Epsilon Factorized Differential Equations for Elliptic Feynman Integrals

Hjalte Frellesvig

Niels Bohr International Academy (NBIA), University of Copenhagen.

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On Epsilon Factorized Differential Equations for Elliptic Feynman Integrals

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Abstract: In this paper we develop and demonstrate a method to obtain epsilon factorized differential equations for elliptic Feynman integrals. This method works by choosing an integral basis with the property that the period matrix obtained by integrating the basis over a complete set of integration cycles is diagonal. This method is a generalization of a similar method known to work for polylogarithmic Feynman integrals. We demonstrate the method explicitly for a number of Feynman integral families with an elliptic highest sector.

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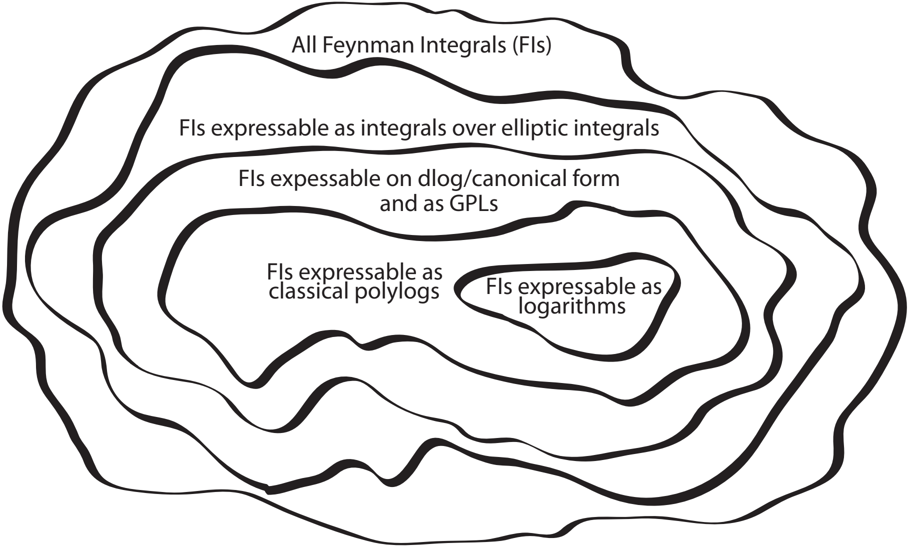
All Feynman Integrals (FIs)
FIs expressable as integrals over elliptic integrals
FIs expessable on dlog/canonical form and as GPLs
FIs expressable as classical polylogs
FIs expressable as logarithms

The method of differential equations is the most fruitful approach to the computation of Feynman integrals

In general the equation system $\partial_s \tilde{J} = \tilde{A}^{(s)} \tilde{J}$ will be hard to solve.

Differential equations in *canonical form* [Henn (2013)]

$$\partial_s \bar{J} = \epsilon A^{(s)} \bar{J} \tag{1}$$

A is free of epsilon dependence, and additionally

$$A^{(s)} = \sum_i B_i \partial_s \log(f_i(s)) \tag{2}$$

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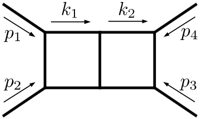
Eq. (2) does not generalize beyond GPLs. But how about eq. (1)?

Let us go through how to obtain the canonical form in a way that generalizes.

Let us start by a non-elliptic example from [Henn (2013)], to motivate our method.

Massless double box:



This integral family has eight master integrals - only two in the highest sector

$$I_{\{a\}} = \int \frac{u\, x_8^{-a_8}\, \mathrm{d}^8 x}{x_1^{a_1} \cdots x_7^{a_7}} \quad \to \quad I_{7\times\mathrm{cut}} = \int_{\mathcal{C}} u_{7\times\mathrm{cut}}\, \hat{\phi}\, \mathrm{d}z \qquad u_{7\times\mathrm{cut}} = s^{d-6} z^{\frac{d}{2}-3} (z+s)^{2-\frac{d}{2}} (z-t)^{d-5}$$

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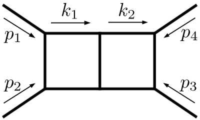

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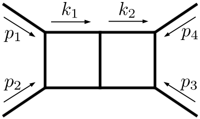

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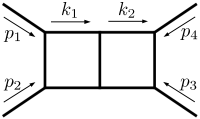

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$a = 0$:	one pole in $z = 0$ of $1/(s^2 t)$	and	one pole in $z = t$ of $-1/(s^2 t)$
$a = -1$:	one pole in $z = t$ of $-1/(s^2)$	and	one pole in $z = \infty$ of $1/(s^2)$

The above prefactors make the integrals pure.

