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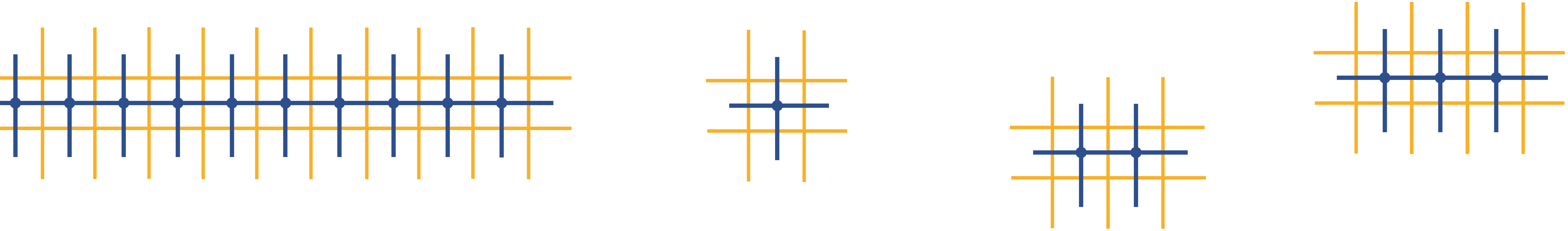
bctp Bethe Center for Theoretical Physics

# Calabi-Yau Geometries and 2D Fishnet Integrals

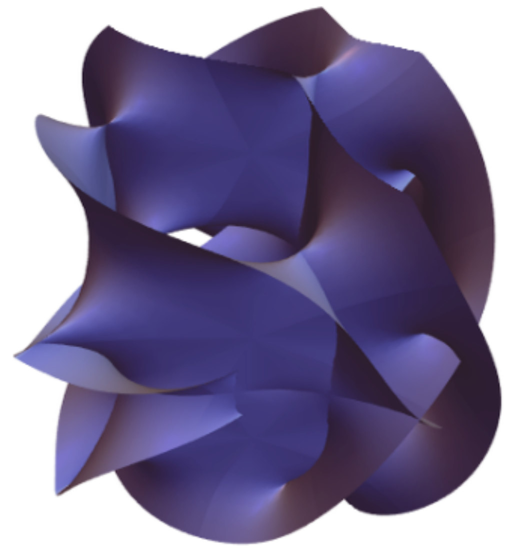
Franziska Porkert



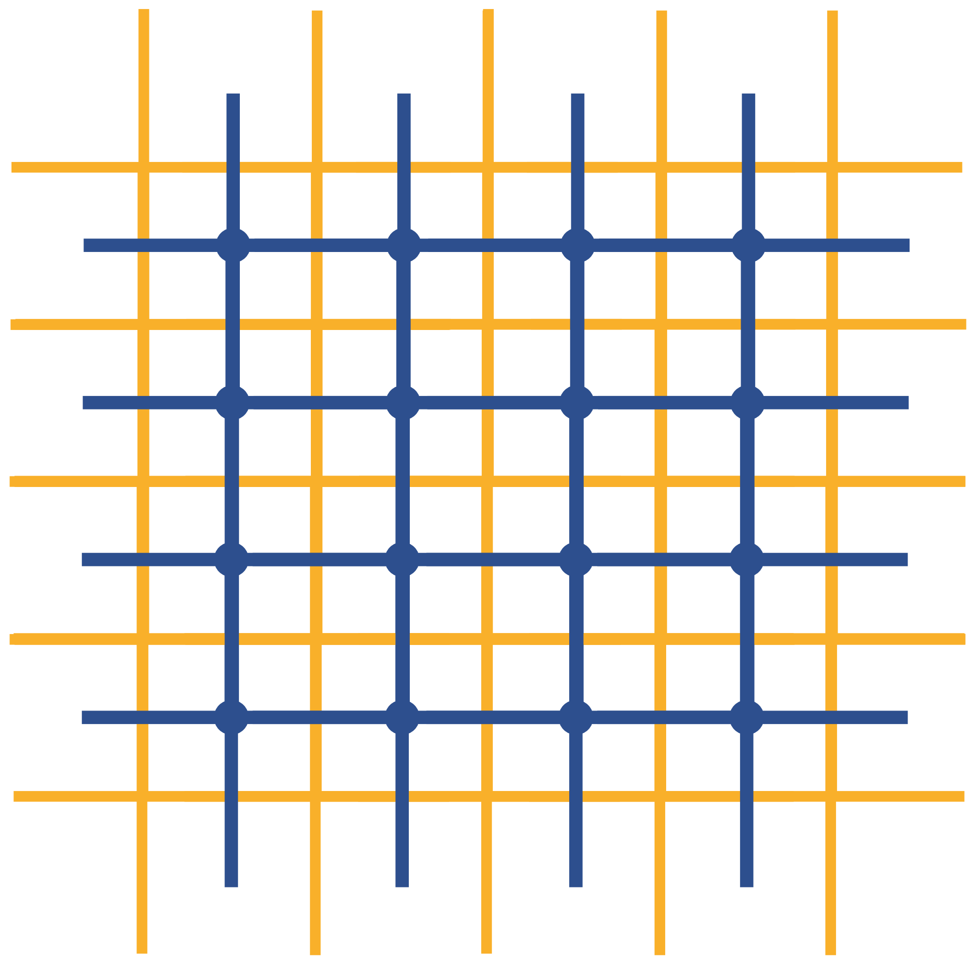
Based on: **2209.05291**
with Claude Duhr, Albrecht Klemm, Florian Loebbert, Christoph Nega

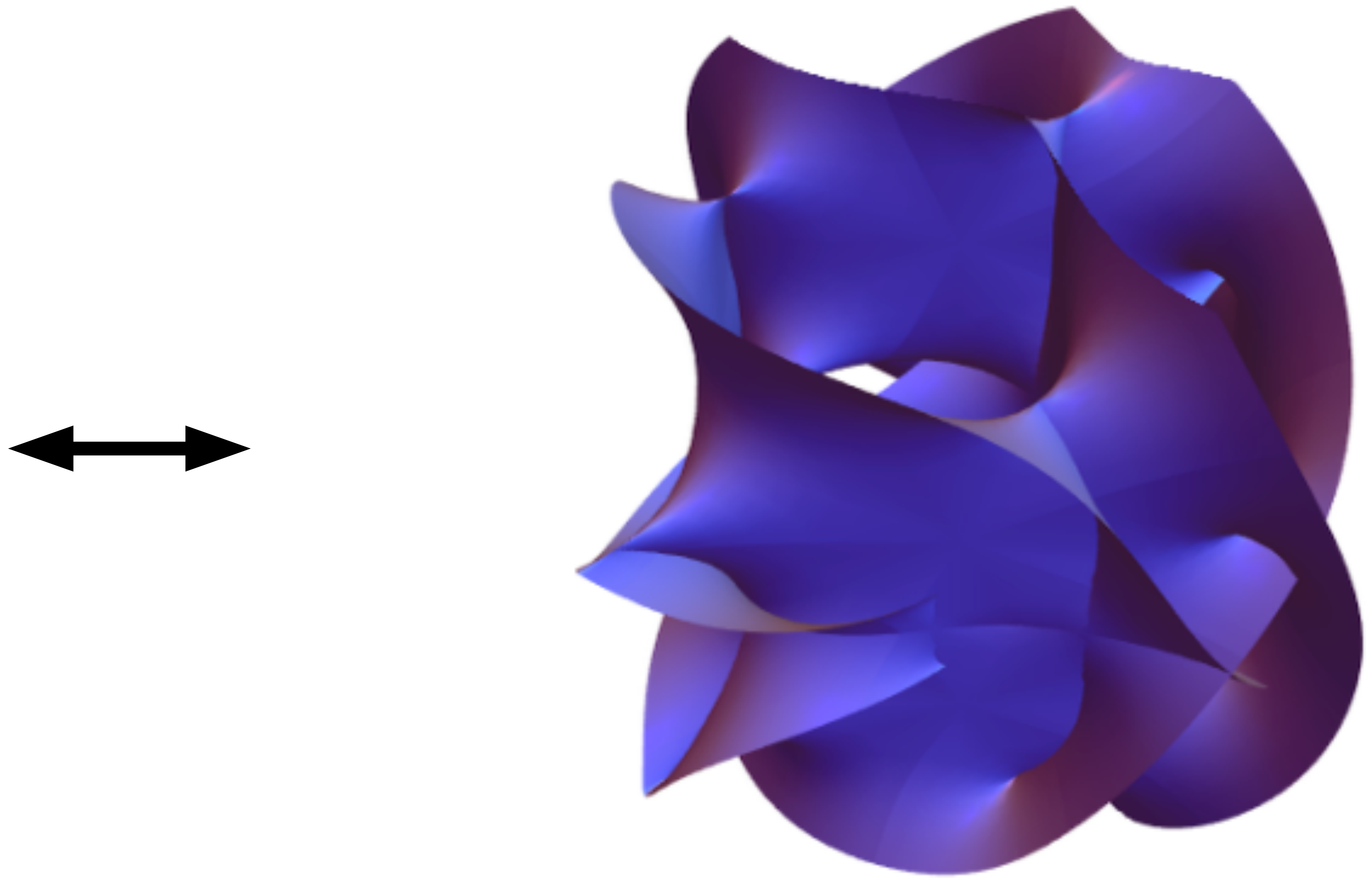
Elliptic Integrals in Fundamental Physics — 14. 9. 2022

