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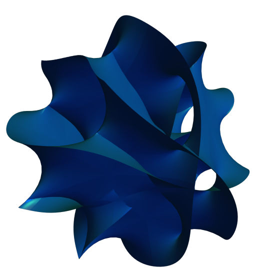
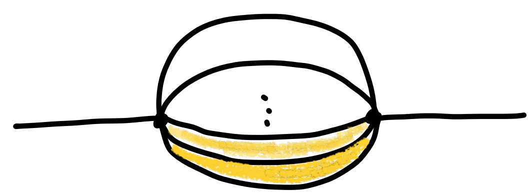
bctp Bethe Center for Theoretical Physics

Calabi-Yau Geometries and Feynman Integrals

Bananas & Ice Cones

Christoph Nega





Joint work with:
Kilian Bönisch, Claude Duhr, Fabian Fischbach, Albrecht Klemm, Reza Safari & Lorenzo Tancredi

"Ice Cone Graphs and Iterated Calabi-Yau Period Integrals" [1], "Feynman Integrals in Dimensional Regularization and Extensions of Calabi-Yau Motives" [2], "Analytic structure of all Banana integrals" [3], "The l-loop Banana amplitude from GKZ systems and relative Calabi-Yau periods" [4]

Elliptic Integrals in Fundamental Physics
Mainz
September 14, 2022

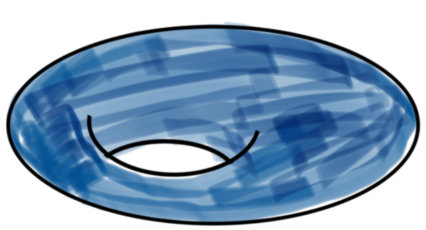
Motivation

- **Feynman integrals** are cornerstone of perturbative QFT and necessary for predictions in collider and gravitational wave experiments.
- High precision measurements require **multi-loop** Feynman integral computations.
- In Feynman integral computations **special functions** and **their properties** are needed.
- There are many examples at two-loop order where **elliptic functions** show up.
- Usually these functions are properly defined on certain **geometries**, e.g. $K(\lambda)$ on elliptic curve $\mathcal{E}$.

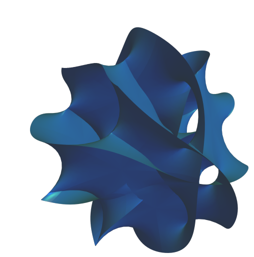
→ **Interplay** between geometry and special functions

- For loop orders $l \geq 2$ also **more complicated** geometries than elliptic curves appear.

→ a natural candidate is a **Calabi-Yau geometry**



Generalization →



Today

World of Calabi-Yaus

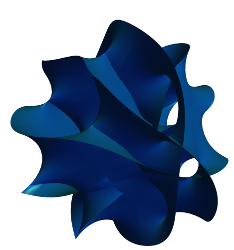
$\int_\gamma \Omega$

Mirror Symmetrie

moduli spaces

$\int_M \Omega \wedge \bar{\Omega}$

$\mathrm{Vol}_\Omega(W)$



$\hat{\Gamma}$-class

Picard-Fuchs equations

M

γ

period integrals

$\pi^\dagger \Sigma \pi$

$c_1 = 0$

$\int_M \Omega \wedge \partial_z^k \Omega$

Feynman Integrals

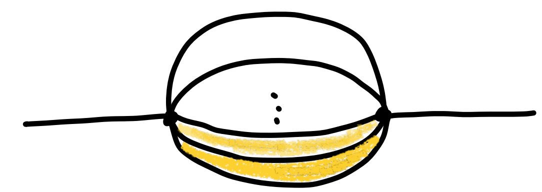


Table of Content

Calabi-Yau Manifolds

Definition:

A **Calabi-Yau** (CY) n-fold X is a complex n-dimensional Kähler manifold equipped with a Kähler $(1,1)$-form ω. There are the (equivalent) additional properties:

- the first Chern class vanishes: $c_1(T_X) = 0$
- there exists a Ricci flat metric g: $R_{i\bar{j}}(g) = 0$
- there exists a no-where vanishing holomorphic $(n,0)$-form Ω
- the holonomy group of X is $\mathrm{SU}(N)$
- on X there exist two covariant constant spinors.

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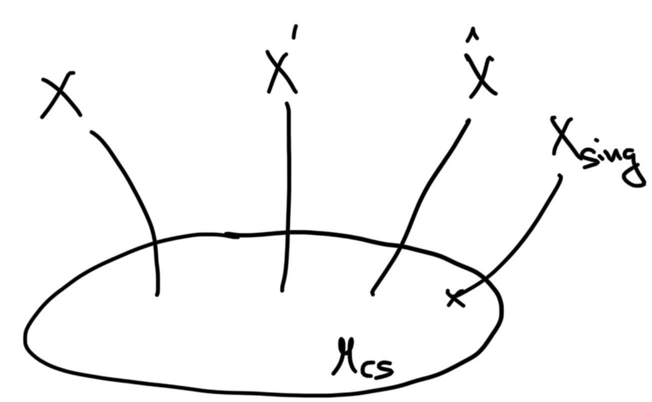
- Forms Ω and ω are both **characteristic** for a CY X $\longrightarrow$ (X, Ω, ω) cf. $(\mathcal{E},\ \mathrm{d}x/y,\ \mathrm{d}x \wedge \mathrm{d}y)$
- The **tangent space** of the **complex structure deformation space** of a CY $\mathcal{M}_{\mathrm{cs}}$ is given by $H^{n-1,1}(X)$.
- It is natural to consider **families** of CYs:



Constructions of CYs

How can we construct CYs?

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- CYs can be defined via **polynomial constraints**:

> "Vanishing of the first Chern class $c_1(T_X)$ gives relation between ambient space and degree of the constraints."

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- By a single polynomial constraint: **Hypersurface CY**

Cubic one-fold: $\{Y^2Z - 4X^3 + g_2(t)XZ^2 + g_3(t)Z^3 = 0\} \subset \mathbb{P}^2$

Quintic three-fold: $\{X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5 - \psi X_0X_1X_2X_3X_4 = 0\} \subset \mathbb{P}^4$

- By many polynomial constraints: **Complete Intersection CY**

One-fold as two quadrics: $\{X^2 + Y^2 - \lambda ZW = Z^2 + W^2 - \lambda XY = 0\} \subset \mathbb{P}^3$

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- CY ambient spaces can be very general: projective spaces, weighted projective spaces, toric spaces, ...

Comments on Constructions of CYs

- Also more general constructions are possible, e.g. non-linear sigma models.