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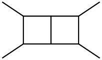
Let us do  again in a different way

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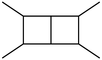
Let us do  again in a different way

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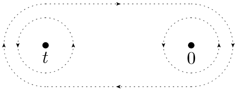

$J_1 = f_1 I_{1111111;0}$, $J_2 = f_2 I_{1111111;-1}$ and also $\gamma_1 = \mathcal{C}_0, \gamma_2 = \mathcal{C}_\infty$

We write down the *period matrix* $P_{ij} = \int_{\gamma_j} \hat{\Phi}_i \mathrm{d}z$

$$\hat{\Phi}_i = \frac{-z^{i-1}}{s^2 z(z-t)} \quad \Rightarrow \quad P = 2\pi i \begin{bmatrix} \frac{f_1}{s^2 t} & 0 \\ 0 & \frac{f_2}{s^2} \end{bmatrix}$$

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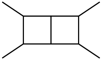
Let us do  again in a different way

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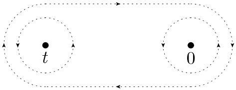

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$$J_i = f_{i1} I_{1111111;0} + f_{i2} I_{1111111;-1} \quad \text{gives} \quad P = 2\pi i \begin{bmatrix} \frac{f_{11}}{s^2 t} & \frac{f_{12}}{s^2} \\ \frac{f_{21}}{s^2 t} & \frac{f_{22}}{s^2} \end{bmatrix}$$

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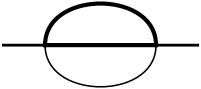
Another non-elliptic example:  around $d = 2$

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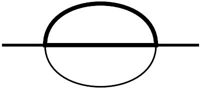
Another non-elliptic example:  around $d = 2$

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2 MIs: "precanonicals" $I_{111;0}$ and $I_{111;\text{-}1}$ correspond to $\hat{\phi}_1 = 1$, $\hat{\phi}_2 = z$

so in $d = 2$ we have the integrand $\hat{\Phi}_i = \dfrac{z^{i-1}}{\sqrt{z(z-4m^2)}(z-s)}$

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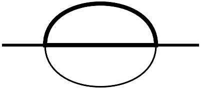
Another non-elliptic example:  around $d=2$

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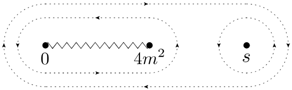

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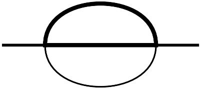
Another non-elliptic example:  around $d=2$

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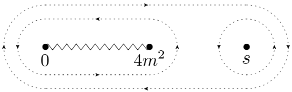

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$$P = 2\pi i I \;\text{ gives }\; J_1 = \sqrt{s(s-4m^2)} I_{111;0}\,,\;\; J_2 = s I_{111;0} - I_{111;\text{-}1}$$

and indeed we get $\partial_s \bar{J} = \epsilon A \bar{J}$

The algorithm

For $J_i = \int_{\mathcal{C}} u\hat{\varphi}_i \mathrm{d}^n x$ write $u\hat{\varphi}_i = \sigma\hat{\Phi}_i$

where σ is *pure* and $\hat{\Phi}$ free of ϵ exponents

(In our previous example

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then I claim: The set of J_i will have epsilon factorized diff-eqs if

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P will be square as the sets of γ and ϕ are dual:

They are bases for (twisted de Rahm) homology and cohomology groups.

This basis choice is a freedom in the algorithm.

