Elliptic polylogarithms and iterated integrals in QCD

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Elliptics 2022, Mainz,
14 September 2022

Part I

Introduction: QCD corrections

precision at colliders

- improved experimental precision at colliders

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- *master equation* for hadron colliders:

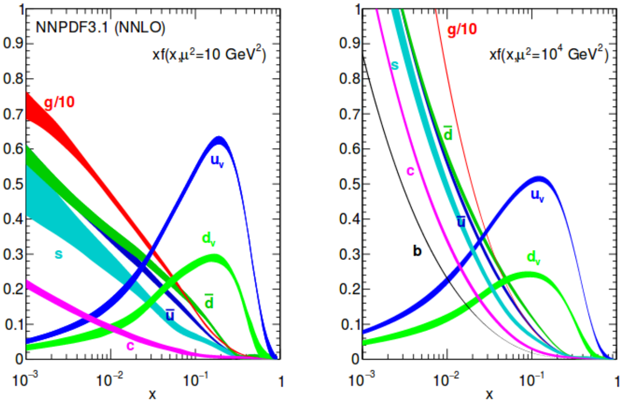
$$\boxed{d\sigma = \sum_{i,j} \int dx_1 dx_2\, f_{i/p}(x_1)\, f_{j/p}(x_2)\, d\hat{\sigma}_{ij}(x_1 P_1, x_2 P_2)} \tag{1}$$

  - $f_{i/p}$: parton distribution function (PDF)
  - $d\hat{\sigma}_{ij}$: partonic cross-section

parton distribution function

$$d\sigma = \sum_{i,j} \int dx_1 dx_2 \boxed{f_{i/p}(x_1)\, f_{j/p}(x_2)}\, d\hat{\sigma}_{ij}(x_1 P_1, x_2 P_2) \tag{2}$$

- parton distribution function contain information about the content of the proton



[NNPDF collaboration]

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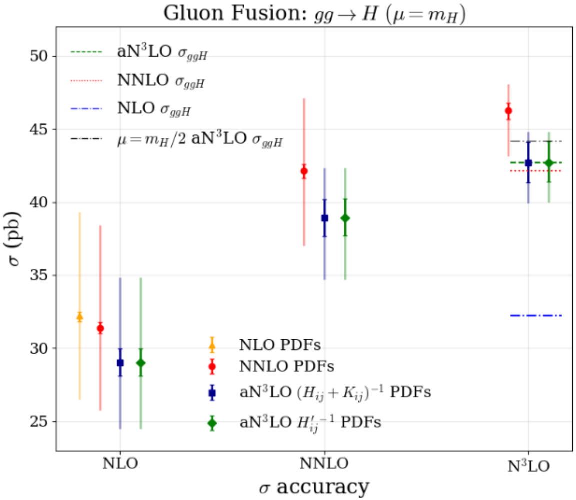

[McGowan, Cridge, Harland-Lang, Thorne, 2207.04739]

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  $\rightarrow$ sole PDF effect (NNLO/aN^{3}LO) for N^{3}LO Higgs production is about $\sim$ **7.3%**

  $\rightarrow$ relevant for precision studies

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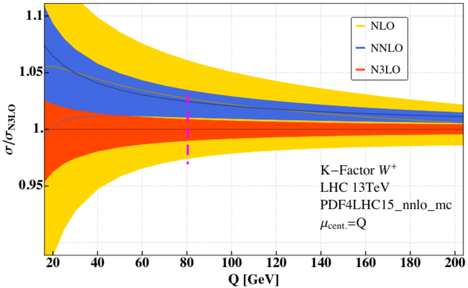

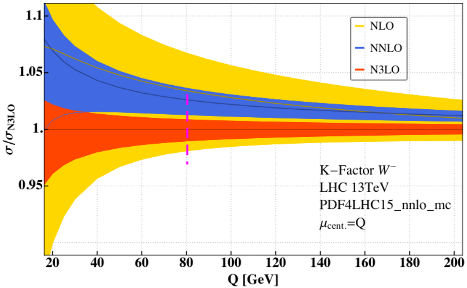

Drell-Yan N^{3}LO $\sim 1\%$ correction

[Duhr, Dulat, Mistlberger; JHEP 11 (2020) 143]

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$\rightarrow$ non-perturbative corrections may become relevant!

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- calculation of higher-order corrections require knowledge of special functions *(see Lorenzo's talk)*

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  [Brown, Levin, Broedel, Duhr, Dulat, Tancredi, Weinzierl, ...]
- in contrast to MPL functions, elliptic integrals still remain to be fully explored ...
  (see other talks here at workshop)

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  - how can the elliptic expression be evaluated numerically in an efficient way?

elliptic integrals and QCD

- collection of processes or sub-processes that involve elliptic functions:
  - three-loop contributions to the ρ-parameter
    [Abreu, Becchetti, Duhr, Marzucca, 1912.02747]
  - leading-colour contribution to Hgg form-factor at NNLO
    [Prausa, Usovitsch, 2008.11641]
  - leading-colour contribution to $t\bar{t}$ production at two loops
    [Badger, Chaubey, Hartanto, Marzucca, 2102.13450]
  - full-colour contribution to pseudo-scalar quarkonium form-factors at two loops
    [Abreu, Becchetti, Duhr, Ozcelik, 2206.03848, 22XX.XXXXX]
  - linear power corrections to e^+e^- shape variables in the three-jet region
    [Caola, Ferrario Ravasio, Limatola, Melnikov, Nason, Ozcelik, 2204.02247]

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Part II

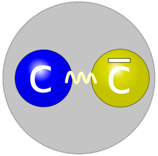
perturbative QCD:

two-loop master integrals and form-factors for pseudo-scalar quarkonia

[Abreu, Becchetti, Duhr, Ozcelik, 2206.03848, 22XX.XXXXX]

Introduction: What is a Quarkonium?

- similar to positronium bound state $\mathrm{e}^{+}\mathrm{e}^{-}$ in QED
- bound state of heavy quark and its anti-quark in QCD, e.g. Charmonium (charm quark) and Bottomonium (bottom quark)



[Figure from Wikipedia 'Quarkonium']

- Toponium ($t\bar{t}$) bound state: high mass of top quark $\rightarrow$ decays via weak interaction before formation of bound state
- for light quarks: mixing between (u,d,s) quarks due to low mass difference $\rightarrow$ π-meson, the ρ-meson and the η-meson

Motivation: Why study Quarkonia?

- charmonium production allows us to probe QCD at its interplay between the perturbative and non-perturbative regimes
- deeper understanding of confinement (production mechanism)
- access to spin/momentum distribution of gluons in protons
  $\rightarrow$ use quarkonia to constrain the gluon PDFs in
  the proton
- it is interesting to assess the convergence of perturbative expansion in α_s where $\alpha_s(m_c) \sim 0.34$ and $\alpha_s(m_b) \sim 0.22$

the η_c - a good gluon probe

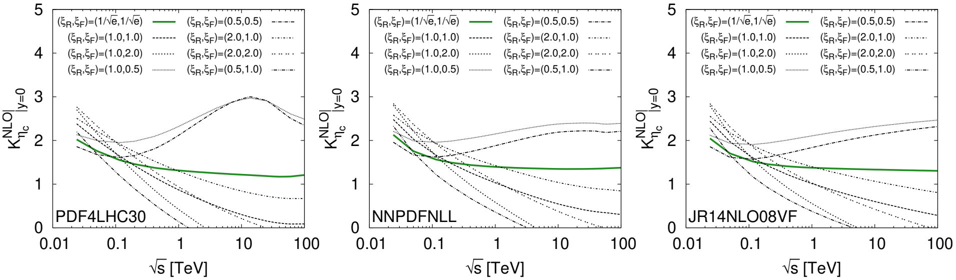
- η_c is a gluon probe at low scales at $M_{\eta_c} = 3$ GeV
- is a pseudo-scalar particle and simplest of all quarkonia as far as computation of hadro-production
- η_c cross section computation known
  - at NLO since 1992 in collinear factorisation

    [J. Kühn, E. Mirkes, Phys.Lett. B296 (1992) 425-429]
  - at LO since 2012 and at NLO since 2013 in TMD factorisation

    [D. Boer, C. Pisano, Phys.Rev. D86 (2012) 094007]

    [J.P. Ma, J.X. Wang, S. Zhao, Phys.Rev. D88 (2013) no.1, 014027]

scale variations and negative cross-sections



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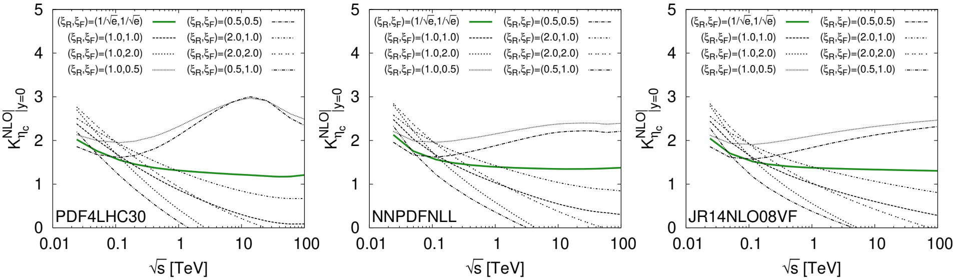