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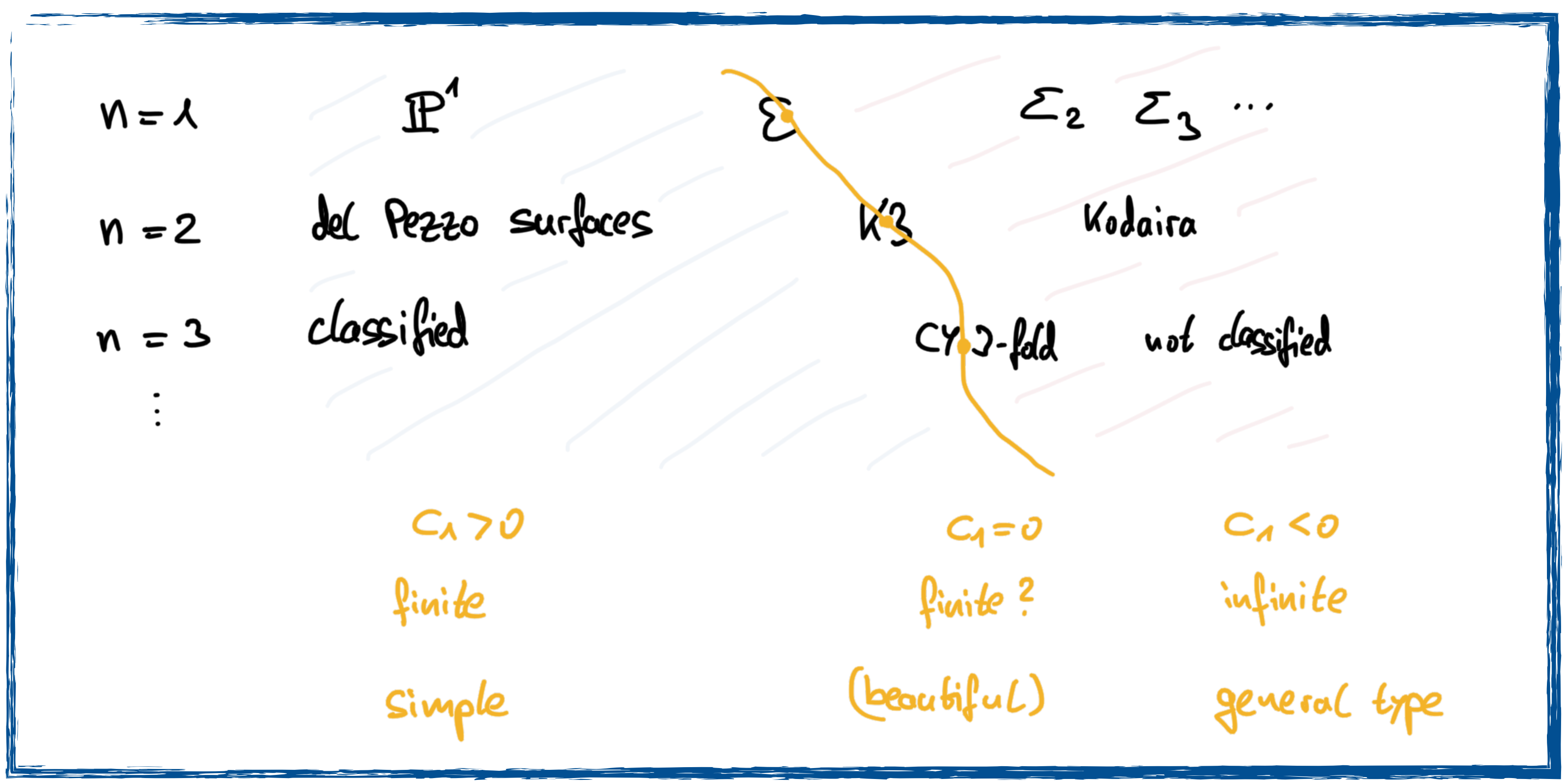

[Kodaira]

[Iskovskih, Mori, Mukai]

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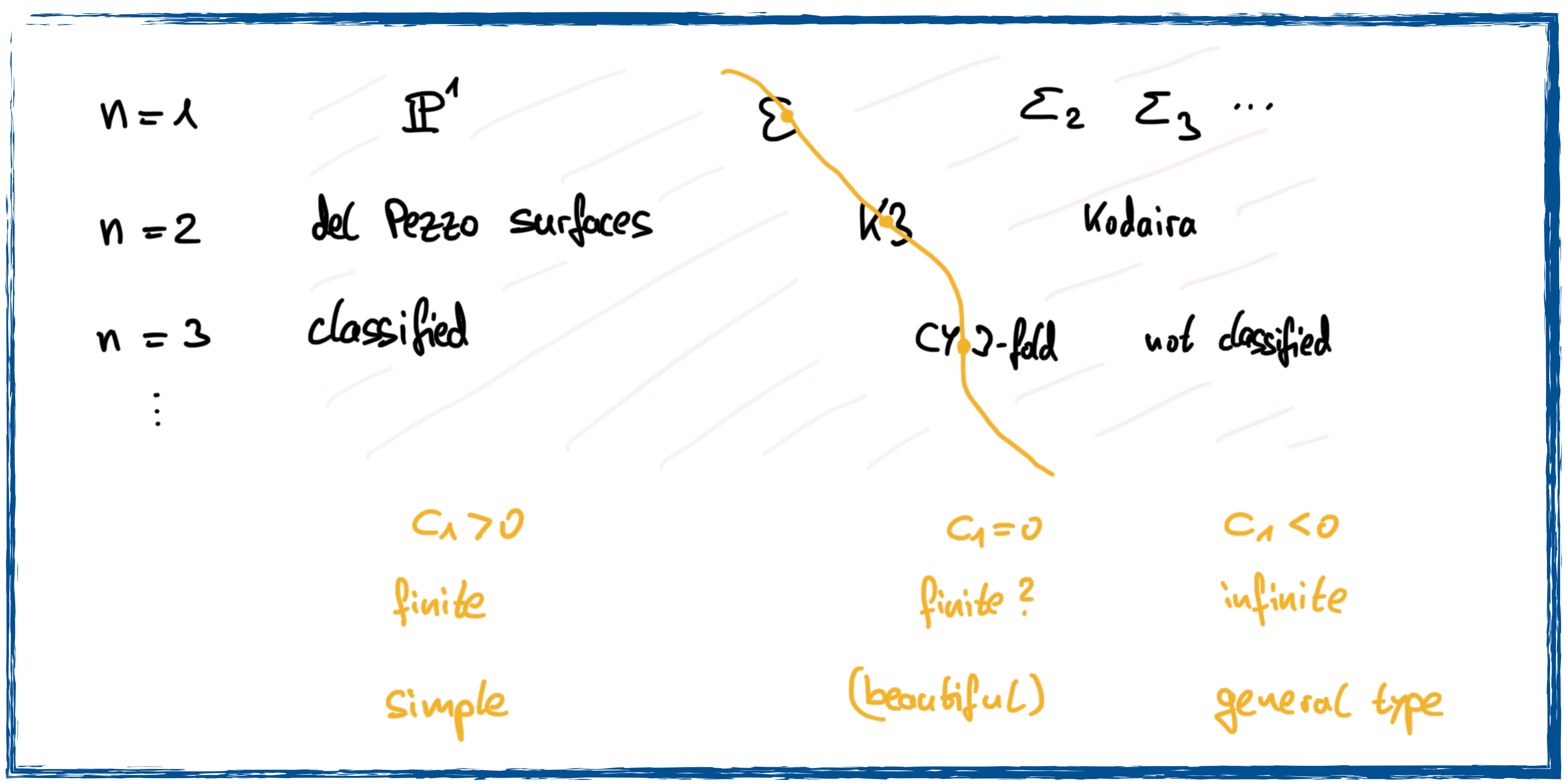

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[Iskovskih, Mori, Mukai]

→ CYs live between "simple" and "general type" varieties.

Periods of a CY

Definition:

Periods define a pairing between the homology $H_n(X,\mathbb{Z})$ and the cohomology $H^n(X,\mathbb{C})$ of the CY X:

$$\begin{aligned} \Pi: \quad H_n(X,\mathbb{Z}) \times H^n(X,\mathbb{C}) &\longrightarrow \mathbb{C} \\ (\Gamma,\alpha) \qquad\qquad &\longmapsto \int_\Gamma \alpha \end{aligned}$$

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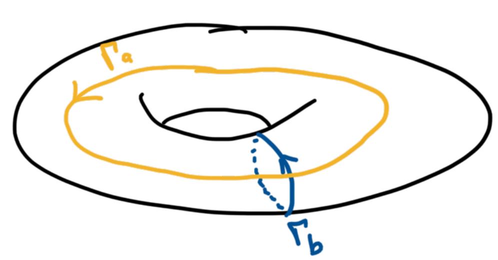

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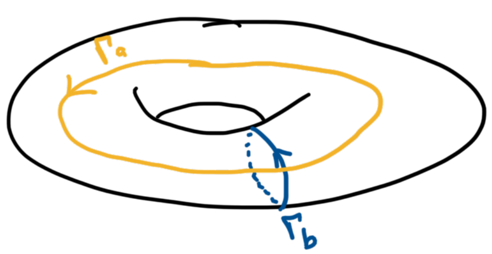

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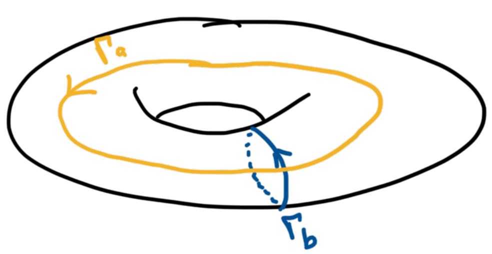

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Elliptic integrals
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"Periods describe the shape of a CY."