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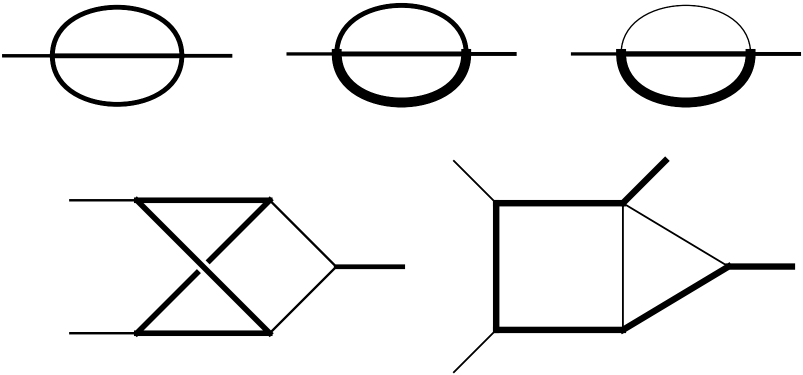
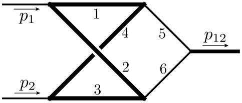
I have done a number of examples:



First elliptic example: 

The non-planar double triangle

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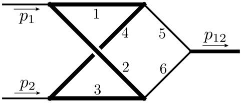
First elliptic example:  The non-planar double triangle

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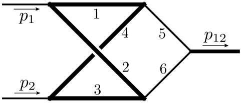
Factorizing out the pure part we get integrals of the form

$$\int_{\mathcal{C}} \frac{\hat{\phi}\mathrm{d}z}{Y} \qquad \text{with} \qquad Y = \sqrt{z(z+s)(z^2+sz-4m^2s)}$$

Examples: npt

First elliptic example: 

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$$\int_{\mathcal{C}} \frac{\hat{\phi}\mathrm{d}z}{Y} \qquad \text{with} \qquad Y = \sqrt{z(z+s)(z^2+sz-4m^2s)}$$

We pick intermediate basis $I_{111111;0}$ and $I_{211111;0}$ corresponding to

$$\hat{\phi}_1 = \frac{1}{s} \qquad \text{and} \qquad \hat{\phi}_2 = \frac{(1+2\epsilon)(z+s)}{s\,(z^2+sz-4m^2s)}$$

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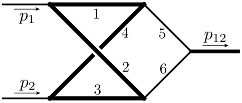
First elliptic example: 

The non-planar double triangle

Again two integrals in the highest, elliptic sector. Around $d = 4$

$$u|_{6\times\mathsf{cut}} = s^{-1+2\epsilon}\big(z(z+s)(z^2+sz-4m^2s)\big)^{-\frac{1}{2}-\epsilon}$$

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$$Y^2 = (z-r_{\mathsf{i}})(z-r_{\mathsf{ii}})(z-r_{\mathsf{iii}})(z-r_{\mathsf{iv}}) \qquad \text{with}$$

$$r_{\mathsf{i}} = -\tfrac{1}{2}\sqrt{s}(\sqrt{s}+\sqrt{16m^2+s})\,, \quad r_{\mathsf{ii}} = -s\,, \quad r_{\mathsf{iii}} = 0\,, \quad r_{\mathsf{iv}} = -\tfrac{1}{2}\sqrt{s}(\sqrt{s}-\sqrt{16m^2+s})$$

Examples: npt

What are the independent contours for

$$\int_{\mathcal{C}} \frac{\hat{\phi}\mathrm{d}z}{Y} \qquad \text{with} \qquad Y = \sqrt{(z-r_{\mathsf{i}})(z-r_{\mathsf{ii}})(z-r_{\mathsf{iii}})(z-r_{\mathsf{iv}})}$$