## Christoph's talk:

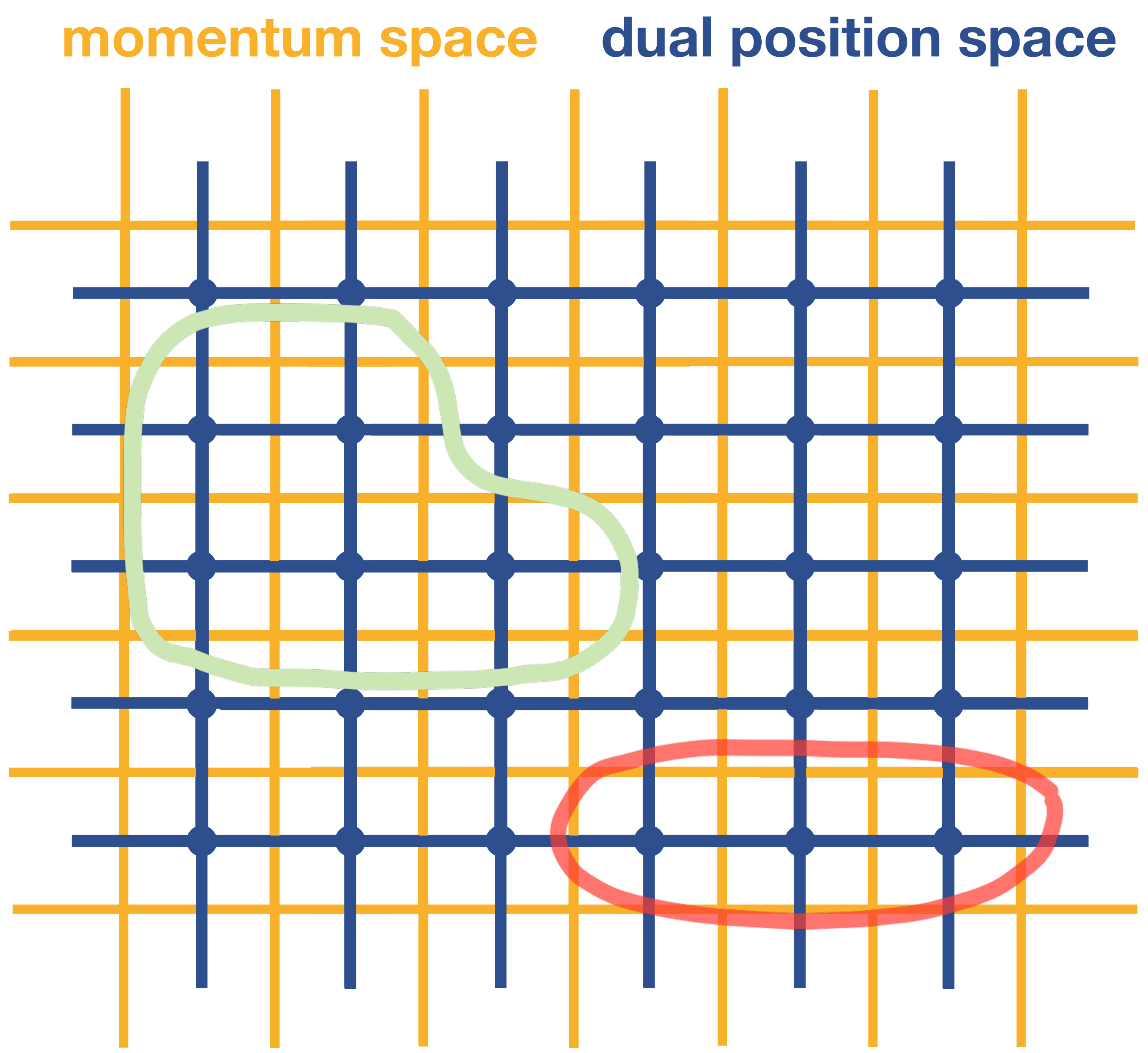


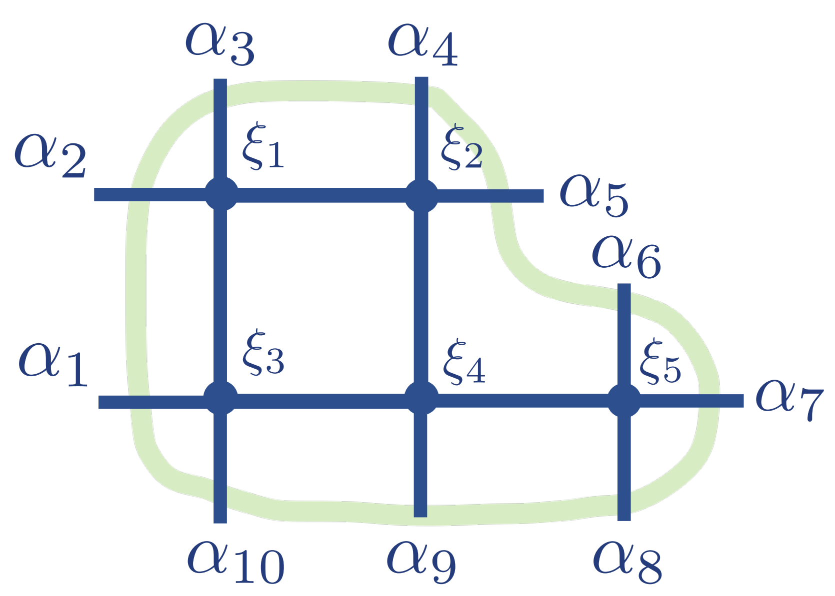
## Next Goal: Understand fishnet integrals from a geometric viewpoint



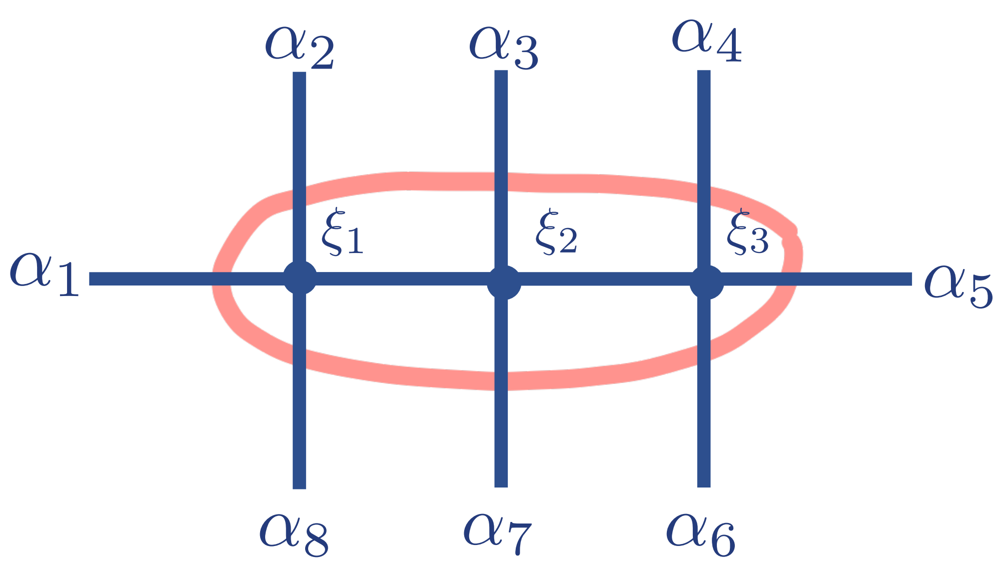


# Massless Fishnet Integrals — General definition



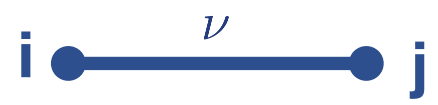

**Random example**




**Three-loop traintrack**

## Feynman rules:
## (Position space, D real dimensions)


$$= \int d^D \xi_i$$




$$= \frac{1}{\xi_{ij}^{2\nu}} = \frac{1}{(\xi_i - \xi_j)^{2\nu}}$$

$\nu = D/4$ for conformal fishnet integrals

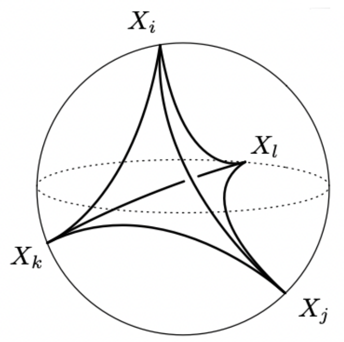
## Fishnet integral:

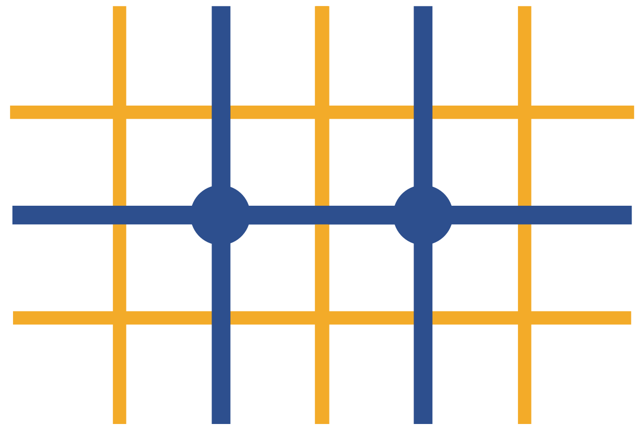
$$I_G(\alpha) \sim \int \left( \prod_{j=1}^{l} d^D \xi_j \right) \frac{1}{P_G(\underline{\xi}, \underline{\alpha})}$$

$$P_G(\underline{\xi}, \underline{\alpha}) = \left[ \prod_{i,j} (\xi_i - \xi_j)^{2\nu_{ij}} \right] \left[ \prod_{i,j} (\xi_i - \alpha_j)^{2\nu_{ij}} \right]$$

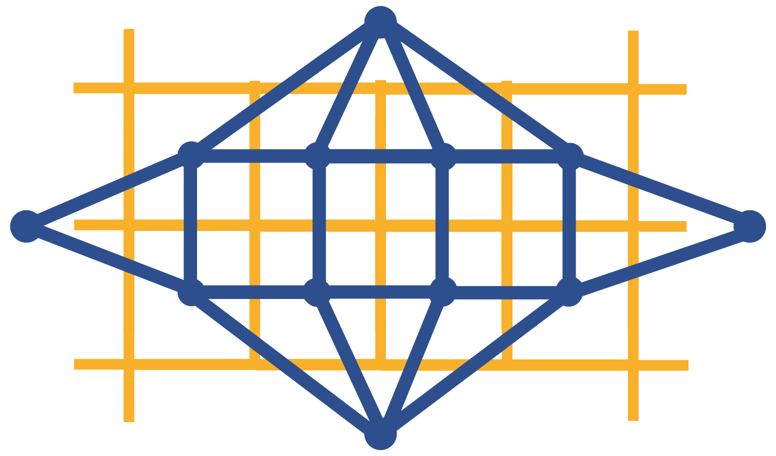
momentum space to dual position space: $p_j^\mu = \xi_j^\mu - \xi_{j+1}^\mu$

Conformal inversion: $x^\mu \mapsto \frac{x^\mu}{x^2}$

**One loop:**
Bloch-Wigner Dilogarithm $\sim \frac{1}{\sqrt{\Delta}} \mathrm{Vol}\Big($   $\Big)$
[Davydychev, Delbourgo; 1997]

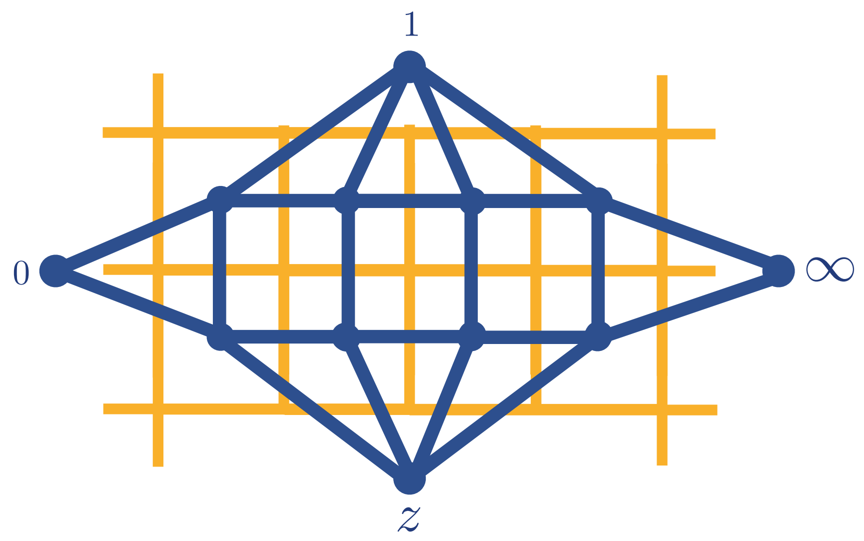


**Two loops:**
Elliptic Polylogarithms
[Kristensson, Wilhelm, Zhang; 2021]



**One parameter:**
Basso Dixon formula:
Recursive relations for all-loop 4-point fishnets in terms of ladders
[Basso, Dixon; 2017]

## Can we understand fishnet integrals from a geometric viewpoint?




**One parameter:**

Recursive relations for all-loop 4-point fishnet integrals in 2 dimensions

[Derkachov, Kazakov, Olivucci; 2019]

**Can we understand fishnet integrals in 2 dimensions from a geometric viewpoint?**

## Goal: Understand 2D fishnet integrals from a geometric viewpoint

**Introduction:** Massless fishnet integrals in 2 dimensions and their symmetries

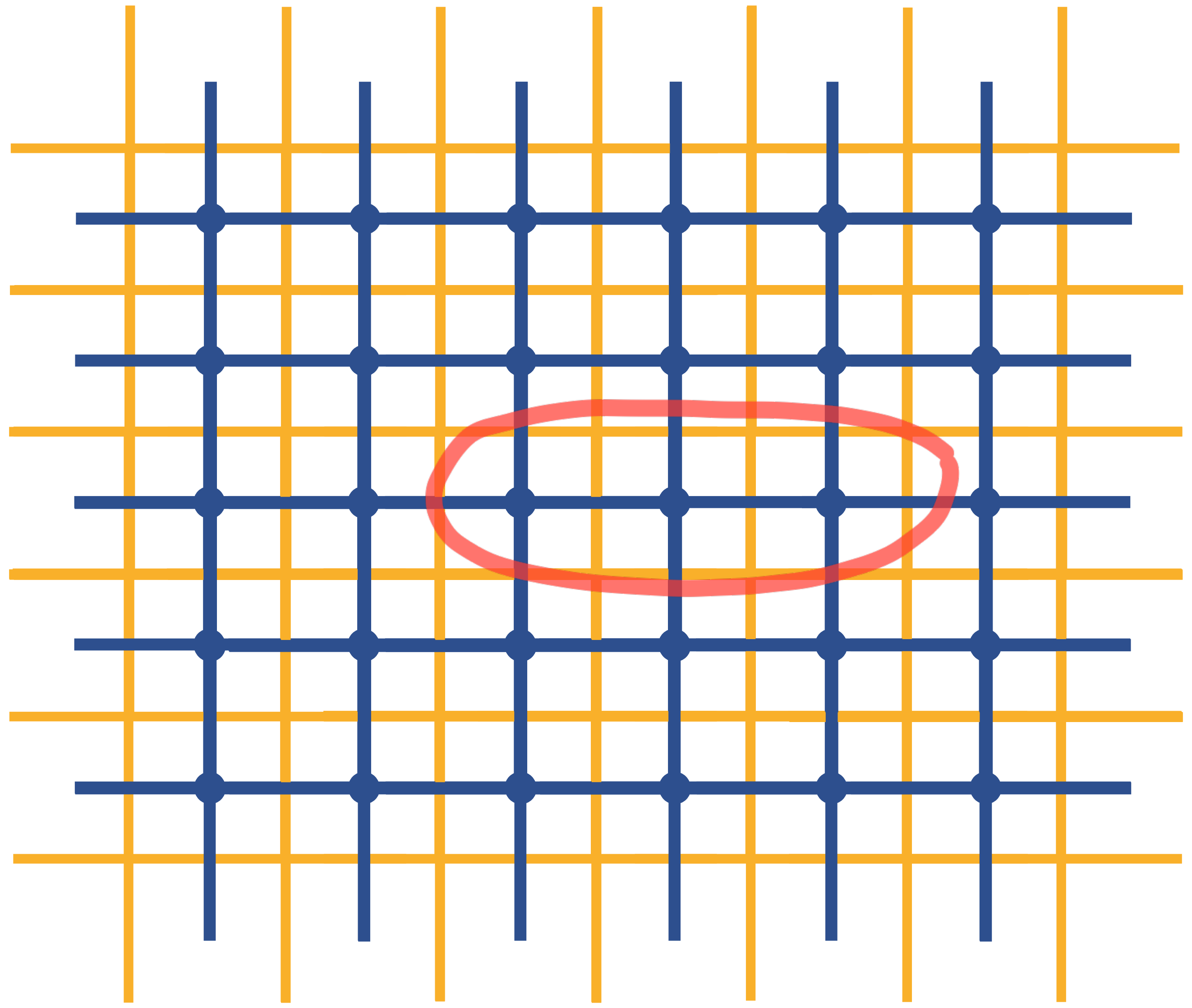
**Question 1:** Can we relate fishnet integrals and Calabi-Yau geometries?