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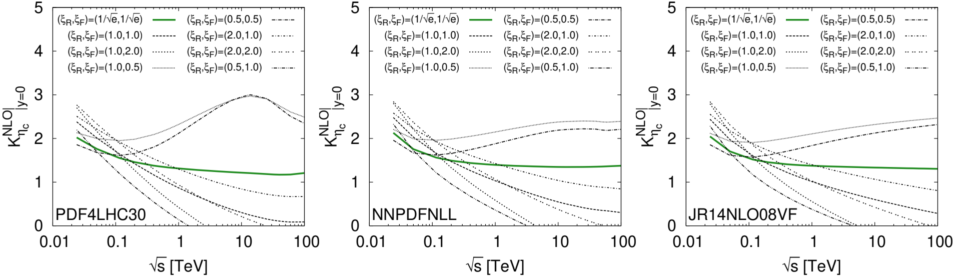

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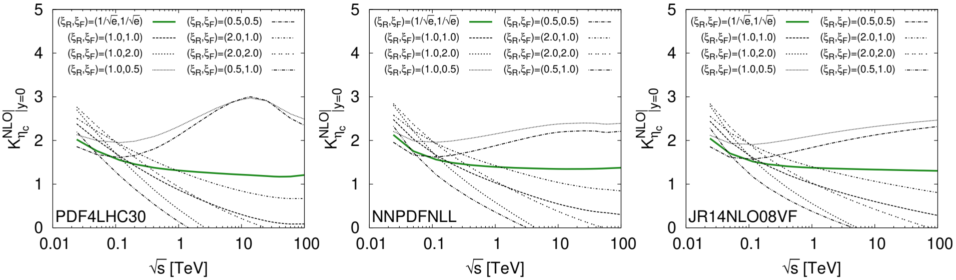

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  - resolved with new scale prescription for μ_F (green curve)

[J.-P. Lansberg, Melih A. Ozcelik, Eur.Phys.J.C 81 (2021) 6, 497 (arXiv:2012.00702)]

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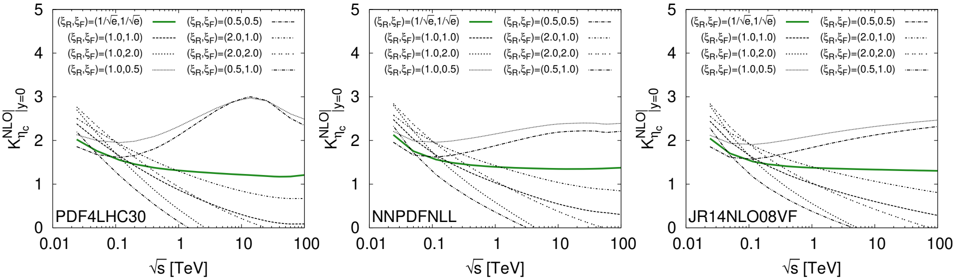

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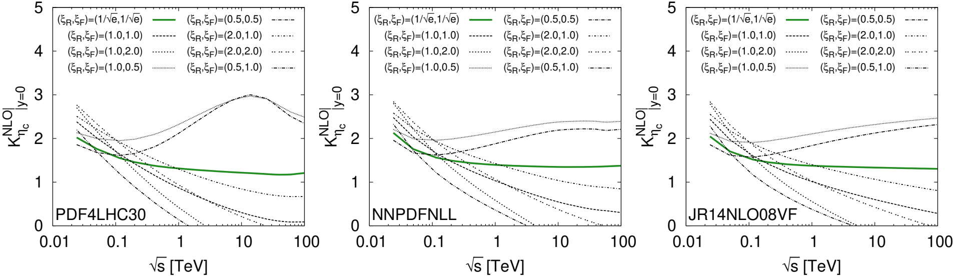

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- for general scale reduction need NNLO calculation $\rightarrow$ need two-loop form-factors

Form-factors

- compute two-loop form-factors analytically in different channels that contribute at NNLO accuracy
  - $\gamma\gamma \leftrightarrow \eta_Q \left({}^1S_0^{[1]}\right) \rightarrow$ exclusive/inclusive decay
  - $gg \leftrightarrow \eta_Q \left({}^1S_0^{[1]}\right) \rightarrow$ **hadro-production** and hadronic decay width
  - $\gamma g \leftrightarrow {}^1S_0^{[8]} \rightarrow$ colour-octet contribution
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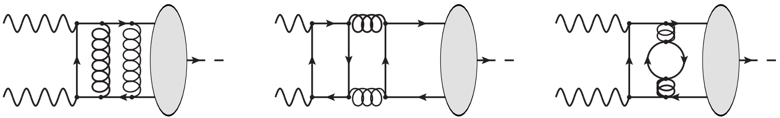
- form-factors applicable to both production and decay

- in the past form-factors have been computed only in numerical form
  - $\eta_Q \rightarrow \gamma\gamma$ **[A. Czarnecki, K. Melnikov, Phys.Lett.B 519 (2001) 212-218] [F. Feng, Y. Jia, W.-L. Sang, Phys.Rev.Lett. 115 (2015) 22, 222001]**
  - para-Positronium $\rightarrow \gamma\gamma$ **[A. Czarnecki, K. Melnikov, A. Yelkhovsky, Phys.Rev.A 61 (2000) 052502]**

Amplitude generation & partial fraction



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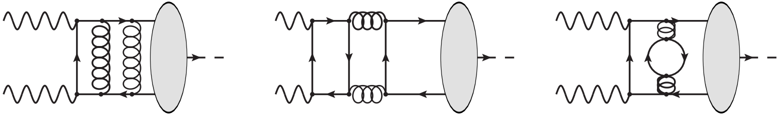
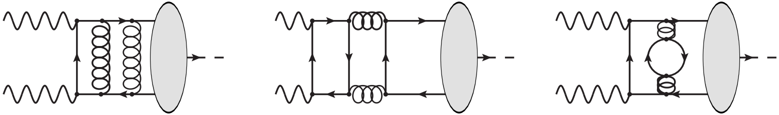


$$\gamma(k_1) + \gamma(k_2) \to Q(p_1)\overline{Q}(p_2) \tag{4}$$

- $p^2 = m_Q^2$ for final-state heavy quarks with $\boxed{p = p_1 = p_2}$
- $k_1^2 = k_2^2 = 0$ for initial-state photons
- threshold kinematics with $\boxed{\hat{s} = M_{\mathcal{Q}}^2 = 4m_Q^2}$ where $M_{\mathcal{Q}} = 2m_Q$

Amplitude generation & partial fraction



$$\gamma(k_1) + \gamma(k_2) \to Q(p_1)\overline{Q}(p_2) \tag{4}$$

- $p^2 = m_Q^2$ for final-state heavy quarks with $\boxed{p = p_1 = p_2}$
- $k_1^2 = k_2^2 = 0$ for initial-state photons
- threshold kinematics with $\boxed{\hat{s} = M_{\mathcal{Q}}^2 = 4m_Q^2}$ where $M_{\mathcal{Q}} = 2m_Q$

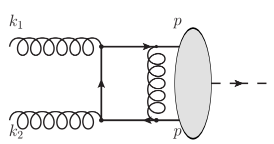
- generate Feynman diagram with `FeynArts` ($\sim$ 450 diagrams for $gg \leftrightarrow \eta_Q$ case)

Amplitude generation & partial fraction

The fact that the two heavy-quark momenta are equal allows us to simplify some integrals beforehand via the procedure of partial fractioning

Example

Feynman diagram:



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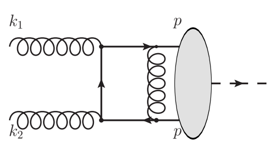
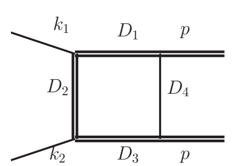


$$I_{\text{Coul.}} = \int d^D q_1 \frac{1}{D_1 D_2 D_3 D_4} = \tag{5}$$



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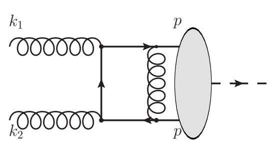

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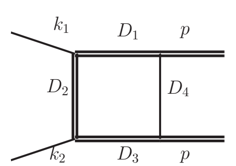

Denominators are linearly dependent: $D_4 = \frac{1}{2}(D_1 + D_3)$

Amplitude generation & partial fraction

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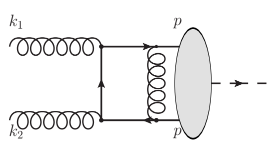
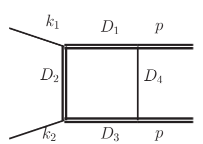
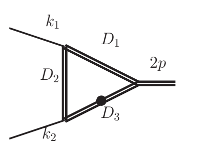


$$I_{\text{Coul.}} = \int d^D q_1 \frac{1}{D_1 D_2 D_3 D_4} = \int d^D q_1 \frac{2}{D_1 D_2 D_3^2} - \int d^D q_1 \frac{1}{D_2 D_3^2 D_4} \quad (6)$$