- Particularly interesting are the periods over Ω, which can be defined through the defining constraints:

$$\Omega = \int_{S^1} \frac{\mu}{P} \quad \longrightarrow \quad \Pi_i = \int_{\Gamma_i} \Omega \qquad \text{cf.} \quad \Omega = \int_{S^1} \frac{\mathrm{d}X \wedge \mathrm{d}Y}{P_3} \sim \frac{\mathrm{d}X}{Y}$$

- For generic CYs it is not even simple to explicitly define all cycles $\Gamma_i \in H_n(X,\mathbb{Z})$.

Computing Periods

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- There are different techniques to find these differential equations:
  - **Integration by Parts** identities, **Griffiths reduction method** or **GKZ** approach
  - Via the **torus period**: $\Pi_0 = \int_{T^n} \Omega$

    i) Perform a residue calculation to obtain Π_0.

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e.g. for elliptic curve:

$$\int_{T^1} \frac{\mathrm{d}X}{\sqrt{X(X-1)(X-\lambda)}} = \sum_{m,n} \binom{2m}{m}\binom{2n}{n} \int_{T^1} \frac{\mathrm{d}X}{X} \left(\frac{X}{4}\right)^m \left(\frac{\lambda}{4X}\right)^n = 2\pi i \sum_{n=0}^{\infty} \binom{2n}{n}^2 \left(\frac{\lambda}{4^2}\right)^n \sim K(\lambda)$$

$$\mathcal{L}_{\mathrm{Leg}} = 4(1-\lambda)\theta^2 - 4\lambda\theta - \lambda \qquad \text{with} \qquad \theta = \lambda \frac{\partial}{\partial \lambda}$$

Computing Periods

- A **basis of the solution space** $\{\varpi_i\}$ to these differential equations can be obtained by standard techniques, e.g. **Frobenius Method**.

- This is particularly simple if a **MUM point** (= total degeneration of indicials) exists:

logarithmic structure reflects
the cohomology of the CY

$$\begin{aligned}
\varpi_0 &= \text{power series in z} \\
\varpi_1 &= \varpi_0 \log(z) + \Sigma_1 \\
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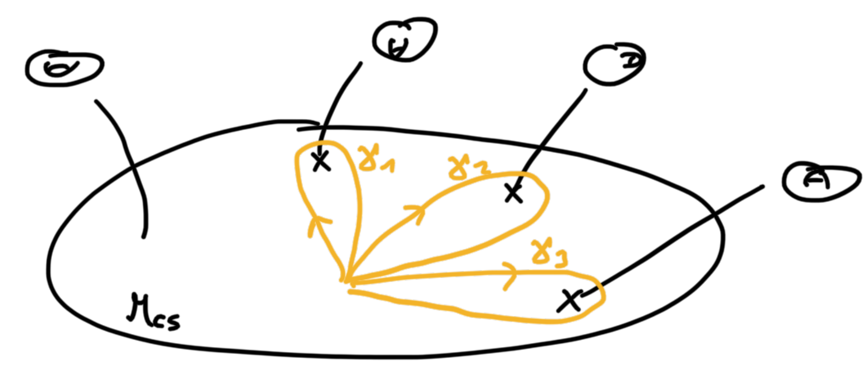
- Finally, a **basis change** from $\{\varpi_i\}$ to $\{\Pi_i\}$ (basis over $\mathbb{Z}$) has to be determined.
This change of basis can be found from **monodromy considerations**:



  - There exist special points in $\mathcal{M}_{\text{cs}}$ where the CY gets singular.

  - Analytic continuation around these points corresponds to a monodromy: $\Pi \longmapsto M_{\gamma_i}\Pi$

  - All monodromies have to respect the intersection pairing Σ between the periods.

  → In a good basis $\{\Pi_i\}$ all monodromies M_{γ_i} have to be "integral", i.e. $M_{\gamma_i} \in \mathcal{O}(\Sigma, \mathbb{Z})$

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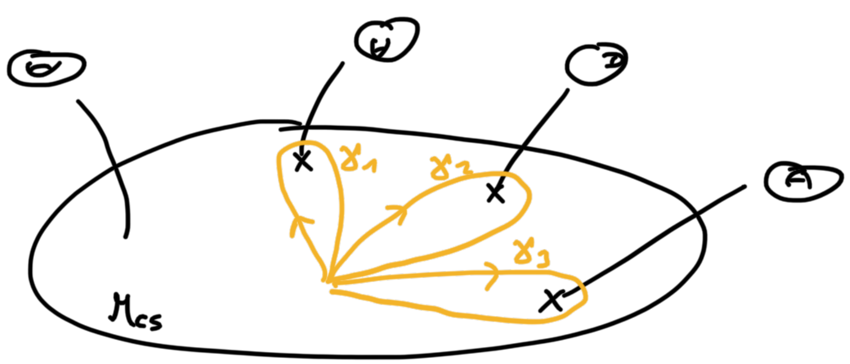

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cf. $\begin{pmatrix} K(\lambda) \\ K(1-\lambda) \end{pmatrix} = \begin{pmatrix} \pi/2 & 0 \\ 2\log(2) & -1/2 \end{pmatrix} \begin{pmatrix} \varpi_0(\lambda) \\ \varpi_1(\lambda) \end{pmatrix}$

Griffiths Transversality

- On a CY there exists the phenomenon of **Griffiths transversality**:

$$\begin{aligned}
\Omega &\in H^{n,0}(X) \\
\partial_z \Omega &\in H^{n,0}(X) \oplus H^{n-1,1}(X) \\
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\boxed{\int_X \Omega\wedge\partial_z^k\Omega = \Pi^T\,\Sigma\,\partial_z^k\Pi = \begin{cases} 0, & k<n\\ C_n, & k=n\end{cases}}
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The rational function C_n is called the Yukawa Coupling.

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The rational function C_n is called the Yukawa Coupling.

- From this we can define a whole matrix $\mathbf{Z}$ of **quadratic relations between the periods**:

$$\mathbf{Z}(z)=\mathbf{W}(z)\Sigma\mathbf{W}(z)^T \qquad \text{with the Wronskian } \mathbf{W}(z)_{i,j}=\{\partial_z^i\varpi_j\}$$ [2]

For $n=1,2$ these relations are known:

$n=1$ Legendre relations

$$\begin{vmatrix} K(\lambda) & K(1-\lambda)\\ K'(\lambda) & K'(1-\lambda)\end{vmatrix} = -\frac{\pi}{4}\frac{1}{(1-\lambda)\lambda}$$

$n=2$ K3 is a symmetric square [Bogner]

$$\{\varpi_0,\varpi_1,\varpi_2\}=\{f_1^2,f_1f_2,f_2^2\}$$

$$\mathcal{L}^{(3)}\varpi_i=0 \qquad\qquad \mathcal{L}^{(2)}f_i=0$$

Kähler Potential

- On a CY there exists a natural real, positiv and **monodromy invariant** object namely the exponential of the **Kähler potential**:

$$\boxed{i^{n^2}\int_X \Omega\wedge\bar{\Omega} = i^{n^2}\Pi^\dagger\Sigma\Pi = e^{-K(z,\bar{z})}}$$

Monodromy invariance follows from:

$$\Pi^\dagger\Sigma\Pi \longrightarrow (M_{\gamma_i}\Pi)^\dagger\,\Sigma\,M_{\gamma_i}\Pi = \Pi^\dagger M_{\gamma_i}^\dagger\,\Sigma M_{\gamma_i}\Pi = \Pi^\dagger\Sigma\Pi$$

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- So if the periods in an integer basis are known the Kähler potential is easily constructed. Σ follows from Griffiths transversality.