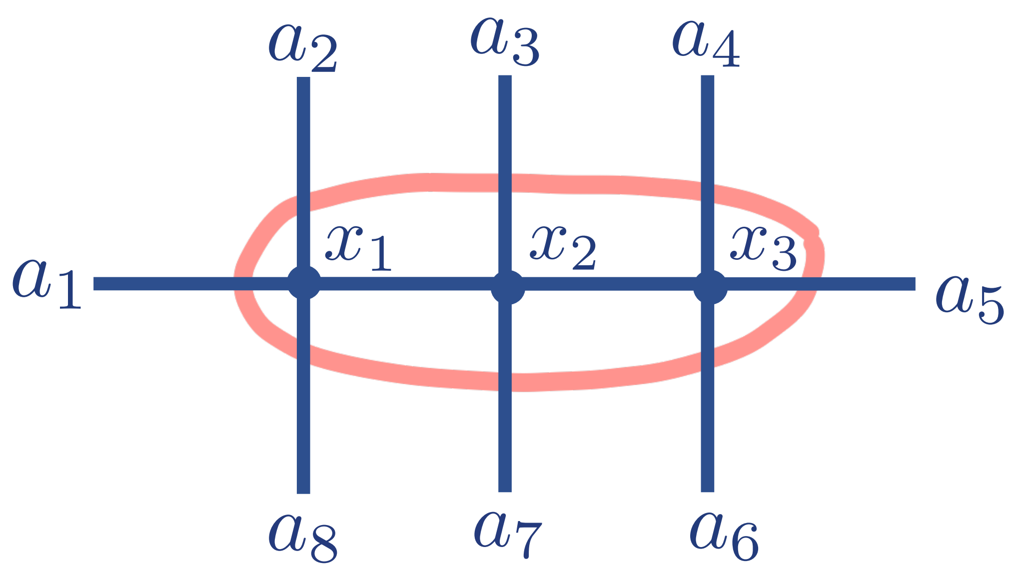
**Question 2:** Is there a connection between the symmetries of fishnet integrals and these geometries?

**Results**

**Question 3:** Can we interpret fishnet integrals as volumes of Calabi-Yau l-folds?

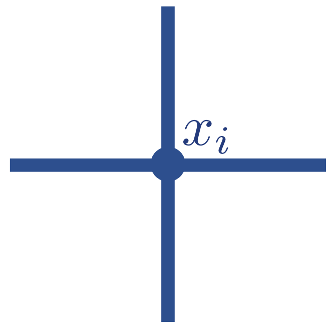
**Summary:** Dictionary






**Three-loop traintrack**

## Feynman rules:
**(D=2, $\nu_i = \frac{1}{2}$ )**




$$= \int dx_i \wedge d\bar{x}_i$$


$$= \frac{1}{|x_{ij}|} = \frac{1}{|x_i - x_j|}$$

$\nu_i = D/4 = 1/2 \rightarrow$ conformal

## Fishnet integrals:
**(D=2, $\nu_i = \frac{1}{2}$ )**

$$I_G(a) = \int_{\mathbb{C}^l} \left( \prod_{j=1}^{l} \frac{d\bar{x}_j \wedge dx_j}{2\pi} \right) \frac{1}{|P_G(\underline{x}, \underline{a})|}$$

$$P_G(\underline{x}, \underline{a}) = \left[ \prod_{i,j} (x_i - x_j) \right] \left[ \prod_{i,j} (x_i - a_j) \right]$$

$\mathbb{R}^2 \longrightarrow \mathbb{C}: \quad x_k = \xi_k^1 + i\xi_k^2$ and $d^2\xi_i / i\pi = dx_i \wedge d\bar{x}_i / 2\pi$

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$$I_G(a) = \int_{\mathbb{C}^l} \left( \prod_{j=1}^{l} \frac{d\bar{x}_j \wedge dx_j}{2\pi} \right) \frac{1}{|P_G(\underline{x}, \underline{a})|} \quad \text{with} \quad P_G(\underline{x}, \underline{a}) = \left[ \prod_{i,j} (x_i - x_j) \right] \left[ \prod_{i,j} (x_i - a_j) \right]$$

## Symmetries:

1. Permutation symmetries of the graph

2. Yangian invariance

# Massless 2D fishnet integrals — Symmetries: Permutation symmetry

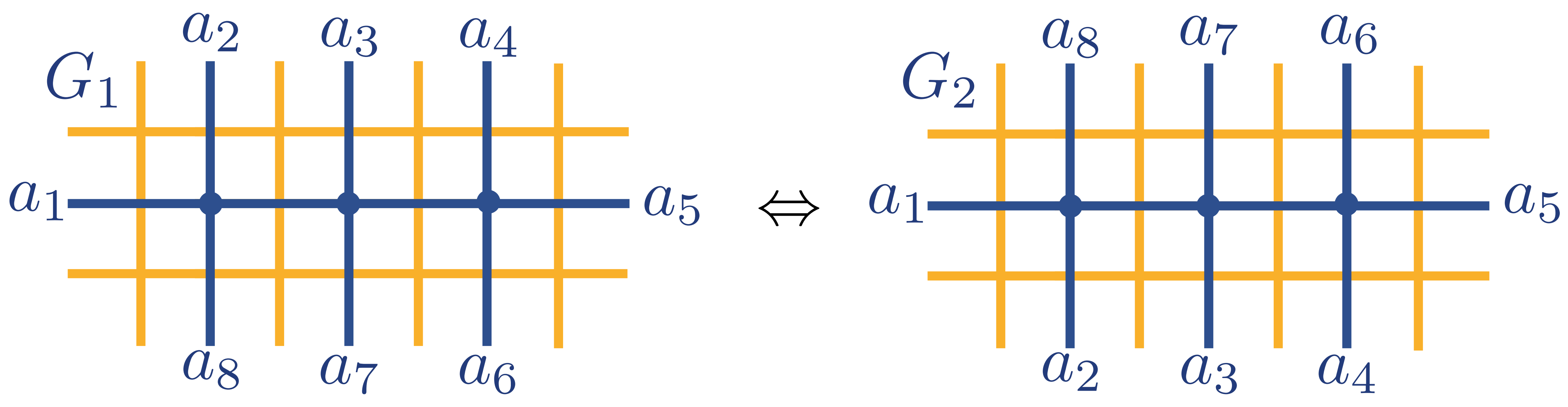
$$I_G(a) = \int_{\mathbb{C}^l} \left( \prod_{j=1}^{l} \frac{d\bar{x}_j \wedge dx_j}{2\pi} \right) \frac{1}{|P_G(\underline{x}, \underline{a})|} \quad \text{with} \quad P_G(\underline{x}, \underline{a}) = \left[ \prod_{i,j} (x_i - x_j) \right] \left[ \prod_{i,j} (x_i - a_j) \right]$$

## Permutation Symmetries:

The topology of the graph $G$ defines a subgroup of permutations $S_G \subseteq S_n$ that exchanges the external points while leaving the value of the integral invariant:

$$I_G(\sigma(\underline{a})) = I_G(\underline{a}) \quad \text{for all} \quad \sigma \in S_G$$

## Example: The three-loop traintrack integral




$$P_{G_1}(\underline{x}, \underline{a}) = P_{G_2}(\underline{x}, \underline{a})$$

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## Level-0 Yangian constraints

$$J^a = \sum_{k=1}^{n} J_k^a \quad \text{with} \quad J_k^a \in \begin{cases} D_k &= -ix_k^\mu \partial^\mu - i\triangle_k \\ P_k^\mu &= -i\partial_k^\mu \\ K_k^\mu &= -i\left(2x_k^\mu x_k^\nu - \eta^{\mu\nu} x_k^2\right)\partial_{k,\nu} - 2i\triangle_k x_k^\mu \\ L_k^{\mu\nu} &= ix_k^\mu \partial_k^\nu - ix_k^\nu \partial_k^\mu \end{cases}$$

↓

conformal cross-ratios

$$I_G(\underline{a}) = V_G(\underline{a})\phi(z_1, z_2, \ldots)$$

prefactor — conformally-invariant function

[Corcoran, Loebbert, Miczajka; 2021]

## Level-1 Yangian constraints

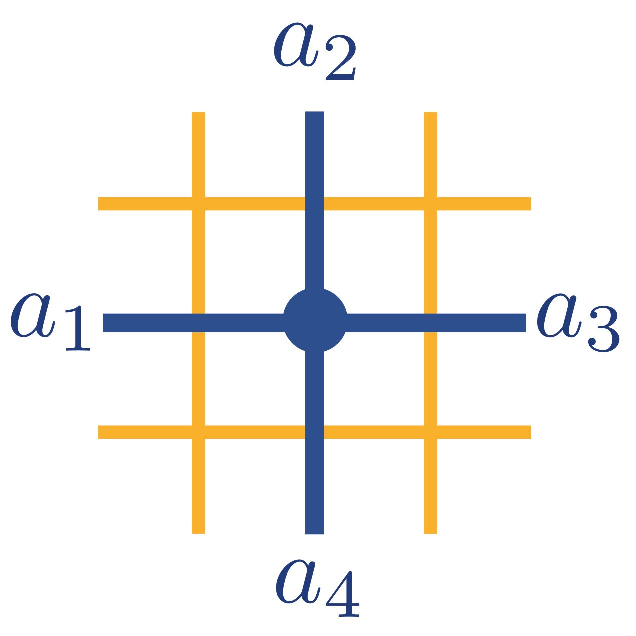
$$\hat{J}^A = \frac{1}{2} f^A_{BC} \sum_{k=1}^{n} \sum_{j=1}^{k-1} J_j^C J_k^B + \sum_{a=1}^{n} s_j J_j^A$$