$= 2$



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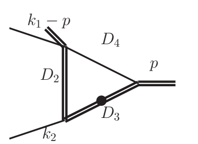

(7)

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  [F. Feng, Comput.Phys.Commun. 183 (2012) 2158-2164]
- perform tensor integral decomposition in new basis
- reduce integrals to master integrals via IBP with `FIRE`

  [A.V. Smirnov, Comput.Phys.Commun. 189 (2015) 182-191]

Amplitude

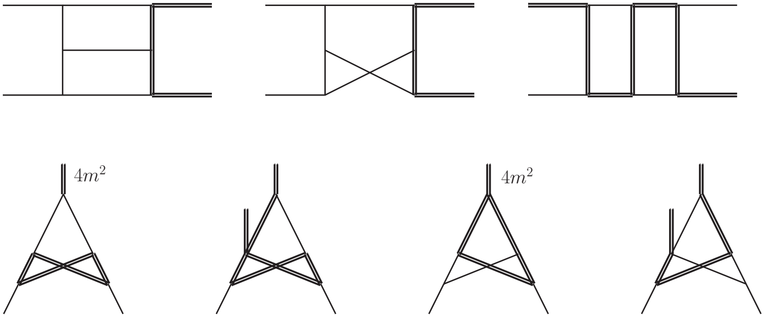
- two-loop Amplitude $\mathcal{A}^{(2)}$:

$$\mathcal{A}^{(2)} = \mathcal{A}^{(0)} \sum_{i=1}^{n_{\text{master}}} c_i(\epsilon)\text{MI}[i] \tag{8}$$

  - tree-level Amplitude $\mathcal{A}^{(0)}$
  - coefficient c_i contains information on:
    - rational factor depending on dimensional regulator ϵ
    - colour factor $(C_A,\ C_F,\ T_F)$
    - number of massive (n_h) and massless (n_l) closed fermion loops (vacuum & light-by-light)
  - need to compute master integrals $\text{MI}[i]$

Topologies and master integrals

Some examples of topologies:



Topologies and master integrals

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  - **Partial Fraction Relations**
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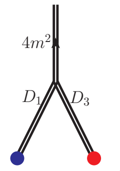
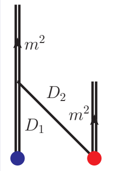
Partial Fraction Identities

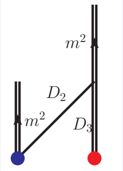
Identity



$= \frac{1}{2}$



$+ \frac{1}{2}$



(9)

relation at *integrand* level:

$$\frac{1}{\underbrace{\left[(q+p)^2-m^2\right]}_{D_1}\underbrace{\left[(q-p)^2-m^2\right]}_{D_3}} = \frac{1}{2}\frac{1}{\underbrace{\left[(q+p)^2-m^2\right]}_{D_1}\underbrace{q^2}_{D_2}} + \frac{1}{2}\frac{1}{\underbrace{q^2}_{D_2}\underbrace{\left[(q-p)^2-m^2\right]}_{D_3}}$$

Partial Fraction Identities

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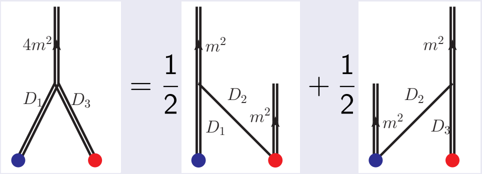
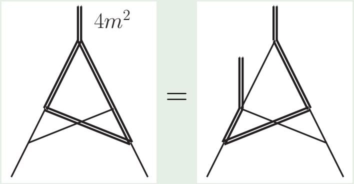


(9)

Example



(10)

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$$m_{45} = \frac{2\,(3d-11)\,m^2}{(d-3)\,(3d-10)}m_{53} - \frac{8m^4}{(d-3)\,(3d-10)}m_{54} + \frac{(d-2)^2}{4\,(d-3)\,(3d-10)\,m^4}m_{76}$$

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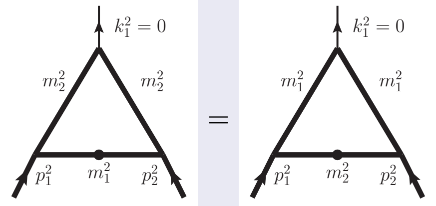
- question for future: can one systematically incorporate partial fraction relations into IBP reduction system (useful for phase-space integrations)?

Triangle Relations

Identity



(11)

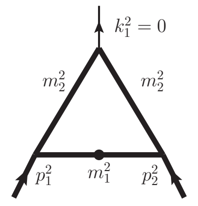
relation at *integral* level:

$$\int d^d q \frac{1}{\left[q^2 - m_1^2\right]^2 \left[(q+p_1)^2 - m_2^2\right] \left[(q-p_2)^2 - m_2^2\right]} = \int d^d q \, (m_1 \leftrightarrow m_2)$$

no constraint for p_1 and p_2 (can involve loop momenta), only constraint is that $k_1^2 = 0$

Triangle Relations

Identity



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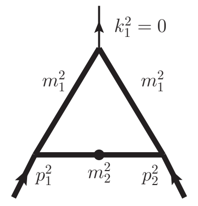
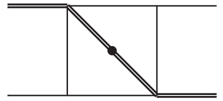


(11)

Example



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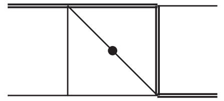
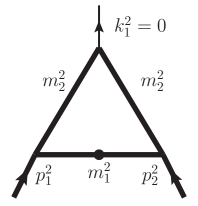


(12)

Triangle Relations

Identity



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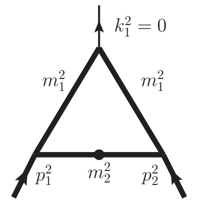

(11)

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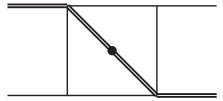
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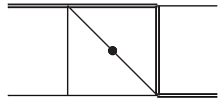
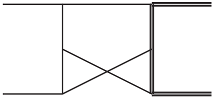

(12)

questions for future: can we systematically incorporate these relations into IBP? And are there more of these relations (box, pentagon integrals)?

Topologies and master integrals

- Appearance of 76 master integrals in total
- some are known in general kinematics (see later) but not usable at special kinematics
- Master integrals are seemingly independent, however we find some interesting equivalence relations beyond IBP
  - Partial Fraction Relations
  - Triangle Relations
- Analytical results for most of the integrals in these topologies are not available in the literature

Example: Topology 2

- some topologies would occur also for open $t\bar{t}$-production
- 

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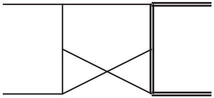
- 

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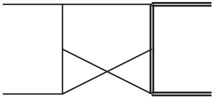
- 

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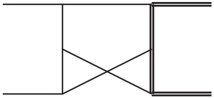
- 
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$\rightarrow$ computed nearly entire topology family via direct integration at threshold

Special functions

- *Multiple Polylogarithms* - points on the *Riemann sphere*
- *elliptic Multiple Polylogarithms* - points on the *torus*
- *iterated integrals of modular forms* - *rational* points on the *torus*

Multiple Polylogarithms (MPLs)

Multiple Polylogarithms (MPLs) [Goncharov,Remiddi,Vermaseren]

$$G(a_1, \ldots, a_n; z) = \int_0^z dt \frac{1}{t - a_1} G(a_2, \ldots, a_n; t) \tag{13}$$

$$G(0; t) = \log t \tag{14}$$

- weight of function corresponds to number of indices $w = n$
- m-loop amplitude usually exhibits functions up to weight of $w = 2m \rightarrow$ will be useful as cross-check of amplitude
- numerical evaluation can be achieved with `GiNaC`-interface

[Vollinga, Weinzierl]

elliptic Multiple Polylogarithms (eMPLs)

elliptic Multiple Polylogarithms (eMPLs) [Brown,Levin;Broedel,Duhr,Dulat,Tancredi;Weinzierl...]

$$E_4\left(\begin{smallmatrix} n_1 \dots n_m \\ c_1 \dots c_m \end{smallmatrix}; x, \vec{q} \right) = \int_0^x dt\, \psi_{n_1}(c_1, t, \vec{q})\, E_4\left(\begin{smallmatrix} n_2 \dots n_m \\ c_2 \dots c_m \end{smallmatrix}; t, \vec{q} \right) \quad (15)$$

$$E_4\left(\begin{smallmatrix} \vec{1} \\ \vec{c} \end{smallmatrix}; x, \vec{q} \right) = G(\vec{c}; x) \quad (16)$$