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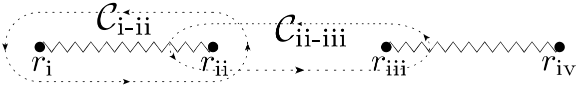
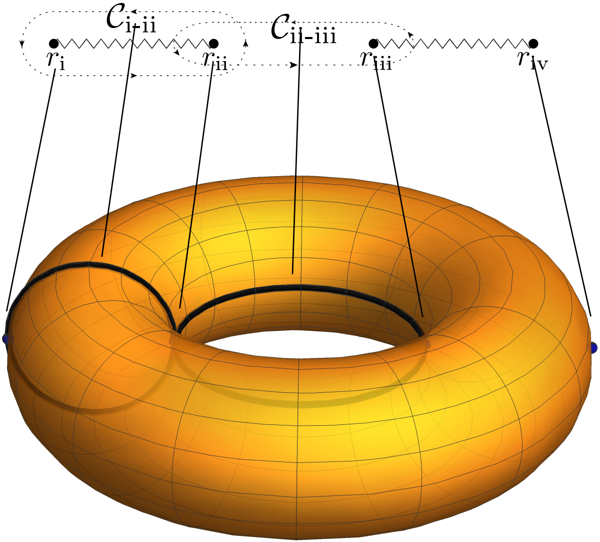
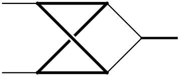



$\mathcal{C}_{\text{i-ii}}$
$\mathcal{C}_{\text{ii-iii}}$
r_{i}
r_{ii}
r_{iii}
r_{iv}

The non-planar double triangle: 

We have

$$\hat{\phi}_1 = \frac{1}{s}\,, \quad \hat{\phi}_2 = \frac{(1{+}2\epsilon)(z{+}s)}{s\,(z^2{+}sz{-}4m^2s)}\,, \quad \gamma_1 = \mathcal{C}_{\text{ii-iii}}\,, \quad \gamma_2 = \mathcal{C}_{\text{i-ii}}\,,$$

and we want

$$P_{ij} = \int_{\gamma_j} \frac{(f_{i1}\hat{\phi}_1 + f_{i2}\hat{\phi}_2)\mathrm{d}z}{Y} = f_{il}\,g_{lj} \qquad \text{with} \qquad Y = \sqrt{z(z{+}s)(z^2{+}sz{-}4m^2s)}$$

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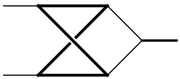
The non-planar double triangle: 

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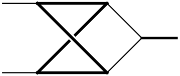

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Complete elliptic integrals of the first, second, and third kind

$$K(k^2) := \int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-k^2x^2)}} \qquad E(k^2) := \int_0^1 \frac{\sqrt{1-k^2x^2}\,\mathrm{d}x}{\sqrt{1-x^2}}$$

$$\Pi(n^2,k^2) := \int_0^1 \frac{\mathrm{d}x}{(1-n^2x^2)\sqrt{(1-x^2)(1-k^2x^2)}}$$

Examples: npt

$$\hat{\phi}_1 = \frac{1}{s}\,,\quad \hat{\phi}_2 = \frac{(1+2\epsilon)(z+s)}{s\,(z^2+sz-4m^2 s)}\,,\quad \gamma_1 = \mathcal{C}_{\text{ii-iii}}\,,\quad \gamma_2 = \mathcal{C}_{\text{i-ii}}$$

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$$g_{11} = \frac{8K(k^2)}{s^{3/2}(\sqrt{16m^2+s}+\sqrt{s})}$$

Examples: npt

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$$g_{21} = \frac{-8(1+2\epsilon)}{s^{3/2}(\sqrt{16m^2+s}+\sqrt{s})} \left(K(k^2) + \frac{\sqrt{16m^2+s}+\sqrt{s}}{\sqrt{16m^2+s}-\sqrt{s}} E(k^2) \right)$$

$$g_{22} = \frac{i(1+2\epsilon)}{s^{3/2}m^2} \left(\frac{K(1-k^2)}{\sqrt{16m^2+s}} - \frac{\sqrt{16m^2+s}+\sqrt{s}}{2(16m^2+s)} E(1-k^2) \right)$$

Examples: npt

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Imposing $P = 2\pi i I$ fixes the f_{il} uniquely, for instance

$$f_{11} = \tfrac{1}{2} i s^{3/2} (\sqrt{16m^2+s} + \sqrt{s}) E(1-k^2) - i s^{3/2} \sqrt{16m^2+s} K(1-k^2)$$

$\dots$

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Examples: npt

So now we have $J_i = f_{i1} I_{111111;0} + f_{i2} I_{211111;0}$ with f_{il} fixed.