↓

**Set of partial differential equations:**

$$\mathrm{PDE}_{jk}\phi = 0 \quad 1 \leq j < k \leq n$$

**D=2**

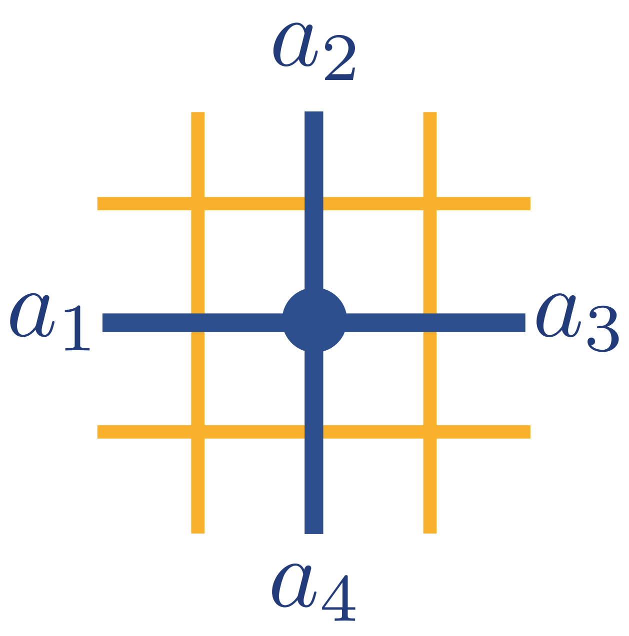


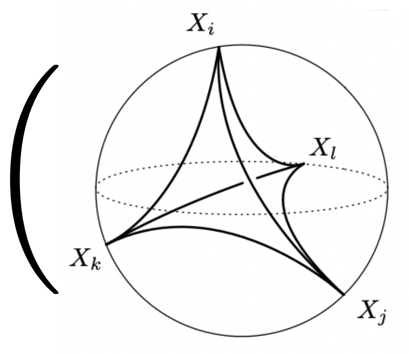

$$= \int \frac{d^2 y}{|y-a_1|\,|y-a_2|\,|y-a_3|\,|y-a_4|} \quad \text{with } z = \frac{(a_1-a_2)(a_3-a_4)}{(a_1-a_3)(a_2-a_4)}$$

$$= \frac{4}{\pi}\frac{1}{|a_2-a_4||a_1-a_3|}\left[K(z)K(1-\bar{z}) + K(\bar{z})K(1-z)\right]$$ [Corcoran, Loebbert, Miczajka; 2021]

periods of the torus — a Calabi-Yau 1-fold

**D=4**

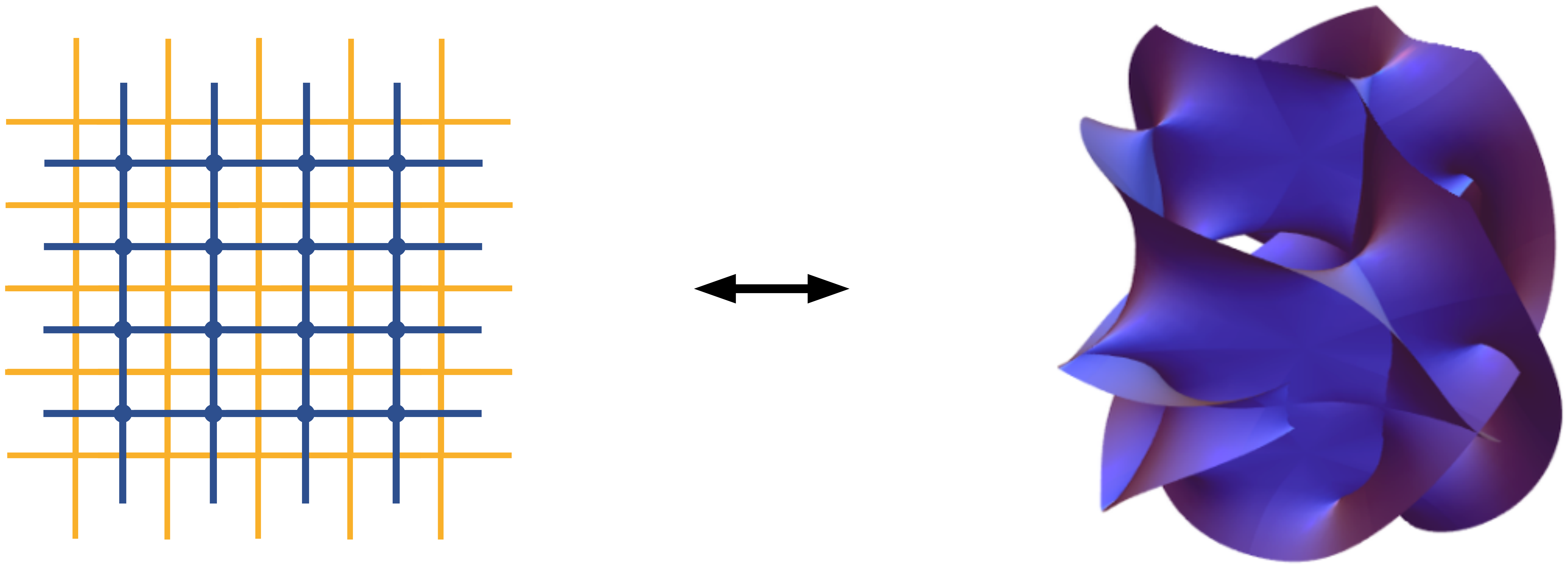



$$\sim \frac{1}{\sqrt{\Delta}} \mathrm{Vol}\Big($$  $$\Big)$$

➡ Could we write the higher-loop fishnet integrals in terms of periods of Calabi-Yau l-folds? Monodromy invariance?

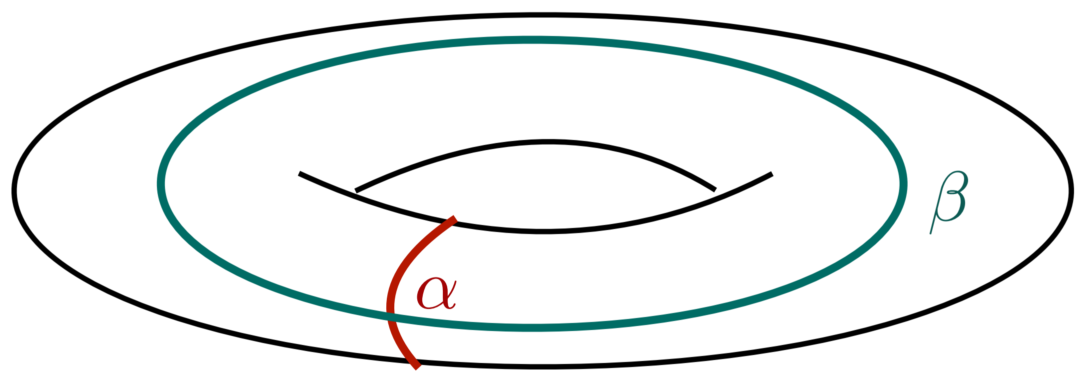
➡ Is there a volume interpretation for general fishnets?

## Question 1: Can we relate fishnet integrals and Calabi-Yau geometries?



$$I_G(\underline{a}) = \int_{\mathbb{C}^l} \left( \prod_{j=1}^{l} \frac{d\bar{x}_j \wedge dx_j}{2\pi} \right) \frac{1}{\sqrt{P_G(\underline{x}, \underline{a})}\sqrt{P_G(\bar{\underline{x}}, \underline{a})}} \quad \text{with} \quad P_G(\underline{x}, \underline{a}) = \left[ \prod_{i,j} (x_i - x_j) \right] \left[ \prod_{i,j} (x_i - a_j) \right]$$

## Torus

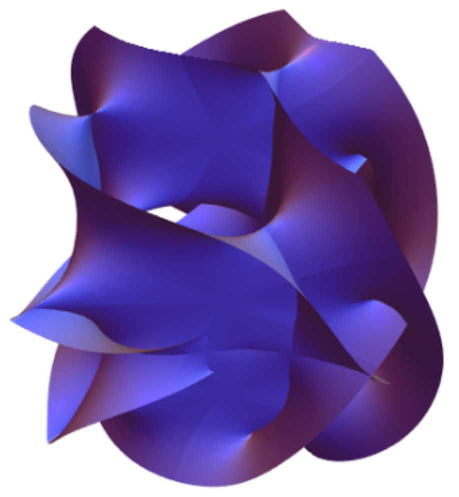



Elliptic curve defined by $y^2 = f(x)$.

⟶ Differential form: $\omega = \frac{dx}{y} = \frac{dx}{\sqrt{f(x)}}$

⟶ Periods: $\int_\alpha \omega$ and $\int_\beta \omega$

## Calabi-Yau geometry



Calabi-Yau l-fold defined by $y^2 = P_G(\underline{x}, \underline{a})$.