- $\vec{q}$ are the roots of the elliptic curve defined by

$$y^2 = (t - q_1)(t - q_2)(t - q_3)(t - q_4) \quad (17)$$

- $\psi_{n_1}(c_1, t, \vec{q})$ are the elliptic kernels
  - e.g. $\psi_0(0, t, \vec{q}) = \frac{c_4}{y}$ where $c_4 = \frac{1}{2}\sqrt{(q_1 - q_3)(q_2 - q_4)}$
  - e.g. $\psi_1(c, t, \vec{q_r}) = \frac{1}{t-c}$

- define weight as $w = \sum_i^m |n_i|$ and length as $l = m$

elliptic Multiple Polylogarithms (eMPLs)

eMPLs in torus representation **[Brown,Levin;Broedel,Duhr,Dulat,Tancredi;Weinzierl...]**

$$\tilde{\Gamma}\left(\begin{smallmatrix} n_1 \dots n_m \\ z_1 \dots z_m \end{smallmatrix}; z, \tau\right) = \int_0^z dz' \, g^{(n_1)}\left(z' - z_1, \tau\right) \tilde{\Gamma}\left(\begin{smallmatrix} n_2 \dots n_m \\ z_2 \dots z_m \end{smallmatrix}; z', \tau\right) \tag{18}$$

- a torus is double-periodic and can be defined as a two-dimensional lattice

$$\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\,\tau = \{m + n\,\tau | m, n \in \mathbb{Z}\} \tag{19}$$

- τ characterises the shape of the torus
- z are the points on the torus within Λ_τ

Iterated integrals of modular forms

if all z_i are rational points on the torus of the form

$$z_i = \frac{r}{N} + \frac{s}{N}\tau \text{ with } 0 \leq r, s < N \text{ and } r, s, N \in \mathbb{N} \tag{20}$$

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$\rightarrow$ can rewrite them in terms of iterated integrals of modular forms

$$I(f_1, \ldots, f_n; \tau) = \int_{i\infty}^{\tau} \frac{d\tau'}{2\pi i} f_1 \, I(f_2, \ldots, f_n; \tau) \tag{21}$$

$$f_i = h^{(n)}_{N,r,s}(\tau) = - \sum_{\substack{(a,b)\in\mathbb{Z}^2 \\ (a,b)\neq(0,0)}} \frac{e^{2\pi i \frac{(bs-ar)}{N}}}{(a\tau + b)^n} \tag{22}$$

Direct Integration

Feynman integral can be represented via two graph polynomials $\mathcal{U}$ and $\mathcal{F}$ which are the first and second Symanzik polynomial respectively.

$$I = (-1)^a \left(e^{\epsilon\gamma_E}\right)^h \Gamma\left(a - h\frac{D}{2}\right) \int_0^\infty dx_1 \ldots \int_0^\infty dx_m \delta(1-\Delta_H) \times \\ \times \prod_{i=1}^{m} \left(\frac{x_i^{a_i-1}}{\Gamma(a_i)}\right) \frac{\mathcal{U}^{a-(h+1)\frac{D}{2}}}{\mathcal{F}^{a-h\frac{D}{2}}} \tag{23}$$

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\end{aligned}
\tag{23}
$$

- each x_i corresponds to a edge/propagator in a graph
- the second Symanzik polynomial $\mathcal{F}$ distinguishes between massive and massless propagators
  - each **massless** propagator/edge contributes **linearly** to $\mathcal{F}$
  - each **massive** propagator/edge contributes **quadratically** to $\mathcal{F}$

Direct Integration

Feynman integral can be represented via two graph polynomials $\mathcal{U}$ and $\mathcal{F}$ which are the first and second Symanzik polynomial respectively.

$$I = (-1)^a \left(e^{\epsilon\gamma_E}\right)^h \Gamma\left(a - h\frac{D}{2}\right) \int_0^\infty dx_1 \ldots \int_0^\infty dx_m \delta(1-\Delta_H) \times \\ \times \prod_{i=1}^{m} \left(\frac{x_i^{a_i-1}}{\Gamma(a_i)}\right) \frac{\mathcal{U}^{a-(h+1)\frac{D}{2}}}{\mathcal{F}^{a-h\frac{D}{2}}} \tag{23}$$

- each x_i corresponds to a edge/propagator in a graph
- the second Symanzik polynomial $\mathcal{F}$ distinguishes between massive and massless propagators
  - each **massless** propagator/edge contributes **linearly** to $\mathcal{F}$
  - each **massive** propagator/edge contributes **quadratically** to $\mathcal{F}$
- need to integrate out each single edge x_i; one done via Cheng-Wu delta function $\delta(1-\Delta_H)$.

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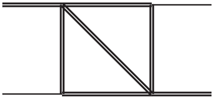

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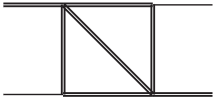

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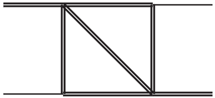

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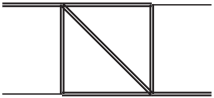

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$\rightarrow$ with suitable choice of $\tilde{\Delta}_H$ can make the integral *elliptic linear reducible* $\rightarrow$ no rationalisation necessary anymore

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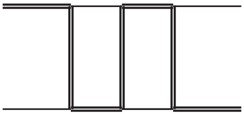

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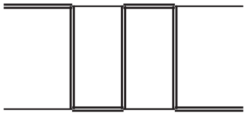

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- apply new approach with 2nd delta function $\rightarrow$ square-roots are no longer entangled $\rightarrow$ can apply *elliptic next-to-linear* reducibility approach

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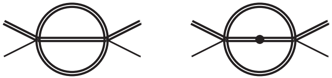
- all spurious roots disappear one by one and we are left with a single elliptic curve $\rightarrow$ eMPLs

Master Integrals - Elliptic Curves

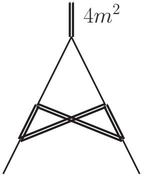
We encounter two different types of elliptic curves,

- one is associated to the elliptic sunrise



$$\vec{q} = \left(\frac{1}{2}\left(1-\sqrt{1+2i}\right), \frac{1}{2}\left(1-\sqrt{1-2i}\right), \frac{1}{2}\left(1+\sqrt{1+2i}\right), \frac{1}{2}\left(1+\sqrt{1-2i}\right)\right) \tag{25}$$

- the other is associated to the master integral



$$\vec{q} = \left(1-\sqrt{5}, 0, 2, 1+\sqrt{5}\right) \tag{26}$$

and appears only in light-by-light scattering contribution

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- QED corrections to para-Positronium result, agreement with existing numerical results in literature **[A. Czarnecki, K. Melnikov, A. Yelkhovsky, Phys.Rev.A 61 (2000) 052502]**

Application: para-Positronium decay width to di-photon

The decay width of para-Positronium to di-photon can be expressed as

$$\Gamma_{\text{p-Ps}\to\gamma\gamma} = \Gamma_0 \left[1 + \left(\frac{\alpha_{em}}{\pi}\right) K_1 + 2\alpha_{em}^2 \log\frac{1}{\alpha_{em}} + \left(\frac{\alpha_{em}}{\pi}\right)^2 (K_2 + K_{2,\text{soft}}) \right.$$
$$\left. - \frac{3\alpha_{em}^3}{2\pi}\log^2\frac{1}{\alpha_{em}} + \frac{\alpha_{em}^3}{\pi} C_2 \log\frac{1}{\alpha_{em}} + \mathcal{O}\left(\alpha_{em}^3\right)\right] \tag{27}$$

- we computed two-loop coefficient K_2 analytically and have numerics up to > 1000 digits accuracy
  $\rightarrow$ in a position to provide ultra-precise predictions for total decay width (2γ, 4γ)

$$\Gamma^{\text{theory, NNLO}}_{\text{p-Ps decay}} = 7989.618221(4)\,(\mu s)^{-1} \tag{28}$$

$$\Gamma^{\text{exp.}}_{\text{p-Ps decay}} = 7990.9(1.7)\,(\mu s)^{-1} \tag{29}$$

$\rightarrow$ experimental precision studies in future, e.g. J-PET

Summary: Form-factors

- computed all two-loop master integrals **analytically**
- produced high-precision numerics (> 1000 digits)
- find some interesting equivalence relations
- have complete **analytical** results for form-factors available
- form-factors are finite after UV and IR renormalisation

  $\rightarrow$ ready for phenomenological applications

Part III

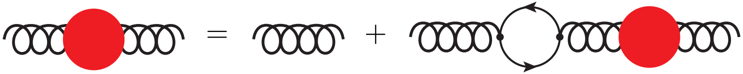
non-perturbative QCD:

renormalons & power corrections to event shapes

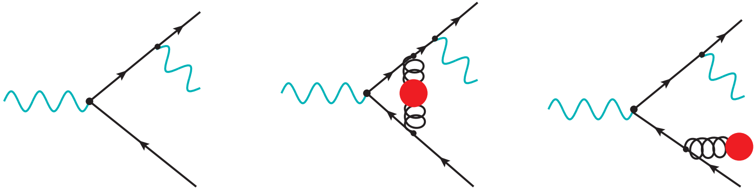
[Caola, Ferrario Ravasio, Limatola, Melnikov, Nason, Ozcelik, 2204.02247]

non-perturbative physics: Renormalons

- Renormalon model identifies simple class of diagrams that dominate in the large n_f limit
  [Beneke, Braun, Dokshitzer, Marchesini, Smye, Webber, etc.]



