.$$

Recall that

$$\frac{\mathcal{E}_4\left(\begin{smallmatrix}-2\\ c\end{smallmatrix};\infty\right)}{2\pi i} = \Omega^{(2)}(w_\infty^+ - w_c^+) - \Omega^{(2)}(w_0^+ - w_c^+) - \Omega^{(2)}(w_\infty^+ - w_c^-) + \Omega^{(2)}(w_0^+ - w_c^-)\,.$$

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$$+ \left[\frac{1}{2\pi i}\left(2\mathcal{E}_4\left(\begin{smallmatrix} -2 \\ -1 \end{smallmatrix}; \infty\right) - \mathcal{E}_4\left(\begin{smallmatrix} -2 \\ 0 \end{smallmatrix}; \infty\right) - \mathcal{E}_4\left(\begin{smallmatrix} -2 \\ \infty \end{smallmatrix}; \infty\right)\right) + \log \frac{t_3^2}{t_2^2}\right] \otimes (2\pi i \tau)\,.$$

Recall that

$$\frac{\mathcal{E}_4\left(\begin{smallmatrix} -2 \\ c \end{smallmatrix}; \infty\right)}{2\pi i} = \Omega^{(2)}(w_\infty^+ - w_c^+) - \Omega^{(2)}(w_0^+ - w_c^+) - \Omega^{(2)}(w_\infty^+ - w_c^-) + \Omega^{(2)}(w_0^+ - w_c^-)\,.$$

Remark:

- Last entries are elliptic integrals: $\Omega^{(0)}(w_c) = \frac{2\pi i}{\omega_1} \int_{-\infty}^{c} \frac{dx}{y}$, $2\pi i \tau = \Omega^{(0)}(w_\infty^+)$.
- Logarithms at the first entries: indicate the branch cuts at $m_i = 0$.

Example: the sunrise integral

Recall that

$$T^{(1)}_{\ominus} = \frac{m_1^2}{\omega_1}$$

$$= \mathcal{E}_4\left(\begin{smallmatrix}0 & -1\\ 0 & -1\end{smallmatrix}; \infty\right) - \mathcal{E}_4\left(\begin{smallmatrix}0 & -1\\ 0 & 0\end{smallmatrix}; \infty\right) + \mathcal{E}_4\left(\begin{smallmatrix}0 & -1\\ 0 & r\end{smallmatrix}; \infty\right) - \mathcal{E}_4\left(\begin{smallmatrix}0 & -1\\ 0 & \infty\end{smallmatrix}; \infty\right)$$

$$+ 4\pi i \mathcal{E}_4\left(\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}; \infty\right) - \mathcal{E}_4\left(\begin{smallmatrix}0\\ 0\end{smallmatrix}; \infty\right) \log \frac{t_2^2}{t_3^2}, \qquad (\text{with } r = -t_3^2/t_1^2)$$

Apply the above identities gives

$$\mathcal{S}\left(2\pi i T^{(1)}_{\ominus}\right) = \log \frac{t_2^2}{t_1^2} \otimes \Omega^{(0)}(w_0^+) + \log \frac{t_1^2}{t_3^2} \otimes \Omega^{(0)}(w_{-1}^+)$$

$$+ \left[\frac{1}{2\pi i}\left(2\mathcal{E}_4\left(\begin{smallmatrix}-2\\ -1\end{smallmatrix}; \infty\right) - \mathcal{E}_4\left(\begin{smallmatrix}-2\\ 0\end{smallmatrix}; \infty\right) - \mathcal{E}_4\left(\begin{smallmatrix}-2\\ \infty\end{smallmatrix}; \infty\right)\right) + \log \frac{t_3^2}{t_2^2} \right] \otimes (2\pi i \tau) .$$

Recall that

$$\frac{\mathcal{E}_4\left(\begin{smallmatrix}-2\\ c\end{smallmatrix}; \infty\right)}{2\pi i} = \Omega^{(2)}(w_\infty^+ - w_c^+) - \Omega^{(2)}(w_0^+ - w_c^+) - \Omega^{(2)}(w_\infty^+ - w_c^-) + \Omega^{(2)}(w_0^+ - w_c^-) .$$

A simpler function for the sunrise integral:

$$T^{(1)}_{\ominus} = 2\mathcal{E}_4\left(\begin{smallmatrix}0 & -1\\ 0 & -1\end{smallmatrix}; \infty\right) - \mathcal{E}_4\left(\begin{smallmatrix}0 & -1\\ 0 & 0\end{smallmatrix}; \infty\right) - \mathcal{E}_4\left(\begin{smallmatrix}0 & -1\\ 0 & \infty\end{smallmatrix}; \infty\right) - \left(2 \log \frac{t_2}{t_3} + \mathcal{E}_4\left(\begin{smallmatrix}-1\\ -1\end{smallmatrix}; \infty\right)\right) \mathcal{E}_4\left(\begin{smallmatrix}0\\ 0\end{smallmatrix}; \infty\right) ,$$

No dependency on r anymore.