Examples: npt

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So now we have $J_i = f_{i1} I_{111111;0} + f_{i2} I_{211111;0}$ with f_{il} fixed.

We get $\partial_s \bar{J} = \epsilon A \bar{J}$ with

$$A_{11} = \frac{8(12m^2+s)K(k^2)K(1-k^2)}{\pi\sqrt{s}(16m^2+s)(\sqrt{16m^2+s}+\sqrt{s})} + \frac{2}{\pi s}\left(1-\frac{8m^2}{16m^2+s}+\frac{\sqrt{s}}{\sqrt{16m^2+s}}\right)E(k^2)E(1-k^2)$$
$$+ \frac{-4(12m^2+s)K(k^2)E(1-k^2)}{\pi s(16m^2+s)} + \frac{-2(\sqrt{16m^2+s}+\sqrt{s})E(k^2)K(1-k^2)}{\pi\sqrt{s}(16m^2+s)}$$

$$A_{12} = \frac{-64im^2K(1-k^2)^2}{\pi\sqrt{s}\sqrt{16m^2+s}(\sqrt{16m^2+s}+\sqrt{s})} + \frac{i(\sqrt{16m^2+s}+\sqrt{s})^2E(1-k^2)^2}{\pi s(16m^2+s)}$$
$$+ \frac{4i}{\pi s}\left(\frac{\sqrt{s}}{\sqrt{16m^2+s}} - \frac{8m^2}{16m^2+s}\right)K(1-k^2)E(1-k^2)$$

$$A_{21} = \frac{i(12m^2+s)(\sqrt{16m^2+s}-\sqrt{s})^2K(k^2)^2}{4m^2\pi s(16m^2+s)} + \frac{i(\sqrt{16m^2+s}-\sqrt{s})^2E(k^2)^2}{\pi s(16m^2+s)}$$
$$+ \frac{-4iK(k^2)E(k^2)}{\pi s}$$

$$A_{22} = \frac{2(\sqrt{16m^2+s}-\sqrt{s})^2K(k^2)K(1-k^2)}{\pi s\sqrt{16m^2+s}(\sqrt{16m^2+s}+\sqrt{s})} + \frac{-2}{s\pi}\left(1-\frac{8m^2}{16m^2+s}-\frac{\sqrt{s}}{\sqrt{16m^2+s}}\right)E(k^2)E(1-k^2)$$
$$+ \frac{16m^2K(k^2)E(1-k^2)}{\pi s(16m^2+s)} + \frac{-32m^2E(k^2)K(1-k^2)}{\pi s\sqrt{16m^2+s}(\sqrt{16m^2+s}+\sqrt{s})}$$

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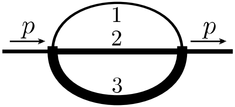
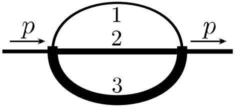
Examples: s3m

Next example: 

The three mass elliptic sunrise

Examples: s3m

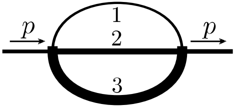
Next example: 

The three mass elliptic sunrise

$$u|_{3\times\text{cut}} = z^{\epsilon}\left(z^2-2(m_1^2+m_2^2)z+(m_1^2-m_2^2)^2\right)^{-\frac{1}{2}-\epsilon}\left(z^2-2(m_3^2+s)z+(m_3^2-s)^2\right)^{-\frac{1}{2}-\epsilon}$$

There are four MIs. We pick intermediate basis $I_{111;00}$, $I_{211;00}$, $I_{111;-10}$, $I_{111;0-1}$

$$\hat{\phi}_1 = 1\,, \quad \hat{\phi}_2 = \frac{(1+2\epsilon)(z+m_2^2+m_3^2)}{z^2-2(m_1^2+m_2^2)z+(m_1^2-m_2^2)^2}\,, \quad \hat{\phi}_3 = z\,, \quad \hat{\phi}_4 \sim \frac{1}{z}$$

Next example: 