- example: 3-jet event $Z^*/\gamma^* \to q\bar{q}\gamma$



- each diagram can be computed perturbatively,

$$d\sigma = d\sigma^{(0)} + \left(\frac{\alpha_s}{\pi}\right) d\sigma^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 n_f \, d\sigma^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 n_f^2 \, d\sigma^{(3)} + \dots \tag{30}$$

non-perturbative physics: Renormalons

- can resum leading-n_f contributions via integral

$$\int_0^Q dk\ k^{p-1}\alpha_s(k) = \alpha_s(Q)\, Q^p \sum_{n=0}^{\infty} \underbrace{\left(\frac{\beta_0}{2\pi}\alpha_s(Q)\right)^n \frac{1}{p^{n+1}} n!}_{\text{factorial growth}}, \tag{31}$$

$$\text{with} \quad \alpha_s(\mu) = \frac{1}{\frac{\beta_0}{2\pi}\log\frac{\mu}{\Lambda_{\text{QCD}}}}, \quad \beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f. \tag{32}$$

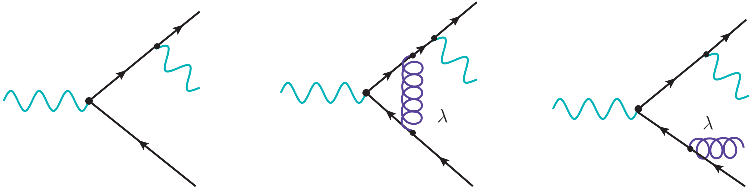
- series is not Borel summable, ambiguity given by

$$\int dk\ k^{p-1}\frac{2\pi}{\beta_0}\frac{\Lambda_{\text{QCD}}}{k-\Lambda_{\text{QCD}}} = \pm i\pi\frac{2\pi}{\beta_0}\Lambda_{\text{QCD}}^p$$

$\rightarrow$ ambiguity removed by non-perturbative power corrections $\Lambda_{\text{QCD}}^p/Q^p$

linear power corrections

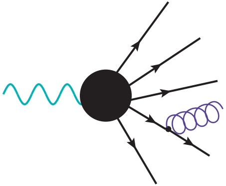
- power corrections can be computed by considering perturbative corrections with massive gluon of mass λ



- direct relation between $\lambda^p \to \Lambda_{\mathrm{QCD}}^p$
- for phenomenological applications only linear terms λ/Q are relevant, higher orders in λ are surpressed by $\mathcal{O}(\Lambda_{\mathrm{QCD}}^2/Q^2)$

Event shapes: The C-parameter

- event shapes describe the *geometry* of the collision (C-parameter, thrust, ...)



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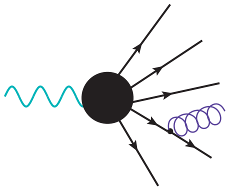

- definition of C-parameter:

$$C = 3 - 3\sum_{i>j}^{N} \frac{(p_i p_j)^2}{(p_i q)(p_j q)}. \tag{33}$$

p_i: momentum of particle i
q: sum of all momenta p_i
N: number of final-state particles

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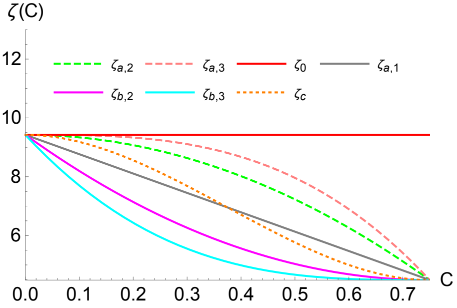

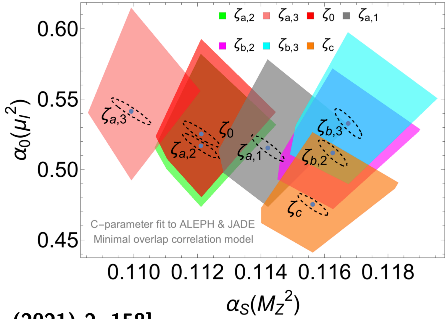

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- for α_s determination, we need **analytic** results in entire 3-jet region!

presence or absence of linear power correction

Statements on presence or absence of linear power corrections: **[Caola et al, JHEP 01 (2022) 093]**

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- *virtual* corrections **do not** induce linear corrections
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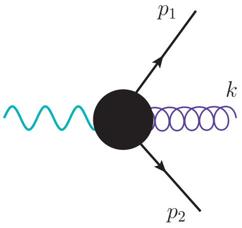
  - soft radiation at next-to-soft approximation may lead to linear corrections

soft radiation at next-to-soft approximation



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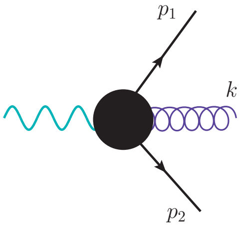

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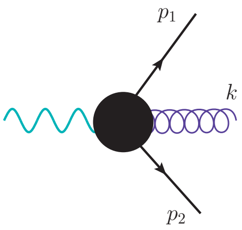

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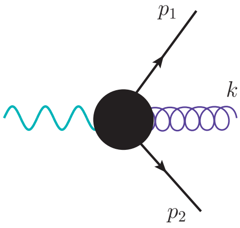

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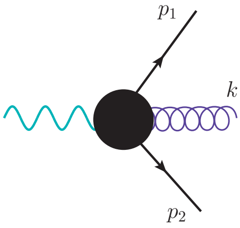

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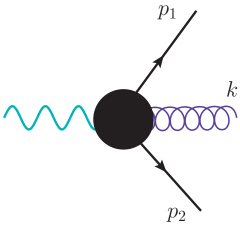

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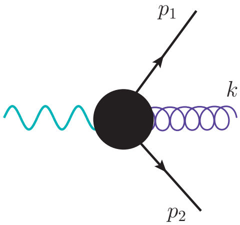

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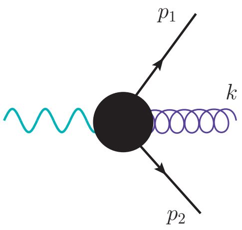

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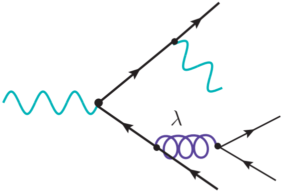
linear power corrections



event shape variables are sensitive to number of final-state particles

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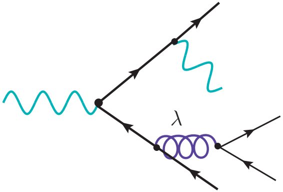

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$\rightarrow$ include $g^*(k) \rightarrow q(l)\overline{q}(\bar{l})$ splitting

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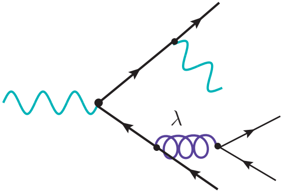

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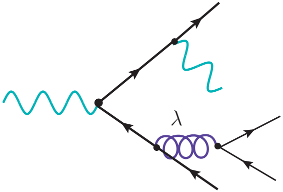

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- can factorise out phase-space of soft partons q, $\bar{q}$
  $\rightarrow$ leads to master equation:

$$\boxed{\begin{aligned} I_C(\{\tilde{p},\lambda\}) = \int [dk] \frac{J^\mu J^\nu}{\lambda^2} \theta\left(\omega_{\max} - \frac{(k\cdot q)}{\sqrt{q^2}}\right) \int [dl][d\bar{l}]\,(2\pi)^4\,\delta^{(4)}(k - l - \bar{l}) \\ \times \operatorname{Tr}\left[\hat{l}\gamma^\mu \hat{\bar{l}}\gamma^\nu\right] \left[C(\{\tilde{p}\}, l, \bar{l}) - C(\{\tilde{p}\})\right] \end{aligned}} \quad (34)$$

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$$J^\mu = \frac{p_1^\mu}{p_1 \cdot k} - \frac{p_2^\mu}{p_2 \cdot k} \tag{36}$$

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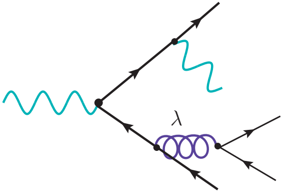
- shift in C-parameter is given by

$$\Delta C = C\big(\{\tilde{p}\},l,\bar{l}\big) - C(\{\tilde{p}\}) = \sum_{i=1}^{3} \frac{(\tilde{p}_i\cdot l)^2}{(\tilde{p}_i\cdot q)\,(l\cdot q)} + \big(l\to\bar{l}\big) \tag{37}$$

linear power corrections



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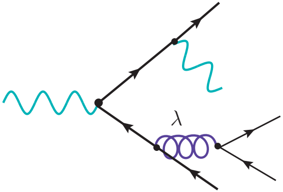