⟶ Differential form: $\Omega = \frac{\mu_B(\underline{x})}{\sqrt{P_G(\underline{x}, \underline{a})}}$

⟶ Periods: $\Pi_i = \int_{\Gamma_i} \Omega$

holomorphic measure on the projective base space

$$I_G(\underline{a}) \sim \int_{M_G} \Omega \wedge \bar{\Omega} \sim \Pi^+ \Sigma \Pi$$

**Question:** Can we relate fishnet integrals and Calabi-Yau geometries?

$$I_G(\underline{a}) \sim \int_{M_G} \Omega \wedge \bar{\Omega} \sim \Pi^{+}\Sigma\Pi \sim e^{-K}$$

**Claim:**

A fishnet integral is related to the Kähler potential $K(z)$ of a specific Calabi-Yau geometry defined by $P_G(\underline{x}, \underline{a})$.

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**Vector of periods** $\Pi_i$ **:** Solutions of the **Picard-Fuchs ideal**

$$I_G(\underline{a}) \sim \int_{M_G} \Omega \wedge \bar{\Omega} \sim \Pi^{+} \Sigma \Pi \sim e^{-K}$$

**Intersection matrix :** Obtained from Griffiths transversality

## Question 2:

**Is there a connection between the symmetries of fishnet integrals and their related geometries?**

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## Torus:

Legendre-family of elliptic curves: $y^2 = x(x-1)(x-z)$ with periods $\int_\alpha \frac{dx}{y}$ and $\int_\beta \frac{dx}{y}$

⟶ Picard-Fuchs differential operator: $\mathcal{L}_{\text{Leg}} = \theta^2 - z\left(\theta + \frac{1}{2}\right)^2$ with $\theta = z\partial_z$

## Calabi-Yau l-fold:

Family of Calabi-Yau l-folds defined by $y^2 = P_G(\underline{x}, \underline{a})$ with periods $\int_{\Gamma_i} \Omega$

⟶ Picard-Fuchs differential operators: **?? — Non-trivial!**

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$$I_G(\underline{a}) = V_G(\underline{a})\phi_G(\underline{a}) \sim \Pi^{+}\Sigma\Pi \sim e^{-K}$$

**Physics** ⟷ **Geometry**

Picard-Fuchs ideal from the **symmetries** ?

**Yangian symmetry** $Y(\mathfrak{sl}_2(\mathbb{R}))$: Differential operators L from the holomorphic part of the Yangian invariance:

$$L\phi_G(z) = 0 \quad \Rightarrow \quad \Pi^{+}\Sigma(L\Pi) = 0$$

**Permutation symmetries** $S_G \subset S_n$: Additional differential operators $\sigma \cdot L$ with $\sigma \in S_G$ : $(\sigma \cdot L)\Pi = 0$

$$Y_S(\mathfrak{sl}_2(\mathbb{R})) = S_G \cdot Y(\mathfrak{sl}_2(\mathbb{R})) \longrightarrow \textbf{Picard-Fuchs ideal}$$

**We find empirically that** $Y_S(\mathfrak{sl}_2(\mathbb{R}))$ **generates the Picard Fuchs ideal of the integral.**

# Question 2 — Connection between the symmetries of fishnet integrals and their related geometries?

## Question 2:

**Is there a connection between the symmetries of fishnet integrals and their related geometries?**

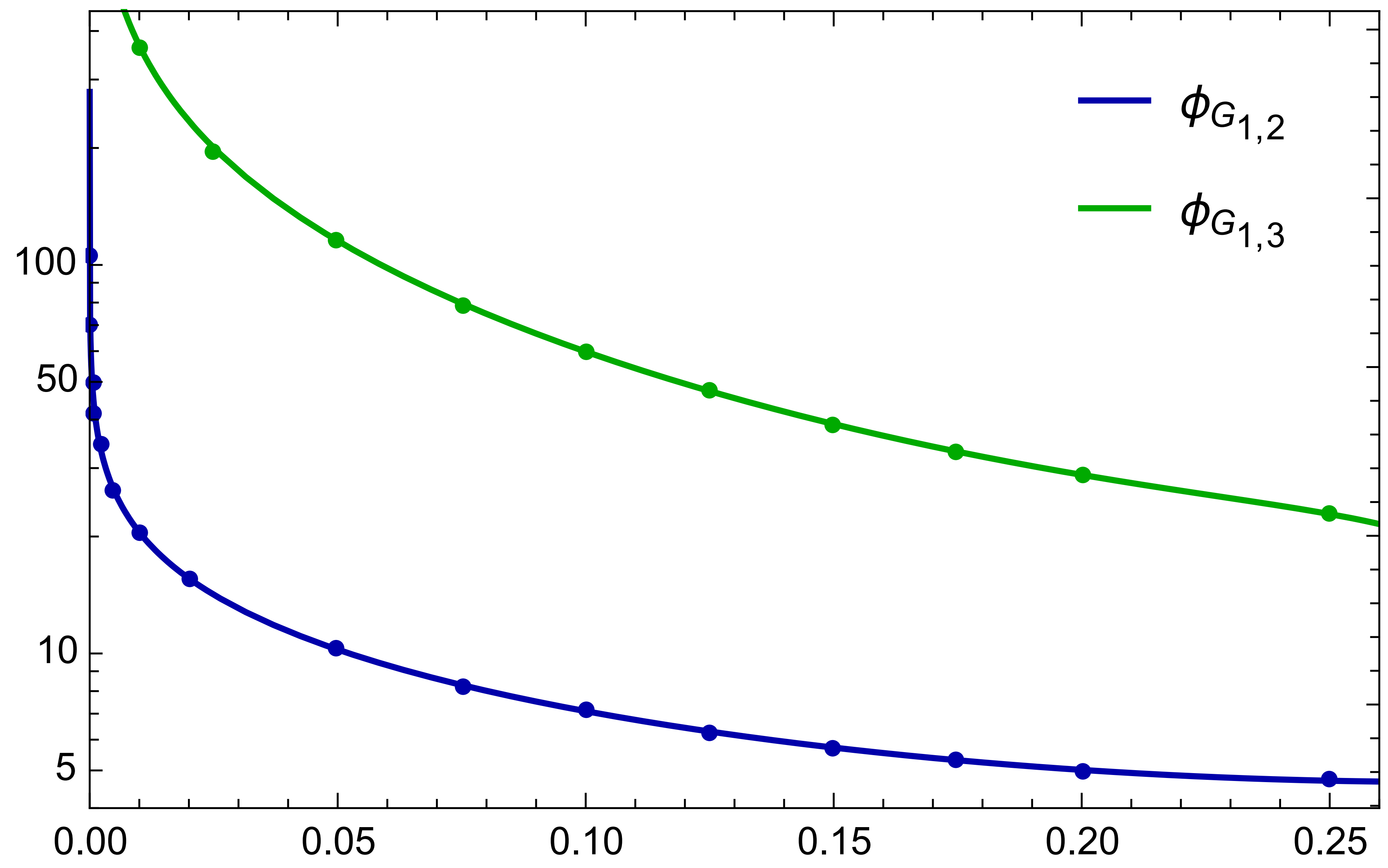
$$I_G(\underline{a}) = V_G(\underline{a})\phi_G(\underline{a}) \sim \Pi^+\Sigma\Pi \sim e^{-K}$$

**Physics** ⇄ **Geometry**

Picard-Fuchs ideal from the **symmetries !**

The ideal of differential operators generated by $\mathcal{S}_G \cdot Y(\mathfrak{sl}_2(\mathbb{R}))$ is equivalent the Picard Fuchs ideal of the periods on $M_G$.