Double periodicity through symbol prime

Denote $\partial_\tau T^{(1)}_{\ominus}$ by $\widetilde{\Omega}$,

$$\mathcal{S}\big(2\pi i T^{(1)}_{\ominus}\big) = \log\frac{t_2^2}{t_1^2}\otimes\Omega^{(0)}(w_0^+) + \log\frac{t_1^2}{t_3^2}\otimes\Omega^{(0)}(w_{-1}^+) + \widetilde{\Omega}\otimes(2\pi i\tau).$$

The application of the symbol prime map gives

$$\mathcal{S}'\Big(\widetilde{\Omega}(w_{-1}^+, w_0^+)\Big) = \Omega^{(0)}(w_0^+)\otimes'\log\frac{t_2^2}{t_1^2} + \Omega^{(0)}(w_{-1}^+)\otimes'\log\frac{t_1^2}{t_3^2}\,,$$

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Consider the shift $w_{-1}^+ \to w_{-1}^+ + \tau$,

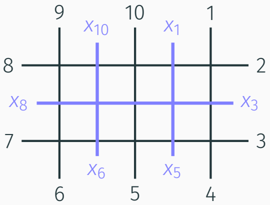
$$\mathcal{S}'\left(\widetilde{\Omega}(w_{-1}^+ + \tau) - \widetilde{\Omega}(w_{-1}^+)\right) = (2\pi i\tau) \otimes' \log\frac{t_1^2}{t_3^2} = \mathcal{S}'\left(-\log\frac{t_1^2}{t_3^2}\right),$$

which cancels the extra term in

$$\log\frac{t_1^2}{t_3^2} \otimes \Omega^{(0)}(w_{-1}^+) \to \log\frac{t_1^2}{t_3^2} \otimes \Omega^{(0)}(w_{-1}^+) + \log\frac{t_1^2}{t_3^2} \otimes 2\pi i\tau$$

Elliptic Double Box

Elliptic double box at 10 point



Kinematics:

$$u_1 = \frac{x_{1,3}^2 x_{5,8}^2}{x_{1,5}^2 x_{3,8}^2}, \quad u_2 = \frac{x_{3,6}^2 x_{8,10}^2}{x_{3,8}^2 x_{6,10}^2}, \quad v_1 = \frac{x_{1,8}^2 x_{5,8}^2}{x_{3,5}^2 x_{3,8}^2}, \quad v_2 = \frac{x_{6,8}^2 x_{3,10}^2}{x_{3,8}^2 x_{6,10}^2},$$

$$u_3 = \frac{x_{1,3}^2 x_{5,10}^2}{x_{1,5}^2 x_{3,10}^2}, \quad u_4 = \frac{x_{1,6}^2 x_{3,5}^2}{x_{1,5}^2 x_{3,6}^2}, \quad u_5 = \frac{x_{1,5}^2 x_{6,10}^2}{x_{1,6}^2 x_{5,10}^2}$$

The planar variable: $x_{i,j}^2 = (p_i + \cdots + p_{j-1})^2$

$$= \int \frac{d^4 x_a d^4 x_b \quad x_{1,5}^2 x_{6,10}^2 x_{3,8}^2}{x_{1,a}^2 x_{3,a}^2 x_{5,a}^2 x_{a,b}^2 x_{6,b}^2 x_{8,b}^2 x_{10,b}^2}$$

- The only contribution to one component of the 2-loop 10-pt N^3MHV amplitude. [Caron-Huot, Larsen]
- The first two-loop Feynman integral which is expected to be some elliptic function in planar $\mathcal{N} = 4$ sYM. [Paulos, Spradlin, Volovich][Nandan, Paulos, Spradlin, Volovich][Caron-Huot, Larsen][Bourjaily, McLeod, Spradlin, von Hippel, Wilhelm][Vergu, Volk]. . .