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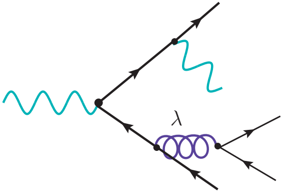

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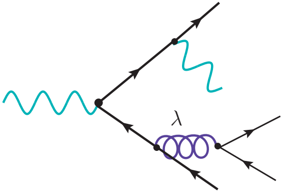
  - $\rightarrow$ one-dimensional integral over energy/velocity remaining:

$$I_C(p_1, p_2, p_3, \lambda) = -\frac{3\lambda}{4\pi^3 q} \sum_{i=1}^{5} \int_0^{\beta_{\max}} d\beta \, G_i(\beta, x, y) \tag{38}$$

linear power corrections



$$G_5 = \frac{\sqrt{1-\beta^2}\ln\left(\frac{1+\beta}{1-\beta}\right)\ln\left(\frac{\sqrt{1-\beta^2c_{12}^2}+\beta s_{12}}{\sqrt{1-\beta^2c_{12}^2}-\beta s_{12}}\right)}{64\beta^8 s_{12}x(x(y-1)+1)(xy-1)\sqrt{1-\beta^2c_{12}^2}}$$
$$\times\Big(\beta^6 x\left[x^2(y-1)y+x\left(-4y^2+4y-5\right)+5\right]+\beta^4\left[x^2\left(54y^2-54y-17\right)\right.$$
$$\left.-21x^3(y-1)y+55x-38\right]+5\beta^2\left[x^2\left(-24y^2+24y+5\right)\right.$$
$$\left.+11x^3(y-1)y-17x+12\right]-35(x-2)\left(x^2(y-1)y+x-1\right)\Big).$$

- integrate out quark and gluon momenta in rest frame of decaying particle
  - $\rightarrow$ one-dimensional integral over energy/velocity remaining:

$$I_C(p_1,p_2,p_3,\lambda) = -\frac{3\lambda}{4\pi^3 q}\sum_{i=1}^{5}\int_0^{\beta_{\max}} d\beta\, G_i(\beta,x,y) \tag{38}$$

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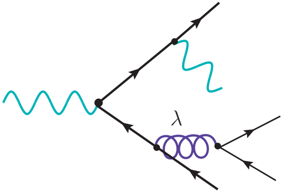

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$$\times\Big(\beta^6 x\left[x^2(y-1)y + x\left(-4y^2+4y-5\right)+5\right] + \beta^4\left[x^2\left(54y^2-54y-17\right)\right.$$
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$\rightarrow$ require class of *elliptic multiple polylogarithms* (eMPLs)

linear power corrections

- first perform a variable transformation in β to rationalise one square-root

$$G_5 = \frac{\sqrt{1-\beta^2}\ln\left(\frac{1+\beta}{1-\beta}\right)\ln\left(\frac{\sqrt{1-\beta^2 c_{12}^2}+\beta s_{12}}{\sqrt{1-\beta^2 c_{12}^2}-\beta s_{12}}\right)}{64\beta^8 s_{12} x(x(y-1)+1)(xy-1)\sqrt{1-\beta^2 c_{12}^2}}$$
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$$\left. +11x^3(y-1)y-17x+12\right] - 35(x-2)\left(x^2(y-1)y+x-1\right)\Big).$$

linear power corrections

- first perform a variable transformation in β to rationalise one square-root

  $\rightarrow$ find $y = \sqrt{(z-q_1)(z-q_2)(z-q_3)(z-q_4)}$ with branch-cuts:

$$q_1 = -\frac{1+s_{12}}{1-s_{12}}, \qquad q_2 = -1, \qquad q_3 = 1, \qquad q_4 = \frac{1+s_{12}}{1-s_{12}}.$$

linear power corrections

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linear power corrections

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- integrate out the expressions using MPL and eMPL kernels

  $\rightarrow$ obtain polylogarithmic G and elliptic E_4 functions

linear power corrections

$$
\begin{aligned}
I_1 =& -\frac{\omega_{\max}}{\lambda}\frac{2(x-1)\left(3x^2(y-1)y+x\left(-4y^2+4y+1\right)-1\right)}{3(x(y-1)+1)^2(xy-1)^2} \\
&-\frac{1}{\beta_{\min}^5}\frac{7}{16(x(y-1)+1)^2(xy-1)^2}\left[x^4(y-1)^2y^2-2x^3y\left(y^3-2y^2+5y-4\right)\right. \\
&\left.+3x^2\left(6y^2-6y-1\right)+x\left(-10y^2+10y+6\right)-3\right] \\
&+\frac{1}{\beta_{\min}^3}\frac{5}{288(x(y-1)+1)^2(xy-1)^2}\left[31x^4(y-1)^2y^2+2x^3y\left(-37y^3+74y^2\right.\right. \qquad \text{(B.6} \\
&\left.\left.-227y+190\right)+141x^2\left(6y^2-6y-1\right)+x\left(-466y^2+466y+282\right)-141\right] \\
&-\frac{1}{\beta_{\min}}\frac{1}{384(x(y-1)+1)^2(xy-1)^2}\left[27x^4(y-1)^2y^2+2x^3y\left(-71y^3+142y^2\right.\right. \\
&\left.-1239y+1168\right)+x^2\left(5286y^2-5286y-817\right) \\
&\left.+x\left(-2950y^2+2950y+1634\right)-817\right],
\end{aligned}
$$

$$
\begin{aligned}
I_2 =& \frac{1}{\beta_{\min}^5}\frac{7(x-1)\left(5x^3(y-1)y+x^2\left(-7y^2+7y+2\right)-3x+1\right)}{8x(x(y-1)+1)^2(xy-1)^2} \\
&-\frac{1}{\beta_{\min}^3}\frac{5\left((x-1)\left(217x^3(y-1)y+x^2\left(-299y^2+299y+82\right)-111x+29\right)\right)}{144\left(x(x(y-1)+1)^2(xy-1)^2\right)} \\
&+\frac{1}{\beta_{\min}}\frac{(x-1)\left(4121x^3(y^2-y)-x^2\left(5875(y^2-y)-1754\right)-2895x+1141\right)}{576x(x(y-1)+1)^2(xy-1)^2} \qquad \text{(B.7} \\
&-\frac{\pi^2(x-1)\left(85x^3(y-1)y+x^2\left(-127y^2+127y+42\right)-83x+41\right)}{256\left(x(x(y-1)+1)^2(xy-1)^2\right)},
\end{aligned}
$$

$$
\begin{aligned}
I_4 =& \frac{1}{\beta_{\min}^5}\frac{7}{16x(x(y-1)+1)^2(xy-1)^2}\left[x^5(y-1)^2y^2-2x^4y\left(y^3-2y^2+10y-9\right)\right. \\
&\left.+7x^3\left(6y^2-6y-1\right)-8x^2\left(3y^2-3y-2\right)-11x+2\right] \\
&-\frac{1}{\beta_{\min}^3}\frac{5}{288\left(x(x(y-1)+1)^2(xy-1)^2\right)}\left[31x^5(y-1)^2y^2-74x^4y\left(y^2(y-2)\right.\right. \\
&\left.\left.+12y-11\right)+x^3\left(1878(y^2-y)-305\right)-x^2\left(1064(y^2-y)-668\right)-421x+58\right] \\
&+\frac{1}{\beta_{\min}}\frac{1}{1152x(x(y-1)+1)^2(xy-1)^2}\left[81x^5(y-1)^2y^2-2x^4y\left(213y^2(y-2)\right.\right. \qquad \text{(B.8} \\
&\left.+7838y-7625\right)+x^3\left(35850(y^2-y)-5959\right)-200x^2\left(103(y^2-y)-71\right) \\
&\left.-10523x+2282\right] \\
&+\frac{w_{\max}}{\lambda}\, 4(1-x)\left((2x^3-3x^2)(y^2-y)+(1-x)^2\right)\left(\log\tfrac{\lambda}{2w_{\max}}+1\right) \cdots
\end{aligned}
$$