# Results

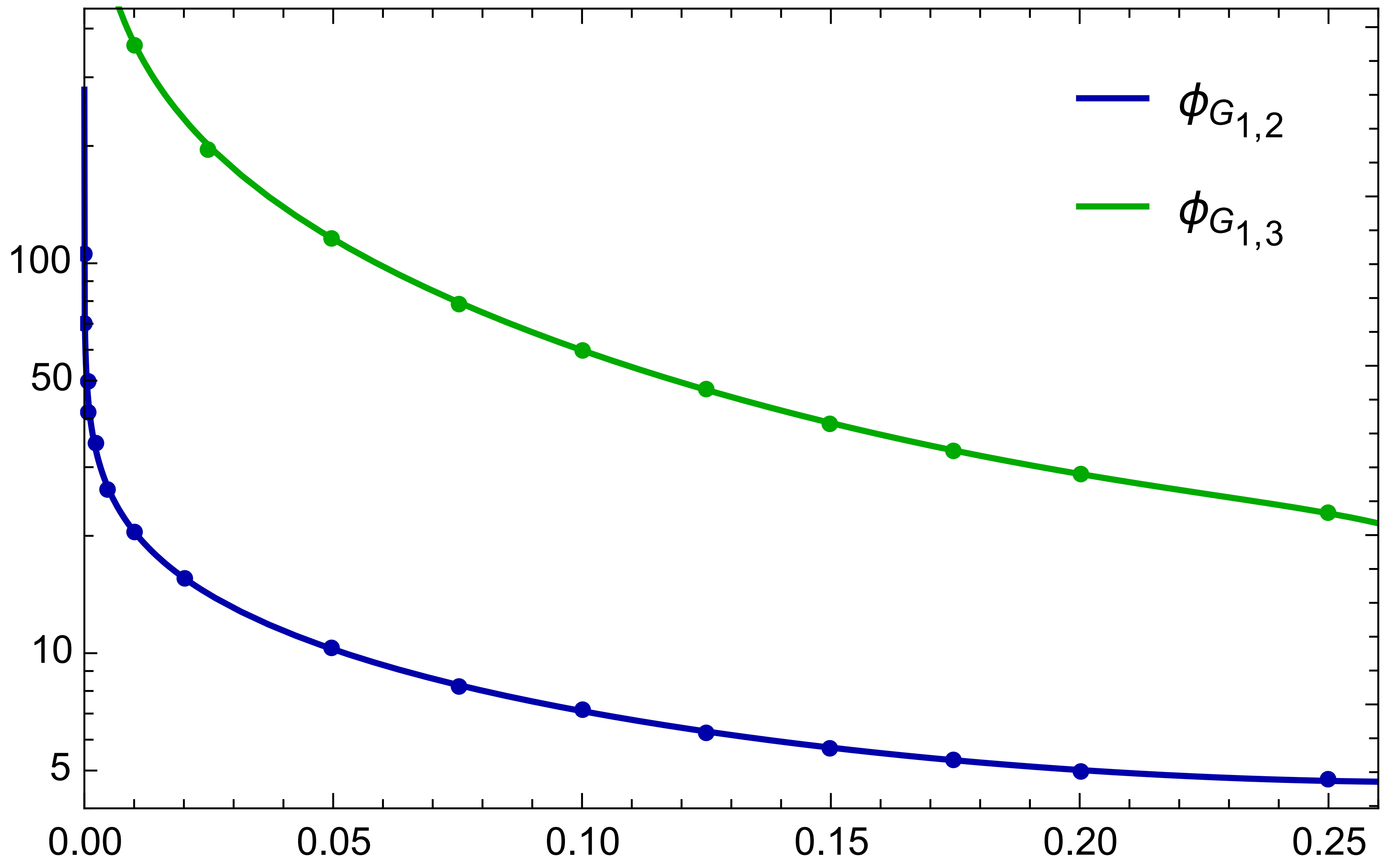


# Results — List

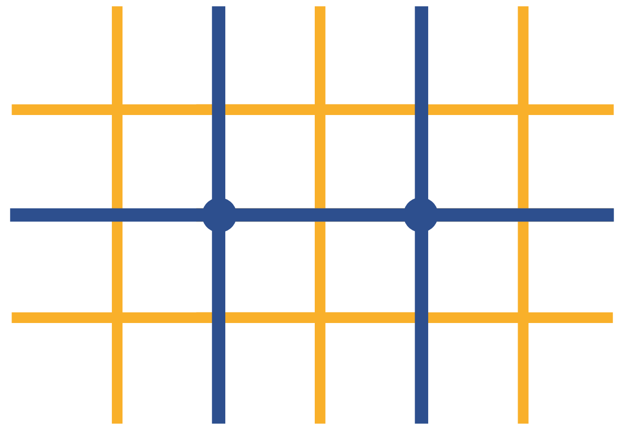
- One-parameter traintracks up to 5 loops ( *agree with known result* [Derkachov, Kazakov, Olivucci; 2018 ] )

- Fully general 2-loop traintrack ( *new result, agrees with numerical evaluation* )
- Fully general 3-loop traintrack ( *new result, agrees with numerical evaluation* )

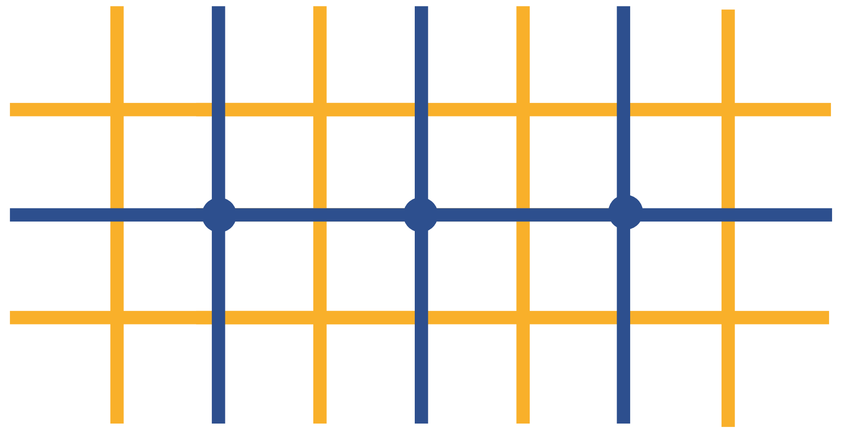
- Relations between fishnet operators




l=2



l=3



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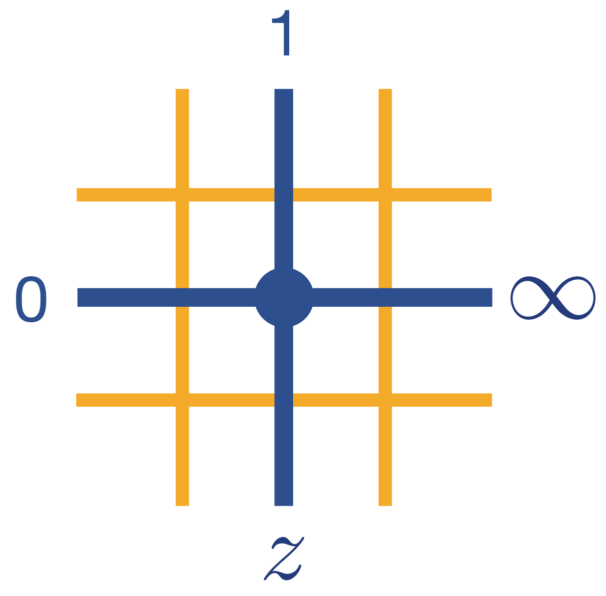
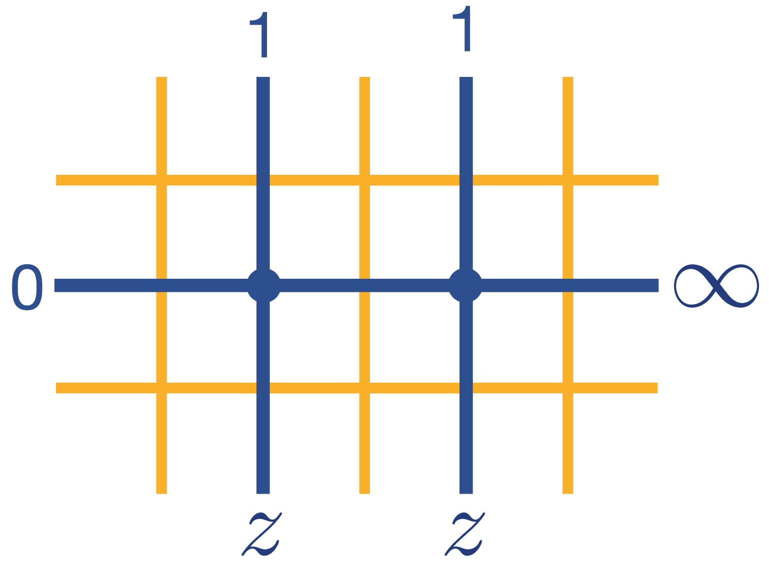
## 1-loop ladder

[Derkachov, Kazakov, Olivucci; 2018 | Corcoran, Loebbert, Miczajka; 2021]




$$I_1(\underline{a}) \sim \frac{1}{|a_{12}||a_{34}|}\left(K(z)K(1-\bar{z}) + K(1-z)K(\bar{z})\right)$$

## 2-loop ladder




$$I_2(\underline{a}) \sim \frac{1}{|a_{12}||a_{34}||a_{24}|}\left(K_+\overline{K}_- + K_-\overline{K}_+\right)^2 \text{ with } \quad K_\pm = K\left(\frac{1}{2}\left(1 \pm \sqrt{1-z}\right)\right)$$

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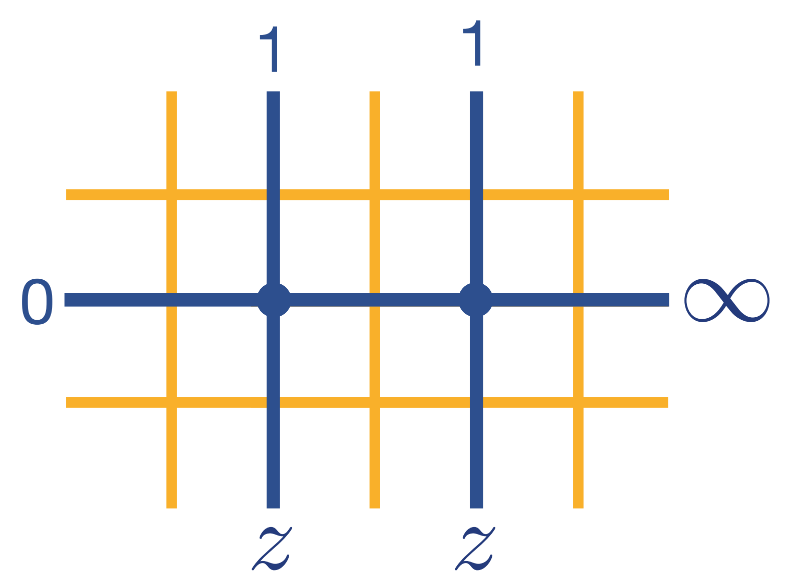
## 2-loop ladder




$$I_2(\underline{a}) \sim \frac{1}{|a_{12}||a_{34}||a_{24}|}(K_+\overline{K}_- + K_-\overline{K}_+)^2 \text{ with } \quad K_\pm = K\left(\frac{1}{2}\left(1 \pm \sqrt{1-z}\right)\right)$$

Every Calabi-Yau operator of degree three is equivalent to the symmetric square of a Calabi-Yau operator of degree two.