Elliptic curve in elliptic double box

1. Using the differential equation related to the one-loop hexagon in 6D [Paulos, Spradlin, Volovich][Nandan, Paulos, Spradlin, Volovich]:

$$\partial_{u_5}\Big($$  $$\Big) = \frac{1}{\sqrt{\Delta_6}}\Big($$  $$\Big)$$

   where the (normalized) Gram determinant Δ_6 is a cubic polynomial in u_5.

2. Directly integrating the Feynman parameter rep [Bourjaily, McLeod, Spradlin, von Hippel, Wilhelm]

 $$= \int_0^\infty \frac{d\alpha}{\sqrt{Q(\alpha)}} H_3(\alpha)$$

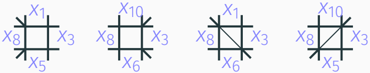
   $Q(\alpha)$ is a quartic polynomial in α.

Remark:  and $H_3(\alpha)$ are polylogarithms of weight 3.

Combining Feynman parameter representation with variable substitutions [Kristensson, Wilhelm, CZ], the double box integral can be evaluated as

$$= \mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & -u_2 & -u_2 \end{smallmatrix}; \infty\right) + \cdots$$

- only consist of $\psi_1(c) = \frac{1}{x-c}$, $\psi_0 = \frac{1}{y}$, $\psi_{-1}(c) = \frac{y_c}{y(x-c)}$, $\psi_{-1}(\infty) = \frac{x}{y}$.
- 26 choices for c, containing 4 square roots from four-mass box sub-diagrams:



In terms of $\widetilde{\Gamma}$ functions (Recall $\int dx/y = \omega_1 \int dw$)

$$= \omega_1 \times \Big(\text{A pure combination of } \widetilde{\Gamma}\text{'s of length 4 and weight 3}\Big).$$

Direct computation gives

$$\mathcal{S}\Big(\frac{2\pi i}{\omega_1}$$  $$\Big) \sim 10^6 \text{ terms, alphabet} \sim 10^3 \text{ letters consisting of } \Omega^{(0,1,2,3)}\text{'s.}$$

Symbol of the elliptic double box

A dramatic simplification happens, 10^6 terms $\to$ 10^4 terms!:

$\mathcal{S}\Big(\frac{2\pi i}{\omega_1}$  $\Big) = \mathcal{S}\Big($  $\Big) \otimes \Omega^{(0)}\Big(w^+_{c_{\text{hex}}} - \frac{w^+_\infty}{2}\Big) + \mathcal{S}(F_-) \otimes \Omega^{(0)}\Big(w^-_\infty - \frac{w^+_\infty}{2}\Big) + \mathcal{S}(F_+) \otimes 2\pi i \tau$

$$+ \Big[\mathcal{S}(F_{d_4}) \otimes \Omega^0\Big(w^+_{d_4} - \frac{w^+_\infty}{2}\Big) + \text{reflections}\Big],$$

where $c_{\text{hex}} = \frac{\langle 9,10,1(7,8)\cap(2,3,5)\rangle}{\langle 1,5,9,10\rangle\langle 2,3,7,8\rangle}$ and $d_4 = z_{1,3,5,8} - 1$.

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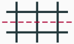
- Manifest the reflection symmetries:  and 

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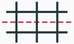
- Manifest the reflection symmetries:  and 
- Manifest the differential equation [Paulos, Spradlin, Volovich][Nandan, Paulos, Spradlin, Volovich]:

$$\omega_1 \partial_{u_5}\Big(w^+_{c_{\text{hex}}}\Big) = \frac{1}{\sqrt{\Delta_6}} \quad \Rightarrow \quad \partial_{u_5}\Big($$  $$\Big) = \frac{1}{\sqrt{\Delta_6}}\Big($$  $$\Big)$$

Symbol alphabet of the elliptic double box

Symbol alphabet for the elliptic double box:

29 DCI rational letters + 24 algebraic letters + $(13+7)$ elliptic letters.

Which are new comparing with the alphabets for the known 2-loop amplitudes [Caron-Huot] [He, Li, CZ]:

- Rational letter: None.
- Algebraic letter:

$$\frac{(z_{1,3,5,8}-z_{1,3,6,8})(\bar{z}_{1,3,5,8}-\bar{z}_{1,3,6,8})}{(\bar{z}_{1,3,5,8}-z_{1,3,6,8})(z_{1,3,5,8}-\bar{z}_{1,3,6,8})}, \qquad \text{and 3 reflection images.}$$

  with

$$z_{i,j,k,l}\bar{z}_{i,j,k,l}=\frac{x_{i,j}^2 x_{k,l}^2}{x_{i,k}^2 x_{j,l}^2}, \quad (1-z_{i,j,k,l})(1-\bar{z}_{i,j,k,l})=\frac{x_{i,l}^2 x_{j,k}^2}{x_{i,k}^2 x_{j,l}^2},$$

- 7 simple elliptic integrals as last entries.
  13 complicated linear independent combinations of $\Omega^{(0,1,2)}$.

Symbol prime and elliptic bootstrap

Now the application of symbol prime gives

$\mathcal{S}\Bigg(\dfrac{2\pi i}{\omega_1}$  $\Bigg) = \sum_{ij} \mathcal{S}(f_i) \otimes \Big(\log a_{ij} \otimes \Omega^{(0)}(w_j) + \widetilde{\Omega}_i \otimes (2\pi i\tau)\Big)$

where

$$f_i \;:\; \text{dilogarithm}\,, \qquad \mathcal{S}'(\widetilde{\Omega}_i) = \sum_j \Omega^{(0)}(w_j) \otimes' \log a_{ij}$$

The same structures as the sunrise integral.

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Rebuilding via Bootstrap [Morales, Spiering, Wilhelm, Yang, CZ (in progress)]

$$\underbrace{\overbrace{(7 \text{ first}) \times (26 \text{ second})}^{\text{First-two entry conditions}} \times (53 \text{ third}) \times (6 \text{ last} \neq 2\pi i \tau)}_{\text{integrability} \to 1}$$

Integrability:

$$d^2 F = 0 \Rightarrow \sum_{I \in (i_1, \ldots, i_p)} c_I\, \phi_{i_1} \otimes \cdots \otimes \phi_{i_{p-1}} \otimes \phi_{i_{p+2}} \otimes \cdots \otimes \phi_{i_n} d\phi_{i_p} \wedge d\phi_{i_{p+1}} = 0$$

The overall constant is fixed by the differential equation $\partial_{u_5}\Big($  $\Big) = \frac{1}{\sqrt{\Delta_6}}\Big($  $\Big)$

Conclusion and Outlook

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Conclusion:

1. Symbology for elliptic polylogarithms:
   - $\Omega^{(0,1)}$: Abel's addition theorem.
   - $\Omega^{(2)}$: Symbol prime.
2. Elliptic bootstrap.

Outlook:

1. Further understanding of the elliptic symbols
   - Understanding the singularity structures of elliptic letters.
   - Simplified symbol → Simplified function?
2. Bootstrapping the 12-point elliptic double box:
   -  (Symbol alphabet from Schubert problem [Morales, Spiering, Wilhelm, Yang, CZ])
3. Compute/Bootstrap general 2-loop amplitudes/Feynman integrals.

Thank You

Backup: Integrability through symbol prime

Suppose the symbol of some eMPL is of the form

$$\mathcal{S}(F) = \sum \mathcal{S}(f_i) \otimes \left(\log a_{ij} \otimes \Omega^{(0)}(w_j) + \widetilde{\Omega}_i \otimes (2\pi i \tau) \right)$$

where f_i are some functions and $\mathcal{S}'(\widetilde{\Omega}_i) = \sum_j \Omega^{(0)}(w_j) \otimes' \log a_{ij}$

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Then, the integrability conditions $[\partial_{w_j}, \partial_\tau]F = 0$ requires

$$\partial_{w_j}\widetilde{\Omega}_i + \partial_\tau \log a_{ij} = 0 \Leftrightarrow \sum_j (\log a_{ij} \otimes \Omega^{(0)}(w_j) + \widetilde{\Omega}_i \otimes (2\pi i \tau)) \text{ is integrable,}$$

which is consequence of

$$\mathcal{S}'(\widetilde{\Omega}_i) = \sum_j \Omega^{(0)}(w_j) \otimes' \log a_{ij}$$

since $\sum_j \Omega^{(0)}(w_j) \otimes \log a_{ij} - \widetilde{\Omega}_i \otimes (2\pi i \tau)$ is integrable by the definition of symbol prime.