$$
\begin{aligned}
I_5 =& -\frac{1}{\beta_{\min}^5}\frac{7(x-2)}{16x}+\frac{1}{\beta_{\min}^3}\frac{5\left(31x^3(y-1)y+x^2\left(-74y^2+74y+23\right)-81x+58\right)}{288x(x(y-1)+1)(xy-1)} \\
&+\frac{1}{\beta_{\min}}\frac{1}{1152x(x(y-1)+1)^2(xy-1)^2}\left[-81x^5(y-1)^2y^2+2x^4y\left(213y^3-426y^2\right.\right. \\
&\left.-404y+617\right)+x^3\left(4134y^2-4134y-1057\right)+x^2\left(-2900y^2+2900y+4396\right) \\
&\left.-5621x+2282\right]-\frac{\pi^2(x-1)^2}{1024s_{12}x(x(y-1)+1)^3(xy-1)^3}\times \\
&\times\left(160x^4(y-1)^2y^2+192x^3(y-1)y+x^2\left(-68y^2+68y+37\right)+45x-82\right) \\
&+\frac{1}{1024x(xy-1)^2(xy-x+1)^2}\left[3x^5y^4-6x^5y^3+3x^5y^2-2x^4y^4+4x^4y^3-196x^4y^2\right. \\
&\left.+194x^4y+430x^3y^2-430x^3y-37x^3-236x^2y^2+236x^2y-8x^2+127x-82\right]\times \\
&\times\left[-4\,\mathrm{E}_4\left(\begin{smallmatrix}-1 & 1\\ 0 & \frac{1+s_{12}}{1-s_{12}}\end{smallmatrix};1,\vec{q}\right)+4\,\mathrm{E}_4\left(\begin{smallmatrix}-1 & 1\\ 0 & \frac{1+s_{12}}{-1+s_{12}}\end{smallmatrix};1,\vec{q}\right)-4\,\mathrm{G}\left(0,\frac{s_{12}+1}{1-s_{12}};1\right)\right. \\
&\left.+4\,\mathrm{G}\left(0,\frac{s_{12}+1}{s_{12}-1};1\right)+\pi^2\right]+\mathrm{E}_4\left(\begin{smallmatrix}0\\ 0\end{smallmatrix};1,\vec{q}\right)\left[\frac{5\pi^2(x-1)^2\left(x^2(y-1)y+1\right)}{8s_{12}x(x(y-1)+1)^2(xy-1)^2}\frac{\eta_1}{\omega_1}\right. \\
&\left.+\frac{5\pi^2(x-1)^2\left(2x^4(y-1)^2y^2+7x^3(y-1)y+x^2\left(-8y^2+8y+3\right)-5x+2\right)}{96s_{12}x(x(y-1)+1)^3(xy-1)^3}\right] \\
&+\left(\frac{5(x-1)^2\left(x^2(y-1)y+1\right)}{8s_{12}x(x(y-1)+1)^2(xy-1)^2}+\frac{5(x-1)^2\left(x^2(y-1)y+1\right)}{8x(x(y-1)+1)^2(xy-1)^2}\right)\mathrm{E}_4\left(\begin{smallmatrix}-1 & 1\\ \infty & \frac{s_{12}+1}{1-s_{12}}\end{smallmatrix};1,\vec{q}\right) \\
&+\left(-\frac{5(x-1)^2\left(x^2(y-1)y+1\right)}{8s_{12}x(x(y-1)+1)^2(xy-1)^2}-\frac{5(x-1)^2\left(x^2(y-1)y+1\right)}{8x(x(y-1)+1)^2(xy-1)^2}\right)\times \\
&\times\mathrm{E}_4\left(\begin{smallmatrix}-1 & 1\\ \infty & \frac{s_{12}+1}{s_{12}-1}\end{smallmatrix};1,\vec{q}\right)+\left(\frac{5(x-1)^2\left(x^2(y-1)y+1\right)}{8s_{12}x(x(y-1)+1)^2(xy-1)^2}\right. \\
&\left.+\frac{-3x^3(y-1)y+x^2\left(2y^2-2y+37\right)-115x+78}{256x(x(y-1)+1)(xy-1)}\right)\mathrm{E}_4\left(\begin{smallmatrix}-1 & 1\\ \infty & -1\end{smallmatrix};1,\vec{q}\right) \\
&+\mathrm{E}_4\left(\begin{smallmatrix}-1 & 1\\ \infty & 1\end{smallmatrix};1,\vec{q}\right)\left(\frac{3x^3(y-1)y+x^2\left(-2y^2+2y-37\right)+115x-78}{256x(x(y-1)+1)(xy-1)}\right. \qquad \text{(B.14)} \\
&\left.-\frac{5(x-1)^2\left(x^2(y-1)y+1\right)}{8s_{12}x(x(y-1)+1)^2(xy-1)^2}\right)+\frac{5(x-1)^2\left(x^2y^2-x^2y+1\right)}{16s_{12}x(xy-1)^2(xy-x+1)^2}\times \\
&\times\left[-\mathrm{E}_4\left(\begin{smallmatrix}2 & 1\\ \frac{s_{12}+1}{1-s_{12}} & \frac{s_{12}+1}{1-s_{12}}\end{smallmatrix};1,\vec{q}\right)+\mathrm{E}_4\left(\begin{smallmatrix}2 & 1\\ \frac{s_{12}+1}{1-s_{12}} & \frac{s_{12}+1}{s_{12}-1}\end{smallmatrix};1,\vec{q}\right)+\mathrm{E}_4\left(\begin{smallmatrix}2 & 1\\ \frac{s_{12}+1}{s_{12}-1} & \frac{s_{12}+1}{1-s_{12}}\end{smallmatrix};1,\vec{q}\right)\right. \\
&\left.-\mathrm{E}_4\left(\begin{smallmatrix}2 & 1\\ \frac{s_{12}+1}{s_{12}-1} & \frac{s_{12}+1}{s_{12}-1}\end{smallmatrix};1,\vec{q}\right)+\mathrm{E}_4\left(\begin{smallmatrix}2 & 1\\ -1 & -1\end{smallmatrix};1,\vec{q}\right)-\mathrm{E}_4\left(\begin{smallmatrix}2 & 1\\ -1 & 1\end{smallmatrix};1,\vec{q}\right)-\mathrm{E}_4\left(\begin{smallmatrix}2 & 1\\ 1 & -1\end{smallmatrix};1,\vec{q}\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
I_3 =& \frac{1}{\beta_{\min}^5}\frac{7(x-2)}{16x}-\frac{1}{\beta_{\min}^3}\frac{5\left(31x^3y^2-31x^3y-74x^2y^2+74x^2y+23x^2-81x+58\right)}{288(x(xy-1)(xy-x+1))} \\
&+\frac{1}{\beta_{\min}}\frac{1}{1152x(x(y-1)+1)^2(xy-1)^2}\left[81x^5(y-1)^2y^2+2x^4y\left(-213y^3+426y^2\right.\right. \\
&\left.+404y-617\right)+x^3\left(-4134y^2+4134y+1057\right)+4x^2\left(725y^2-725y-1099\right) \\
&\left.+5621x-2282\right]+\frac{w_{\max}}{\lambda}\frac{4}{3x}\left[\log s_{12}-\log\frac{\lambda}{2w_{\max}}-1\right] \qquad \text{(B.1} \\
&+\frac{(x-1)^2}{512s_{12}x(xy-1)^3(xy-x+1)^3}\left(160x^4y^4-320x^4y^3+160x^4y^2+192x^3y^2\right. \\
&\left.-192x^3y-68x^2y^2+68x^2y+37x^2+45x-82\right)\left\{\frac{\pi^2}{4}+\mathrm{G}\left(0,\frac{s_{12}+1}{1-s_{12}},1\right)\right. \\
&-\mathrm{G}\left(0,\frac{s_{12}+1}{s_{12}-1},1\right)+\mathrm{E}_4\left(\begin{smallmatrix}-1 & 1\\ \infty & \frac{s_{12}+1}{1-s_{12}}\end{smallmatrix};1,\vec{q}\right)-\mathrm{E}_4\left(\begin{smallmatrix}-1 & 1\\ \infty & \frac{s_{12}+1}{s_{12}-1}\end{smallmatrix};1,\vec{q}\right) \\
&+\mathrm{E}_4\left(\begin{smallmatrix}-1 & 1\\ 0 & \frac{s_{12}+1}{1-s_{12}}\end{smallmatrix};1,\vec{q}\right)-\mathrm{E}_4\left(\begin{smallmatrix}-1 & 1\\ 0 & \frac{s_{12}+1}{s_{12}-1}\end{smallmatrix};1,\vec{q}\right)+\mathrm{E}_4\left(\begin{smallmatrix}-1 & 1\\ \infty & -1\end{smallmatrix};1,\vec{q}\right)
\end{aligned}
$$

linear power corrections

- first perform a variable transformation in β to rationalise one square-root

  $\rightarrow$ find $y = \sqrt{(z - q_1)(z - q_2)(z - q_3)(z - q_4)}$ with branch-cuts:

$$q_1 = -\frac{1 + s_{12}}{1 - s_{12}}, \qquad q_2 = -1, \qquad q_3 = 1, \qquad q_4 = \frac{1 + s_{12}}{1 - s_{12}}.$$