⟶ The solutions space of the Picard-Fuchs operators for the periods in the two-loop fishnet is spanned by $(K_+^2, K_+K_-, K_-^2)$ .

[Michael Bogner; 2013 ]

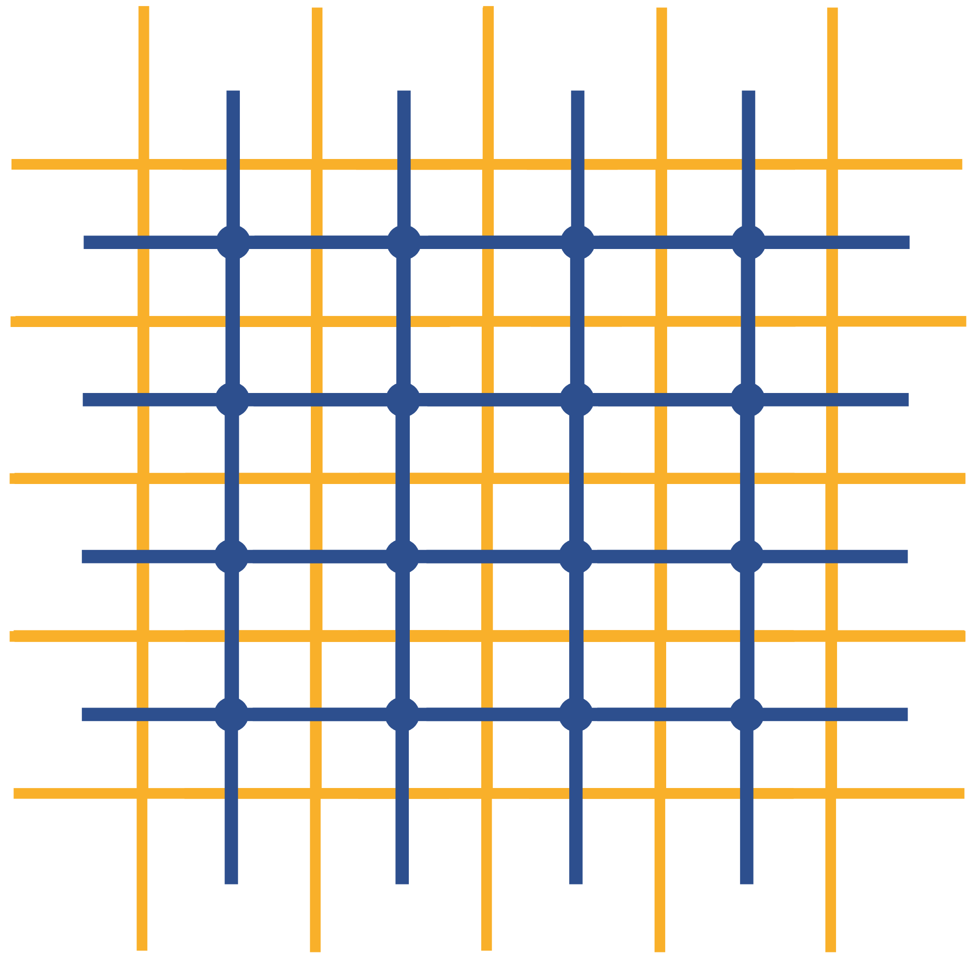
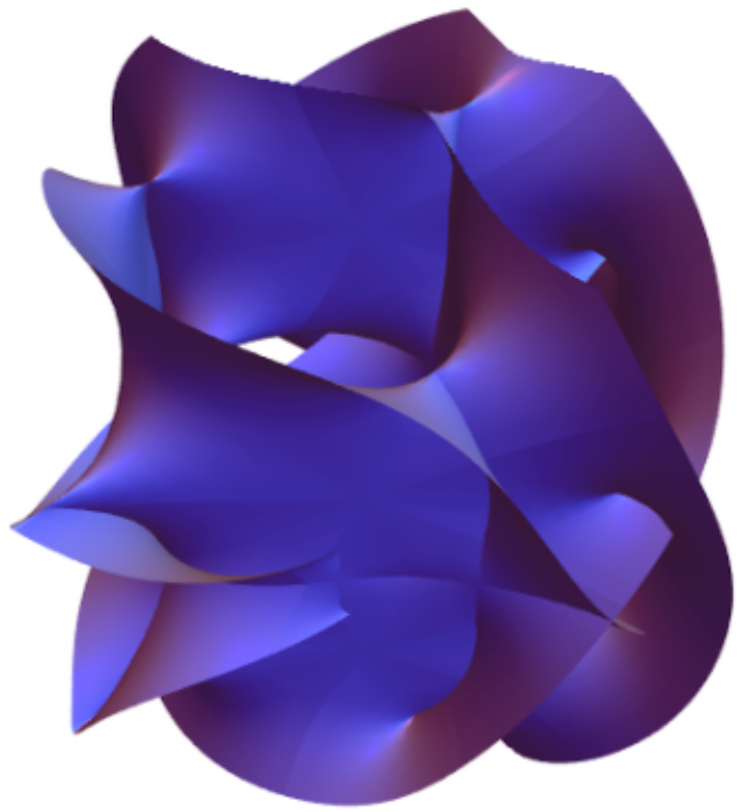
## n-loop ladder with n > 2

It is not possible to express the periods in terms of elliptic integrals anymore but in terms of Hadamard products of elliptic functions.

**Symmetric square:** For a differential operator $L_2 = \partial_z^2 + a\partial_z + b$ : $L_3 = \partial_z L_2 + 2aL_2 + 2(n-1)b\partial_z$

## Question 3:

**Can we interpret fishnet integrals as volumes of Calabi-Yau l-folds?**

 $\longleftrightarrow$ $\mathrm{Vol}_Q\Big($  $\Big)$

There is no canonical metric on the Calabi-Yau $(M_G, \Omega)$.

↓

Consider the mirror $(W_G, \omega)$ with the classical volume:

$$\mathrm{Vol}_{\mathrm{clas}}(W_G) = \int_{W_G} \frac{\omega^l}{l!}$$

Interpretation of the fishnet integral as a classical volume of the mirror Calabi-Yau: **only for l=1,2.**

Interpretation of the fishnet integral as **the quantum volume** of a mirror Calabi-Yau.

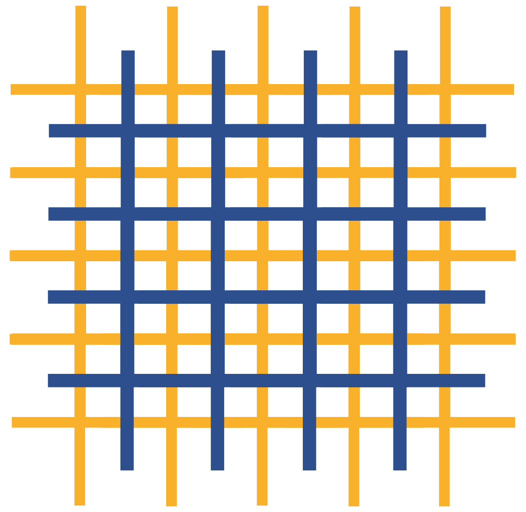
↓

Defined by three axioms:

- The quantum volume is a positive real quantity.
- The quantum volume extends uniquely over the moduli space of the Calabi-Yau $M_G$ .
- The quantum volume approaches the classical volume at the MUM-point.

$$I_G(\underline{a}) \sim \Pi^+ \Sigma \Pi \sim \mathrm{Vol}_Q(W_G)$$

# Summary - Dictionary

| l-loop fishnet integral | Calabi-Yau l-fold |
|---|---|
|  $\dfrac{\prod_{i=1}^{l} dx_i \wedge d\bar{x}_i}{\lvert P_G(\underline{x},\underline{a})\rvert}$ | Calabi-Yau l-fold defined by $y^2 = P_G(\underline{x},\underline{a})$ |
| $Y_S(\mathfrak{sl}_2(\mathbb{R})) = S_G \cdot Y(\mathfrak{sl}_2(\mathbb{R}))$ | Picard-Fuchs ideal |
| $I_G(a) = \displaystyle\int_{\mathbb{C}^l} \left( \prod_{j=1}^{l} \frac{d\bar{x}_j \wedge dx_j}{2\pi} \right) \frac{1}{\lvert P_G(\underline{x},\underline{a})\rvert}$ | $\left\{ \begin{array}{l} e^{-K} \\ \mathrm{Vol}_Q(W_G) \end{array} \right.$ <br> $W_G$ = mirror of $M_G$ |