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- This object will be very important in Franziska Porkert's talk!

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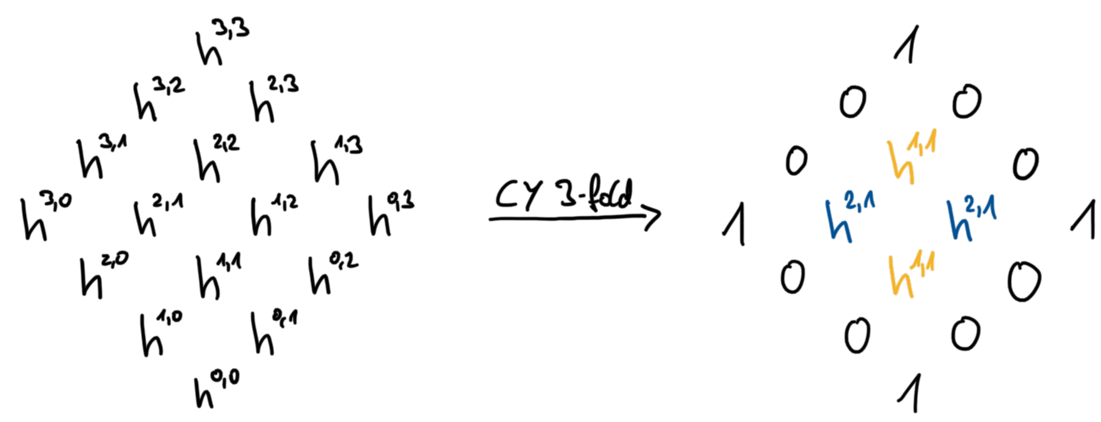
Mirror Symmetry

- Only **two** Hodge numbers for a CY three-fold are undetermined:



- These two Hodge numbers describe the **complex structure** and **Kähler deformations** of a CY:

$$h^{n-1,1} = \dim(\mathcal{M}_{\mathrm{cs}}) \qquad \text{and} \qquad h^{1,1} = \dim(\mathcal{M}_{\mathrm{Ks}})$$

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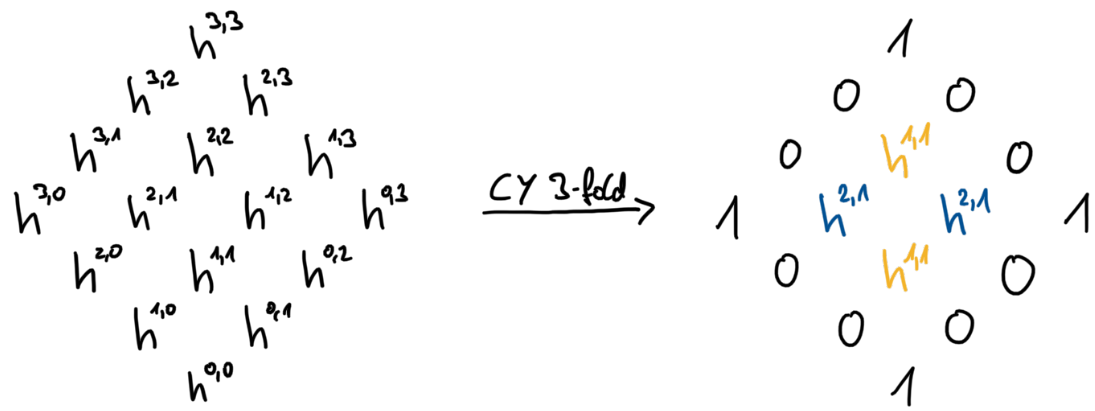

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- **Mirror symmetry** exchanges these two deformation spaces. This means that CYs come generically in **mirror pairs** (M, W) such that:

$$h^{n-1,1}(M) = h^{1,1}(W) \qquad \text{and} \qquad h^{1,1}(M) = h^{n-1,1}(W)$$

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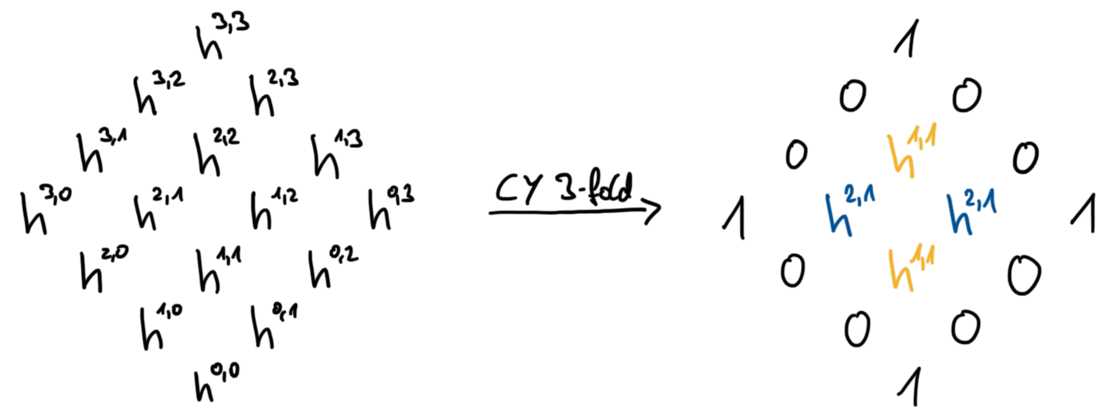

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$$h^{n-1,1}(M) = h^{1,1}(W) \qquad \text{and} \qquad h^{1,1}(M) = h^{n-1,1}(W)$$

- One very general construction of mirror pairs (M, W) is given by **Batyrev's mirror construction**: [Batyrev] [Batyrev-Borison]

$$\begin{tikzcd}
 & (P_\Delta, P_{\hat{\Delta}}) \text{ polynomial} \arrow[dr] & \\
(\Delta, \hat{\Delta}) \text{ captures combinatorial data} \arrow[ur] \arrow[dr] & & (M, W) = (\{P_\Delta = 0\} \subset \mathbb{P}_{\hat{\Delta}}, \{P_{\hat{\Delta}} = 0\} \subset \mathbb{P}_\Delta) \\
 & (\mathbb{P}_\Delta, \mathbb{P}_{\hat{\Delta}}) \text{ ambient space} \arrow[ur] &
\end{tikzcd}$$

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- Via the $\hat{\Gamma}$**-class** one can construct an integer basis of periods:
  - On the mirror CY W an integer basis is given asymptotically through a **topological integral**:

$$\Pi_{\mathcal{G}}(t) = \int_W e^{\omega t}\hat{\Gamma}(TW)\mathrm{ch}(\mathcal{G}) + \mathcal{O}(e^{-t})$$

  - Using the **mirror map** this gives the asymptotics of an integral basis on M:

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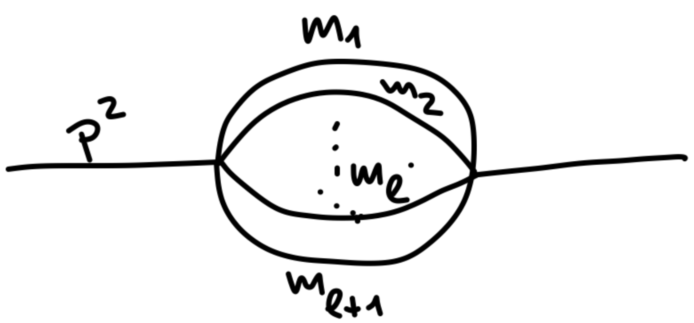
- Mirror symmetry permits an interpretation of the exponential of the Kähler potential as **quantum volume** of the mirror CY W:
  - The mirror map gives a Kähler form on W: $\quad t(z) \longrightarrow \omega_W := \mathrm{Im}(t)\,, \qquad \mathrm{Vol}_{\mathrm{cl}}(W) = \int_W \frac{\omega_W^n}{n!}$
  - The exp. of the Kähler potential is the natural positive and monodromy invariant object that has the classical volume as leading term:

$$e^{-K} = i^{n^2}\Pi^\dagger\Sigma\Pi = |\Pi_0|^2\mathrm{Vol}_q(W) \sim |\Pi_0|^2\mathrm{Vol}_{\mathrm{cl}}(W)$$

CYs and Banana Integrals

- One of the simplest families of Feynman integrals:

Banana integrals



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Which functions show
up in banana integrals?