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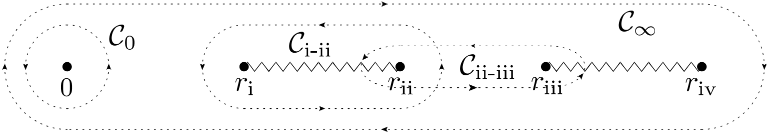
$$\int_{\gamma_j}\frac{\hat{\phi}_i \mathrm{d}z}{Y} \quad \text{with} \quad Y = \sqrt{\left(z^2-2(m_1^2+m_2^2)z+(m_1^2-m_2^2)^2\right)\left(z^2-2(m_3^2+s)z+(m_3^2-s)^2\right)}$$



$$\gamma_1 = \mathcal{C}_{\text{ii-iii}}\,, \quad \gamma_2 = \mathcal{C}_{\text{i-ii}}\,, \quad \gamma_3 = \mathcal{C}_\infty\,, \quad \gamma_4 = \mathcal{C}_0$$

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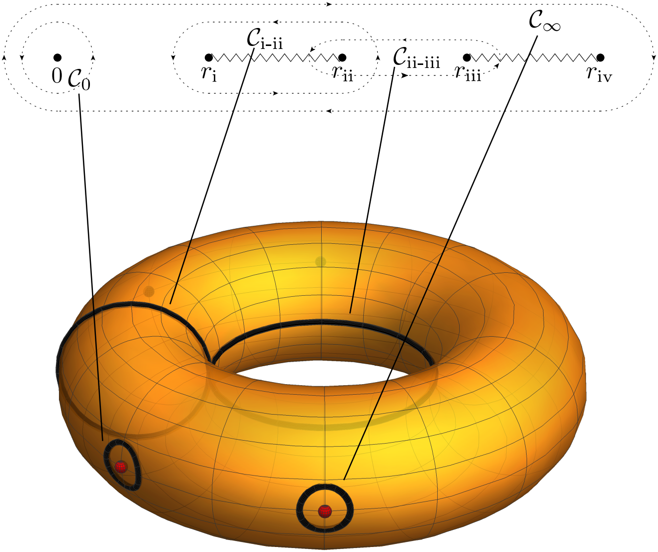

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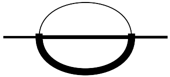
Examples: s3m

The three mass sunrise: 

Integrands $\hat{\phi}_1 = 1, \;\; \hat{\phi}_2 = \dfrac{(1{+}2\epsilon)(z{+}m_2^2{+}m_3^2)}{z^2{-}2(m_1^2{+}m_2^2)z{+}(m_1^2{-}m_2^2)^2}, \;\; \hat{\phi}_3 = z, \;\; \hat{\phi}_4 \sim \dfrac{1}{z}$

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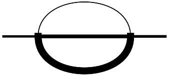
The three mass sunrise: 

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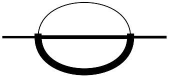
The three mass sunrise: 

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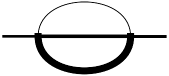
The three mass sunrise: 

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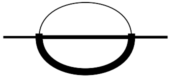
$g_{13} = 0, \quad g_{33} = -2\pi i, \quad \cdots$ We also see $\Pi(\tilde{n}^2, k^2)$

Examples: s3m

The three mass sunrise: 

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$g_{13} = 0, \quad g_{33} = -2\pi i, \quad \cdots \qquad$ We also see $\Pi(\tilde{n}^2, k^2)$

We may then impose $P = 2\pi i I$. 16 constraints fix the f_{il} uniquely.

$\partial_s \bar{J} = \epsilon A^{(s)} \bar{J}$. The expressions are too big to be written here ...

Discussion

The elephant in the room: How do we integrate $\partial_x \bar{J} = \epsilon A \bar{J}$?

We need elliptic generalization of GPLs.