- integrate out the expressions using MPL and eMPL kernels

  $\rightarrow$ obtain polylogarithmic G and elliptic E_4 functions

- next step: rewrite E_4 to $\tilde{\Gamma}$ representation

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    $\rightarrow$ rewrite as iterated integrals of Eisenstein form I

linear power corrections

$$\mathcal{T}_\lambda[I_C] = -\frac{3}{4\pi^3}\frac{\lambda}{q}(t_1+t_2+t_3),$$

with

$$\begin{aligned}t_1 =& \frac{5(x-1)^2\left(x^2(y-1)y+1\right)}{4\pi^2\omega_1 s_{12}x(x(y-1)+1)^2(xy-1)^2}\times\\
&\times\left[I\left(1,h^{(3)}_{4,1,2},\tau\right)\left(-2\omega_1(s_{12}+1)g^{(1)}\left(\frac{1}{4},\tau\right)-g^{(2)}\left(\frac{3}{4},\tau\right)\right.\right.\\
&\left.+g^{(2)}\left(-\frac{1}{4},\tau\right)+\omega_1^2(s_{12}+1)^2\right)\\
&+i\pi I\left(h^{(3)}_{4,1,0},\tau\right)\left(-g^{(1)}\left(\frac{1}{4},\tau\right)+g^{(1)}\left(\frac{3}{4},\tau\right)+\omega_1 s_{12}+\omega_1\right)\\
&\left.+i\pi I\left(h^{(3)}_{4,1,2},\tau\right)\left(g^{(1)}\left(\frac{\tau}{2}+\frac{1}{4},\tau\right)-g^{(1)}\left(\frac{\tau}{2}+\frac{3}{4},\tau\right)+\omega_1(s_{12}-1)\right)\right]\end{aligned}$$

$$\begin{aligned}t_2 =& \frac{1}{384\pi s_{12}x\left(x^2(y-1)y+x-1\right)^3}\times\\
&\times\left[-\frac{6\tilde{C}(x(y-1)+1)(xy-1)}{\omega_1}\left(\omega_1 g^{(1)}\left(\frac{1}{4},\tau\right)(s_{12}(x(y-1)+1)(xy-1)\times\right.\right.\\
&\times(x(x((3x-2)(y-1)y-37)+115)-78)-160(x-1)^2\left(x^2(y-1)y+1\right))\\
&\left.+80(x-1)^2\left(x^2(y-1)y+1\right)\left(g^{(2)}\left(\frac{\tau}{2}-\frac{1}{4},\tau\right)-g^{(2)}\left(\frac{\tau}{2}+\frac{3}{4},\tau\right)\right)\right)\\
&+3\tilde{C}\omega_1(s_{12}+1)(x(y-1)+1)(xy-1)\,(s_{12}(x(y-1)+1)(xy-1)\times\\
&\times(x(x((3x-2)(y-1)y-37)+115)-78)-160(x-1)^2\left(x^2(y-1)y+1\right))\\
&+5\pi^3(x-1)^2\left(12\eta_1(x(y-1)+1)(xy-1)\left(x^2(y-1)y+1\right)\right.\\
&\left.\left.+\omega_1(x(x((y-1)y(x(2x(y-1)y+7)-8)+3)-5)+2)\right)\right],\end{aligned} \tag{B.22}$$

$$\begin{aligned}t_3 =& -\frac{1}{128\pi^2\omega_1 s_{12}x\left(x^2(y-1)y+x-1\right)^2}I\left(1,h^{(3)}_{4,1,0},\tau\right)\times\\
&\times\left[2\omega_1 g^{(1)}\left(\frac{1}{4},\tau\right)(s_{12}(x(y-1)+1)(xy-1)(x(x((3x-2)(y-1)y-37)\right.\\
&+115)-78))-160(x-1)^2\left(x^2(y-1)y+1\right)\times\\
&\times\left(2\omega_1 g^{(1)}\left(\frac{1}{4},\tau\right)+g^{(2)}\left(\frac{\tau}{2}+\frac{3}{4},\tau\right)-g^{(2)}\left(\frac{\tau}{2}-\frac{1}{4},\tau\right)\right)\\
&+\omega_1^2(s_{12}+1)\left(160(x-1)^2\left(x^2(y-1)y+1\right)\right.\\
&\left.\left.-s_{12}(x(y-1)+1)(xy-1)(x(x((3x-2)(y-1)y-37)+115)-78)\right)\right].\end{aligned} \tag{B.23}$$

linear power corrections

- iterated integrals of modular forms with $\Gamma(4)$

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$$g^{(2)}\left(-\frac{1}{4},\tau\right), \quad g^{(2)}\left(\frac{\tau}{2}+\frac{3}{4},\tau\right), \quad g^{(2)}\left(\frac{\tau}{2}-\frac{1}{4},\tau\right)$$

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$$g^{(2)}\left(-\frac{1}{4},\tau\right),\quad \cancel{g^{(2)}\left(\frac{\tau}{2}+\frac{3}{4},\tau\right)},\quad g^{(2)}\left(\frac{\tau}{2}-\frac{1}{4},\tau\right)$$

- find simple relations as:

$$g^{(1)}\left(\frac{1}{4},\tau\right)=-\,g^{(1)}\left(\frac{3}{4},\tau\right),\qquad g^{(2)}\left(-\frac{1}{4},\tau\right)=g^{(1)}\left(\frac{3}{4},\tau\right)$$
$$g^{(2)}\left(\frac{\tau}{2}-\frac{1}{4},\tau\right)=g^{(2)}\left(\frac{\tau}{2}+\frac{3}{4},\tau\right)$$

linear power corrections

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- find more relations as:

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$\rightarrow$ simplicity of result calls for an explanation and suggests deeper structure

Factorisation

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I_C(\{\tilde{p},\lambda\}) = \int [dk] \frac{J^\mu J^\nu}{\lambda^2}\, \theta\left(\omega_{\max} - \frac{(k\cdot q)}{\sqrt{q^2}}\right) \int [dl][d\bar{l}]\,(2\pi)^4\,\delta^{(4)}\big(k - l - \bar{l}\big) \\ \times \mathrm{Tr}\left[\hat{l}\gamma^\mu \hat{\bar{l}}\gamma^\nu\right]\left[C\big(\{\tilde{p}\}, l, \bar{l}\big) - C(\{\tilde{p}\})\right] \tag{40}
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$\longrightarrow$

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The universal factor

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- same **universal** factor appears for different observables in arbitrary N-jet kinematics!
- rigorous derivation and generalisation of similar factor known from 2-jet limit (so-called Milan factor) **[Y. Dokshitzer, A. Lucenti, G. Marchesini, G. Salam, JHEP 05 (1998) 003]** and many more

Event shapes: Thrust

- similarly can compute other observables with factorised formula and can generalise factorisation to N-jet kinematics (4-jet, 5-jet, ...)

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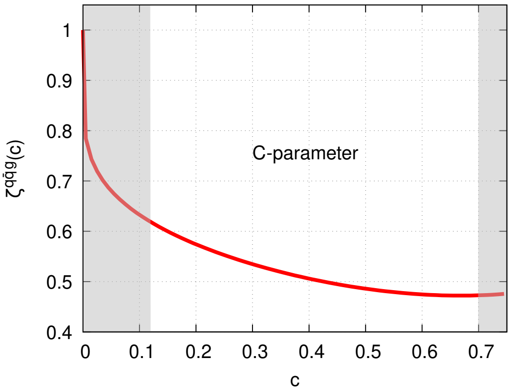
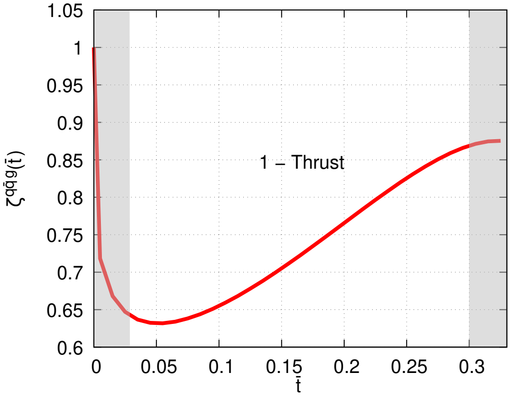
linear power corrections to C-parameter & thrust





→ find agreement previous with results (2-jet limit, ...)

→ have **analytic** results for entire 3-jet region and these are superior to numerical methods

Summary: Power corrections

- used formalism of elliptic polylogarithms and iterated integrals of modular forms to compute power corrections to C-parameter
- improved understanding of analytic structure of linear power corrections
- derived a factorisation formula which allows us to easily compute linear power corrections for different observables
- shown that same universal factor appears for different observables
- computed analytically linear power corrections for C-parameter and thrust T in entire three-jet region

$\rightarrow$ can now be used for pheno and α_s determination

Part IV

Conclusions & Outlook

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- progress in QCD phenomenology thanks to understanding and formalism of elliptic polylogarithms and iterated integrals of modular forms

- in the future:
  - need to optimise and understand relations among elliptic integrals
  - require fast and efficient numerical evaluations for phenomenology
  - ...
  - near/far future: for physical processes or observables that exhibit multiple elliptic curves
    $\rightarrow$ need to generalise elliptic polylogs to higher genus...

Thank you for attention!