Boundary Conditions
How do we have
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Calabi-Yau

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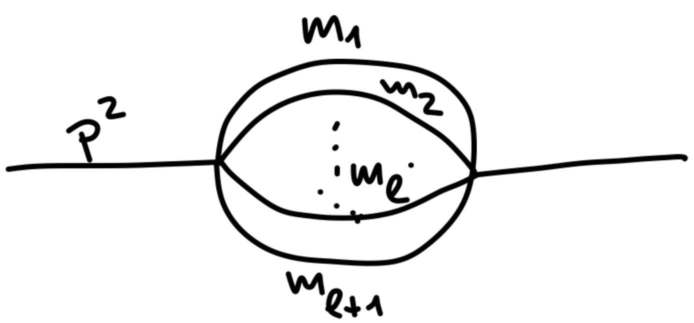

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Calabi-Yau

- **Symanzik** approach:

<u>Graph polynomials:</u>

$$\mathcal{U}(\underline{x}) = \left(\prod_{i=1}^{l+1} x_i\right)\left(\sum_{i=1}^{l+1} \frac{1}{x_i}\right)$$

$$\mathcal{F}(p^2, \underline{m}^2; \underline{x}) = \left(-p^2 + \left(\sum_{i=1}^{l+1} \frac{1}{x_i}\right)\left(\sum_{i=1}^{l+1} m_i^2 x_i\right)\right)\left(\prod_{i=1}^{l+1} x_i\right)$$

$$(x_1 : \ldots : x_{l+1}) \in \mathbb{P}^l$$

$$I_l(p^2, \underline{m}^2, D) = \int_{\sigma_l} \frac{\mathcal{U}^{l+1-\frac{l+1}{2}D}}{\mathcal{F}^{l+1-\frac{l}{2}D}}\,\mu_l$$

$@\ D = 2 - 2\epsilon$

$$= \int_{\sigma_l} \frac{1}{\mathcal{F}}\,\mu_l + \mathcal{O}(\epsilon)$$

In **two dimensions** banana integrals are particularly **simple**.

Hypersurface Calabi-Yau

- Using the second Symanzik polynomial we can associate a **Calabi-Yau variety** to the banana integrals

$$\mathcal{F}(p^2, \underline{m}^2; \underline{x}) \longrightarrow \text{Newton Polytope}$$ [4]

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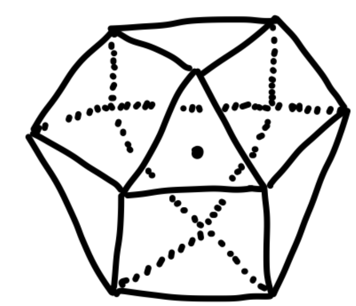
$\mathcal{F}(p^2, \underline{m}^2; \underline{x}) \longrightarrow$ Newton Polytope [4]

	$l=1$	$l=2$	$l=3$
Graph:			
generic polynomial:	$P_1 = a_0 + a_1 \frac{1}{x} + a_2 x$	$P_2 = a_0 + a_1 x_1 + a_2 \frac{1}{x_2} + a_3 \frac{1}{x_1 x_2} + a_4 \frac{1}{x_1} + a_5 x_2 + a_6 x_1 x_2$	P_3
Polyhedron:			

- From the Batyrev mirror construction we get **pairs of Calabi-Yau varieties**:

$$M_{l-1} = \{P_{\Delta_l} = 0 \subset \mathbb{P}_{\Delta_l^\star}\} \xleftrightarrow{\text{Mirror Symmetry}} W_{l-1} = \{P_{\Delta_l^\star} = 0 \subset \mathbb{P}_{\Delta_l}\}$$

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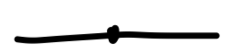
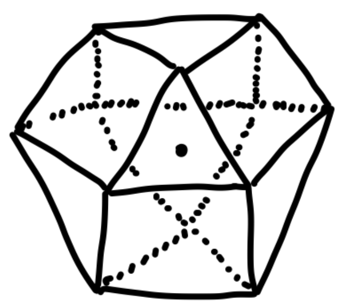
Polyhedron:			

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- Unfortunately these CYs have far **too many parameters**:

Complex moduli: $\#(\{z_i\}) = h^{l-2,1} = l^2$ vs. Physical parameters: $\#(p^2, \underline{m}^2) - 1 = l + 1$

Compete Intersection CY

- Better Approach is to analyze the "**torus period**" carefully:

$$I_l^{\max} = \int_{T^l} \frac{1}{\mathcal{F}}\,\mu_l = \ldots = (2\pi i)^{l+1} \sum_{n=0}^{\infty} \sum_{|k|=n} \binom{n}{k_1,\ldots,k_{l+1}}^2 \prod_{i=1}^{l+1} z_i^{k_i}$$

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  - There exists a nice **complete intersection CY** defined by the following constraints: [Kerr] [3]

$$P_1 = w_2^{(1)} \cdots w_2^{(l+1)} \left(1 - m_1^2 \frac{w_1^{(1)}}{w_2^{(1)}} - \ldots - m_{l+1}^2 \frac{w_1^{(l+1)}}{w_2^{(l+1)}}\right)$$

$$P_2 = w_1^{(1)} \cdots w_1^{(l+1)} \left(-p^2 + \frac{w_2^{(1)}}{w_1^{(1)}} + \ldots + m_{l+1}^2 \frac{w_2^{(l+1)}}{w_1^{(l+1)}}\right)$$

$$(w_1^{(i)} : w_2^{(i)}) \in \mathbb{P}^1_{(i)}, \quad M_{l-1}^{\mathrm{CI}} = \left\{ P_1 = P_2 = 0 \subset F_l \subset \bigtimes_{i=1}^{l+1} \mathrm{P}^1_{(i)} \right\}$$

  - Correct number of parameters: $z_i = \frac{m_i^2}{p^2}$ for $i = 1, \ldots, l+1$

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- The periods follow from a **GKZ system** defined through ℓ-vectors:

Periods of the Calabi-Yau: $\Pi_k = \int_{\Gamma_k} \Omega(z)$ with $\Gamma_k \in H_{l-1}(M_{l-1})$

$\mathcal{D}$-module: $\mathcal{D}_r\ \Pi_k = 0$ for $r = 1,\ldots,\mathrm{rank}(\{\mathcal{D}\})$

CYs and Banana Integrals

- We have still to deal with **simplex** integration domain: [3,4]

$$\partial\sigma_l \neq 0 \qquad \text{which means} \qquad \sigma_l \notin H_{l-1}(M_{l-1})$$

$\rightarrow$

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- For simplicity we now consider the one-parameter **equal-mass case.**

CYs and Banana Integrals

- The **additional special solution** can be interpreted as **iterated Calabi-Yau period**: $\mathcal{L}_l I_l(z) = -(l+1)!z$

Using variation of parameters/constants we find: [2]

$$I_l(z) \sim \underline{\Pi}_l(z)^T \int_0^z \mathrm{d}z' \, \mathbf{W}_l(z')^{-1} \, \underline{\mathrm{Inhom}}_l(z')$$

$$\sim \underline{\Pi}_l(z)^T \boldsymbol{\Sigma}_l \int_0^z \frac{\mathrm{d}z'}{z'^2} \, \underline{\Pi}_l(z')$$