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$$E_4\left({n_1, \ldots, n_k \atop c_1, \ldots, c_k}\right) = \int_0^x \mathrm{d}t\, \psi_{n_1}(c_1, t) E_4\left({n_2, \ldots, n_k \atop c_2, \ldots, c_k}\right)$$

$$\psi_1(c,x) = \frac{1}{x-c}\,, \quad \psi_0(0,x) = \frac{c_4}{y}\,, \quad \psi_{-1}(c,x) = \frac{y_c}{y(x-c)}\,, \quad \psi_{-1}(\infty,x) = \frac{x}{y}\,, \quad \ldots$$

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$$\mathsf{ELi}_{n,m}(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{x^j}{j^n}\frac{y^k}{k^m} q^{jk} \qquad \text{or} \qquad I(f_1,\dots,f_n;\tau) = \int_{\tau_0}^{\tau} \mathrm{d}\tau_1 f_1(\tau_1) I(f_2,\dots,f_n;\tau_1)$$

where f_n are *modular forms* on the lattice defined by the elliptic curve.

Also Γ, $\tilde{\Gamma}$, and many other approaches. It is a booming field.

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Yet none of this is directly suitable.

Numerical integration of the dif-eq will definitely work.

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$P = (2\pi i)^n I$ is sufficient, not necessary.

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$$\tilde{J}_1^{\text{npt}} = f_{11} I^{\text{npt}}_{111111;0}\,, \qquad \tilde{J}_2^{\text{npt}} = f_{21} I^{\text{npt}}_{111111;0} + f_{22} I^{\text{npt}}_{211111;0} \quad \text{with}$$

$$f_{11} = \frac{\epsilon\, s^{3/2}(\sqrt{16m^2{+}s}{+}\sqrt{s})}{4K(k^2)} \qquad\qquad f_{12} = 0$$

$$f_{21} = -s^{3/2}\left((\sqrt{16m^2{+}s}{+}\sqrt{s})E(k^2) + (\sqrt{16m^2{+}s}{-}\sqrt{s})\left(1 + \epsilon\frac{24m^2{+}s}{m^2}\right) K(k^2) \right)$$

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This system gives $\mathrm{d}\tilde{J}_i^{\text{npt}}/\mathrm{d}s = \epsilon \tilde{A}_{ij}\tilde{J}_j^{\text{npt}}$ with

$$\tilde{A}_{11} = \frac{-(8m^2+s)}{s(16m^2+s)} \qquad\qquad \tilde{A}_{12} = \frac{(\sqrt{16m^2+s}+\sqrt{s})^2}{8\sqrt{2}s(16m^2+s)K(k^2)^2}$$

$$\tilde{A}_{21} = \frac{8\sqrt{2}(8m^2+s)^2K(k^2)^2}{s(16m^2+s)(\sqrt{16m^2+s}+\sqrt{s})^2} \qquad\qquad \tilde{A}_{22} = \frac{-(8m^2+s)}{s(16m^2+s)}$$

A variable change to the *period ratio* $\tau = K(1-k^2)/K(k^2)$

might help integrating the system following [Adams, Weinzierl (2018)]

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$$\tilde{P}_{22} = 4\pi i + 2i\epsilon K(k^2) \left(8E(1{-}k^2) + \frac{(8m^2{+}s)\sqrt{16m^2{+}s} - (24m^2{+}s)\sqrt{s}}{m^2(\sqrt{s}{+}\sqrt{16m^2{+}s})} K(1{-}k^2) \right)$$

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See upcoming work by Stefan Weinzierl and I

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Discussion

Number of master integrals can be counted with the Lee-Pomeransky criterion

$$\nu = \text{number of solutions to } \text{``}\omega = 0\text{''} \qquad \text{where} \qquad \omega := \mathrm{dlog}(u)$$

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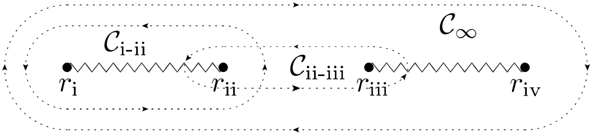
This criterion miscounts for the non-planar double-triangle (it counts three, there are two)

We had intermediate basis $I^{\text{npt}}_{111111;0}$ and $I^{\text{npt}}_{211111;0}$ corresponding to

$$\hat{\phi}_1 = \frac{1}{s}, \quad \hat{\phi}_2 = \frac{(1+2\epsilon)(z+s)}{s\,(z^2+sz-4m^2s)} \quad \text{and} \quad \gamma_1 = \mathcal{C}_{\text{ii-iii}}, \quad \gamma_2 = \mathcal{C}_{\text{i-ii}}$$

Why not a third $I^{\text{npt}}_{111111;\text{-}1}$ corresponding to $\hat{\phi}_3 = z/s$ and $\gamma_3 = \mathcal{C}_\infty$?