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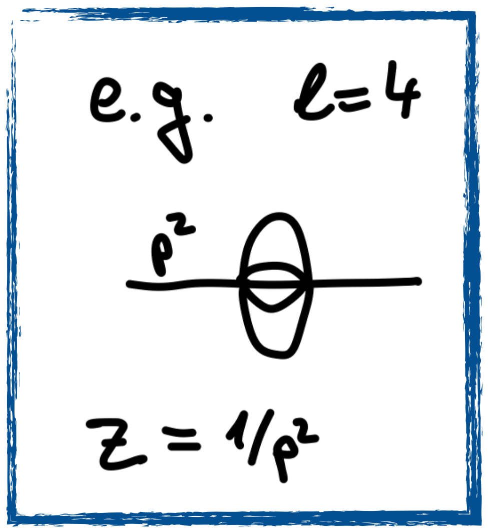
- The coefficients λ_i follow from the $\hat{\mathbf{\Gamma}}$**-class**: [Iritani]

$$\mathrm{Im}(\lambda): \quad \mathrm{Im}(I(T)) = \int_{W_{l-1}} \mathrm{e}^{\omega T} \widehat{\Gamma}(T W_{l-1}) + \mathcal{O}(\mathrm{e}^T)$$

$$\mathrm{Re}(\lambda): \quad \mathrm{Re}(I(T)) = \int_{F_l} \mathrm{e}^{\omega T} \frac{\Gamma(1-c_1)}{\Gamma(1+c_1)} \cos(\pi c_1) + \mathcal{O}(\mathrm{e}^T)$$

Analytic Structure of the Banana Integrals

Equal Mass Case:



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$$\mathcal{L}_4 = 1 - 5z + (-4+28z)\theta + (6-63z+26z^2-225z^3)\theta^2 + (-4+70z-450z^3)\theta^3 + (1-z)(1-9z)(1-25z)\theta^4$$

AESZ 34 [Almquist, Enckefort, van Straten and Zudilin]

$$\mathcal{L}_4 I_4(z) = -5!z$$

2) Frobenius basis:

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$$I_4(z) = (-450\zeta(4) - 80\zeta(3)i\pi)\varpi_0 + (80\zeta(3) - 120\zeta(2)i\pi)\varpi_1 + 180\zeta(2)\varpi_2 + 20i\pi\varpi_3 + \varpi_4$$

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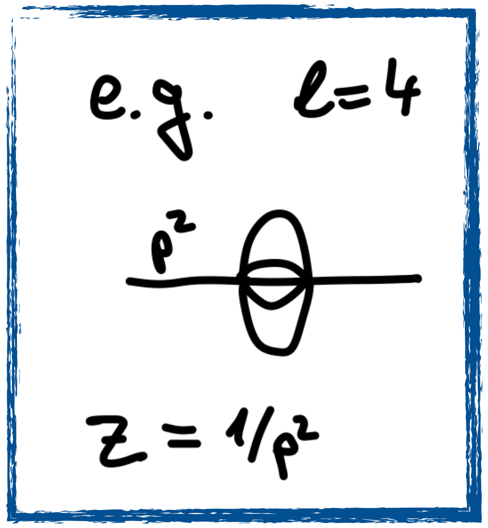

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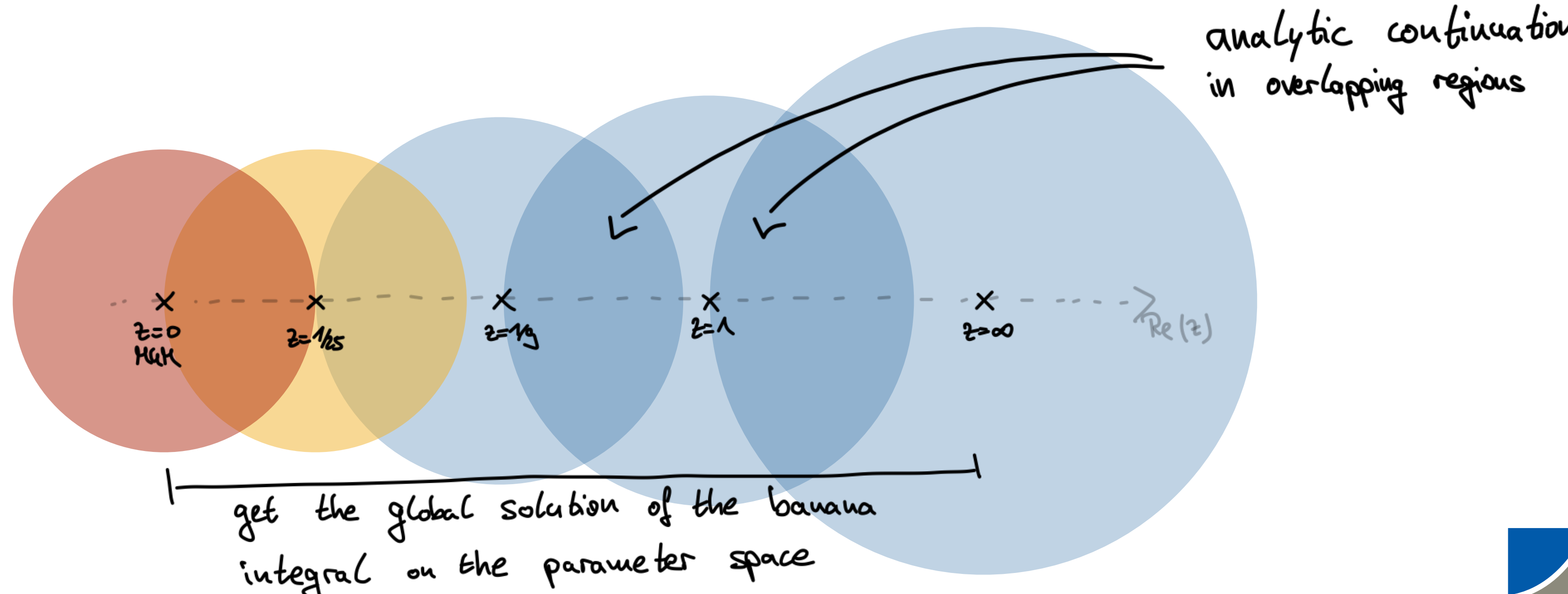

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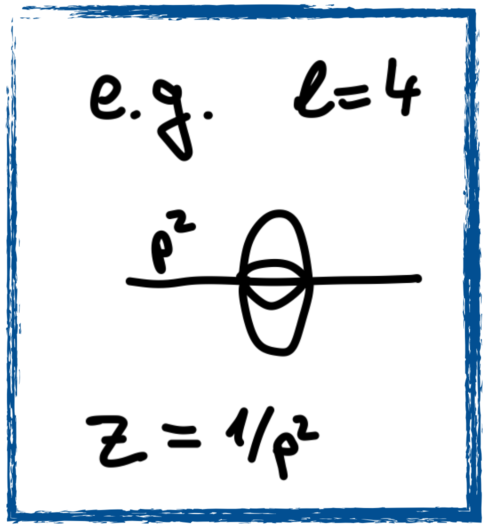

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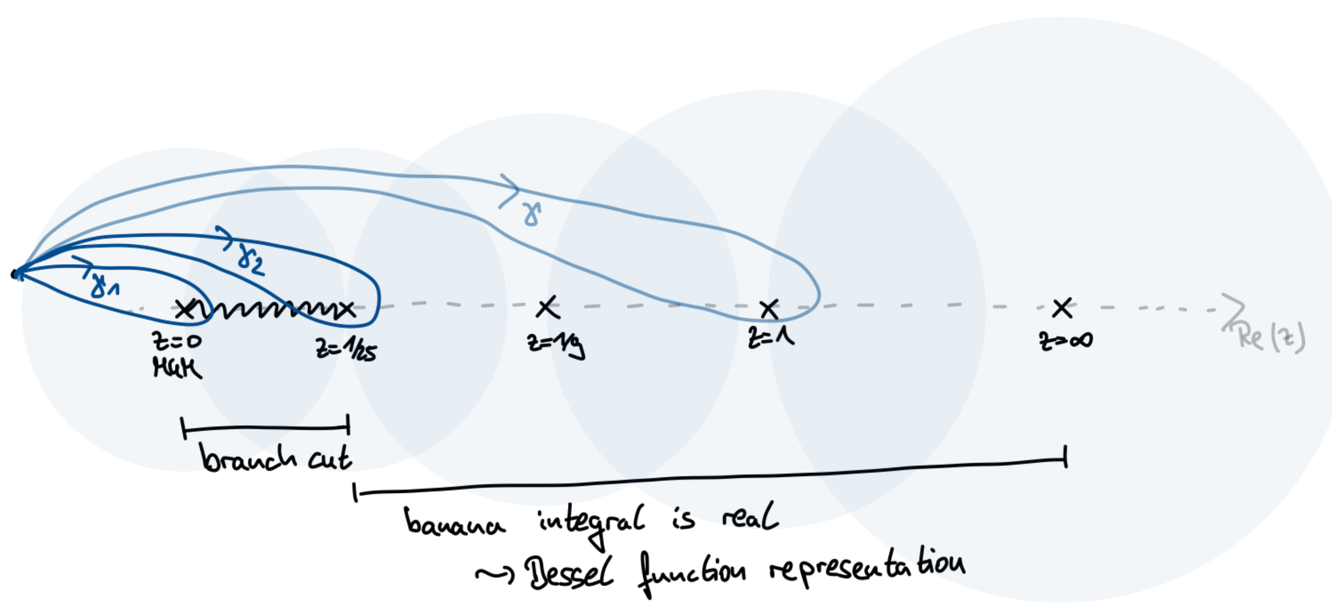

Monodromy action: $I_4 \xrightarrow{\gamma_1, \gamma_2} M_{1,2}\, I_4$

$I_4 \xrightarrow{\gamma} I_4$

as predicted by the optical theorem

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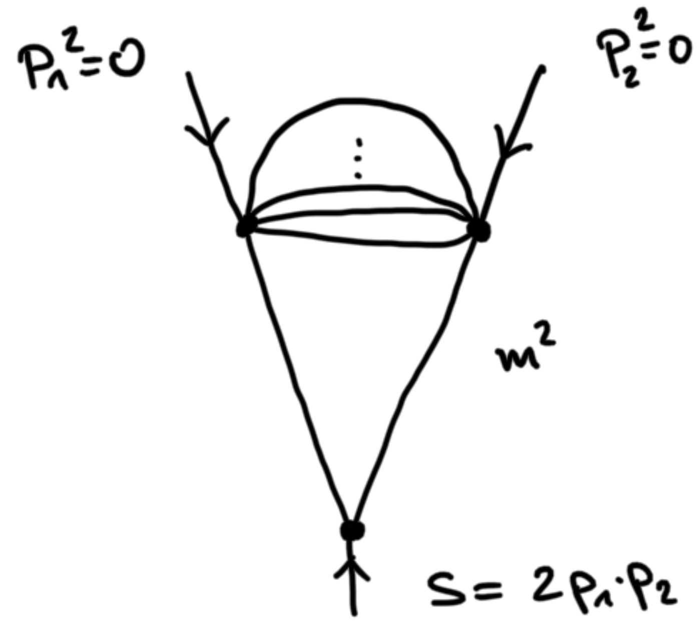
The Ice Cone Family

- Consider the following one-parameter family of **ice cone graphs** in two dimensions:



external parameters: p_1 and p_2 with $p_1^2 = p_2^2 = 0$
so we have only $s = 2p_1 \cdot p_2$

internal masses: all equal to m

$\longrightarrow$ In truth, only **one parameter** given by the ratio s/m^2 .

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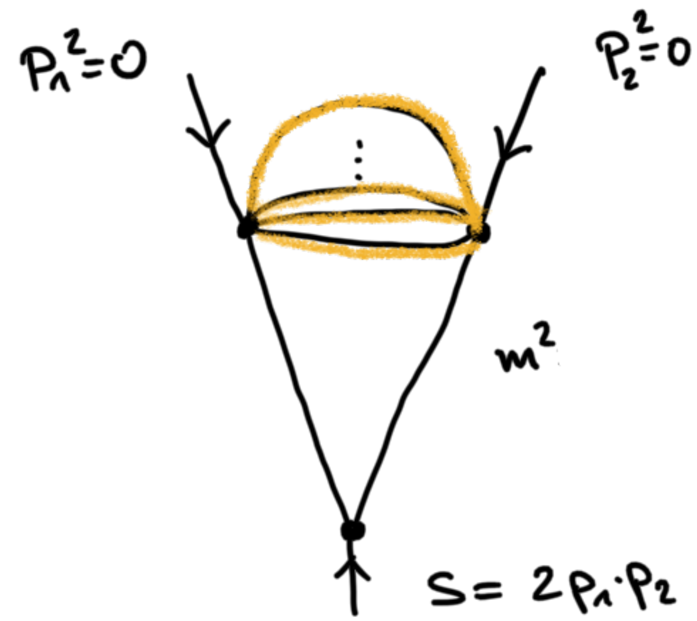

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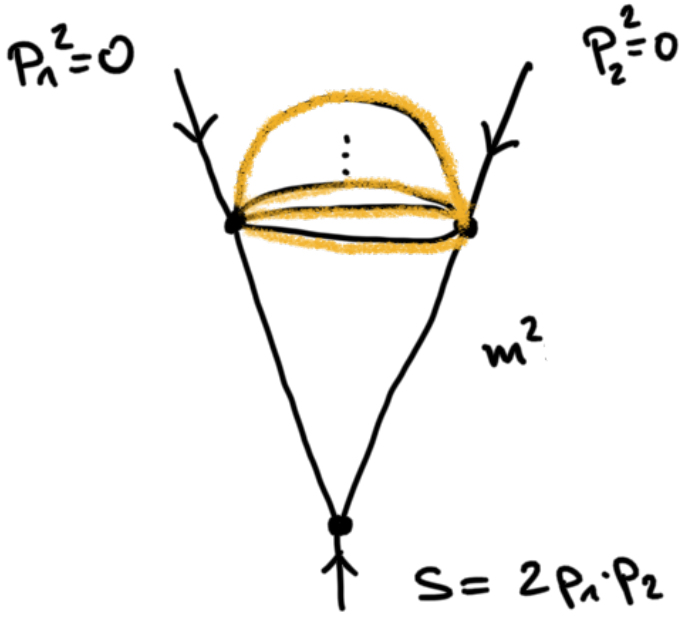

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- Our strategy to compute ice cone integrals has **three steps**:

  i) Find a good basis of master integrals such that the GM connection is simple.

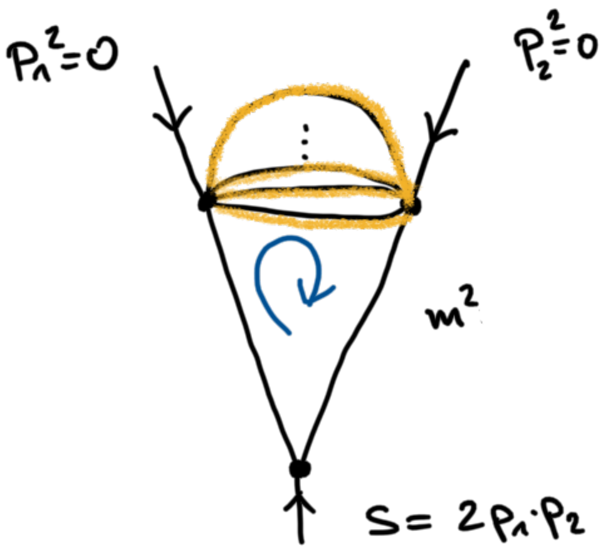
  ii) Solve the GM differential equation in terms of banana integrals.

  iii) Use monodromy considerations to obtain the correct linear combination.