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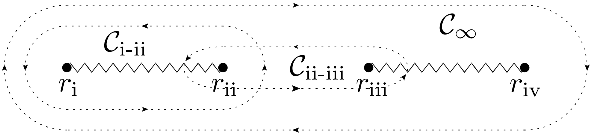

From IBPs we know $I^{\text{npt}}_{111111;\text{-}1} = \frac{-2}{s} I^{\text{npt}}_{111111;0} + \text{lower} \quad \Rightarrow \quad g_{3i} = \frac{-2}{s} g_{1i}$

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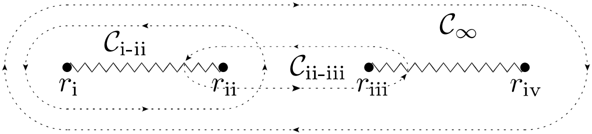

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$g_{33} = 0, \quad g_{13} = -2\pi i, \quad g_{31}$ depends on $\Pi(n^2, k^2)$ but it works on γ_1

$$\Pi\left(\tfrac{4}{(1+x)^2}, \tfrac{-16x}{(x-3)(1+x)^3}\right) = \tfrac{2x}{3(x-1)} K\left(\tfrac{-16x}{(x-3)(1+x)^3}\right)$$

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An elliptic generalization of canonical forms?

Canonical forms are defined by

$$\partial_s \bar{J} = \epsilon A^{(s)} \bar{J} \quad (1) \qquad \text{and} \qquad A^{(s)} = \sum_i B_i \, \partial_s \log(f_i(s)) \quad (2)$$

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While my expressions

- A Reduce to canonical forms in the polylogarithmic case
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I hope my algorithm and expressions can be a step in the generalization of canonical forms to the elliptic case and beyond.

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For more information:

Diff-eqs for Feynman Integrals:

[Kotikov (1991)], [Gehrmann and Remiddi (2000)], [Henn (2013)]

"Canonicalization":

[Henn (2013,15)], [Lee (2015)], [Wasser (2016)], [Gituliar and Magerya (2017)],
[Argeri, Di Vita, Mastrolia, Mirabella, Schlenk, Schubert, Tancredi (2014)],
[Henn, Mistlberger, Smirnov, Wasser (2020)], [Chen, Jiang, Xu, Yang (2021)] ...

Baikov Parametrization:

[Baikov (1997)], [Frellesvig, Papadopoulos (2017)]

Cuts and Integral Relations:

[Bosma, Sogaard, Zhang (2017)], [Primo and Tancredi ($2\times$2017)]

Pure Functions and Prescriptive Unitarity:

[Arkani-Hamed, Bourjaily, Cachazo, Trnka (2012)], [Bourjaily, Herrmann, Trnka (2017)],
[Bourjaily, Kalyanapuram, Langer, Patatoukos (2021)]

Vector Space Structure:

[Lee and Pomeransky (2013)], [Mastrolia and Mizera (2019)],
[Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, Mattiazzi, Mizera ($2\times$2019,2021)],
[Chestnov, Frellesvig, Gasparotto, Mandal, Mastrolia (2022)], Seva's talk on Tuesday

Elliptic Feynman Integrals and Elliptic Polylogs:

[Laporta and Remiddi (2005)], [Brown and Levin (2011)], [Bloch and Vanhove (2015)]
[Remiddi, Tancredi (2016,2017)], [Broedel, Duhr, Dulat, Tancredi (2017,$3\times$18,19)]
[Adams, Bogner, Ekta, Weinzierl (2015,16,17,$2\times$18,21)] ...

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Thank you for inviting me
and thank you for listening!

Hjalte Frellesvig

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