Bananas in Ice Cones

- Consider the following representation of the ice cone:

$$I_{\text{ice}}^{(l)} = \int \frac{\mathrm{d}^d k}{(k^2 - m^2)((k + p_1 + p_2)^2 - m^2)} I_{\text{ban}}^{(l-1)}((k + p_2)^2)$$



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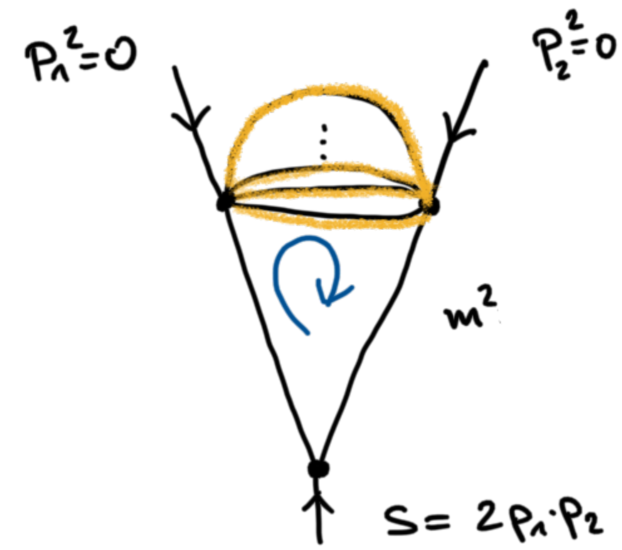

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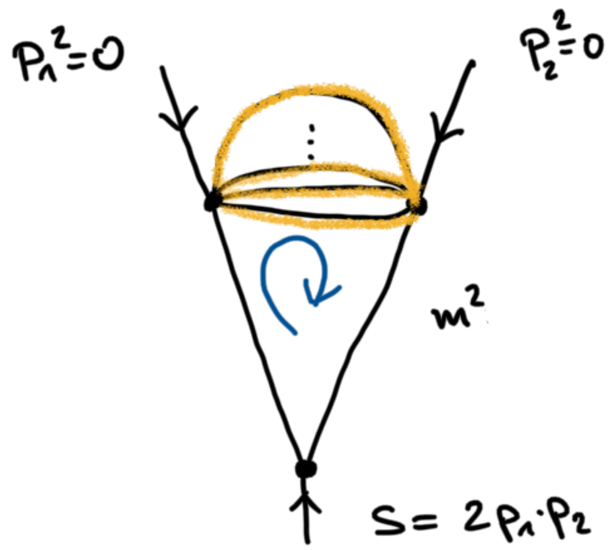

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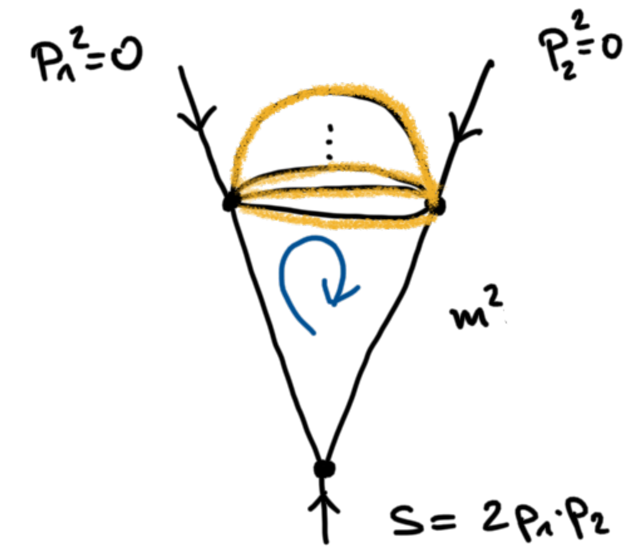

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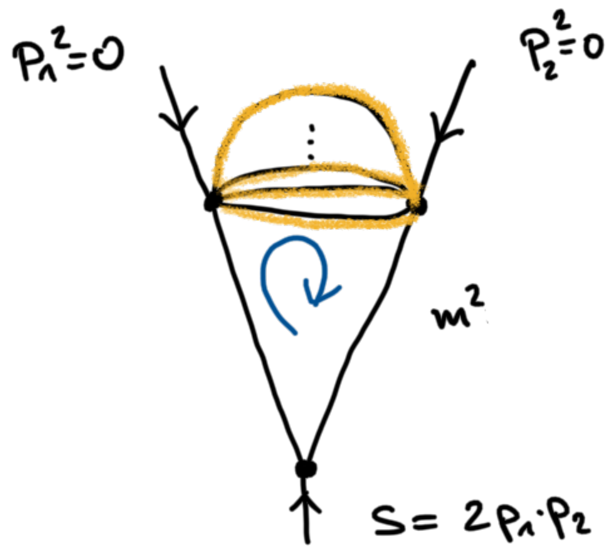

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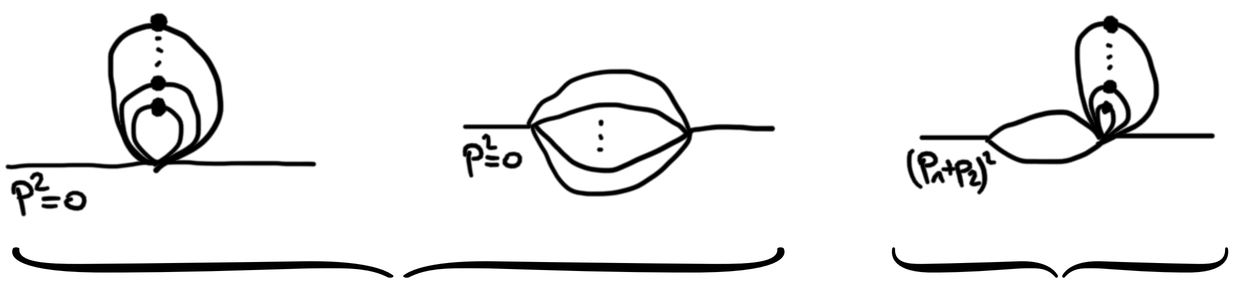
- A good basis of master integrals is now obtained if these **two residues really decouple**.

Master Integrals and Gauss-Manin Connection

- We found that **a good basis** is given by:

trivial master integrals:



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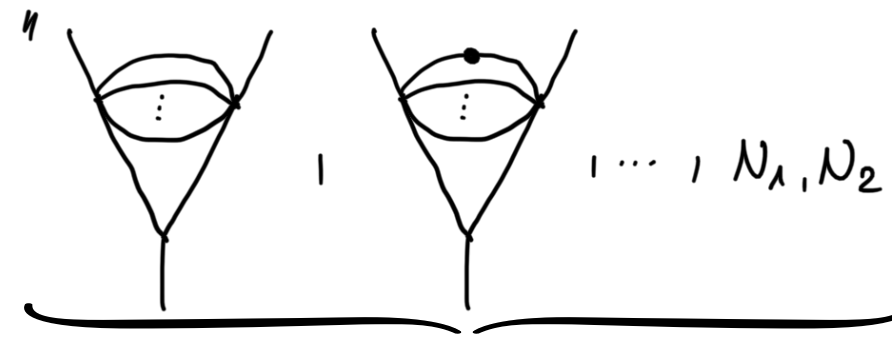

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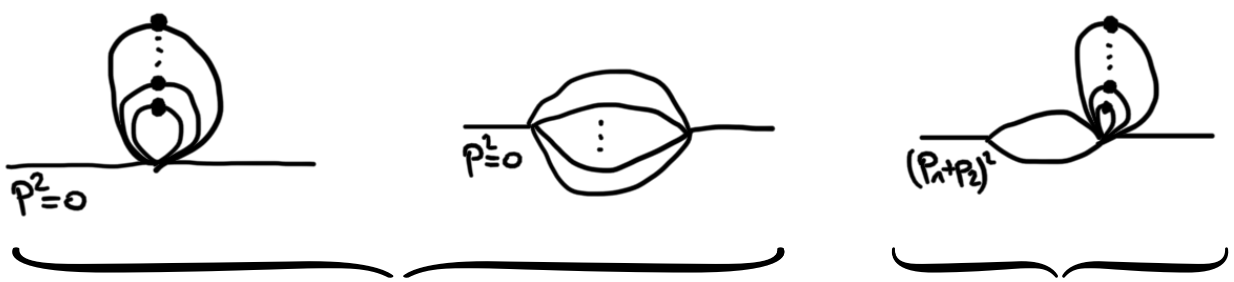

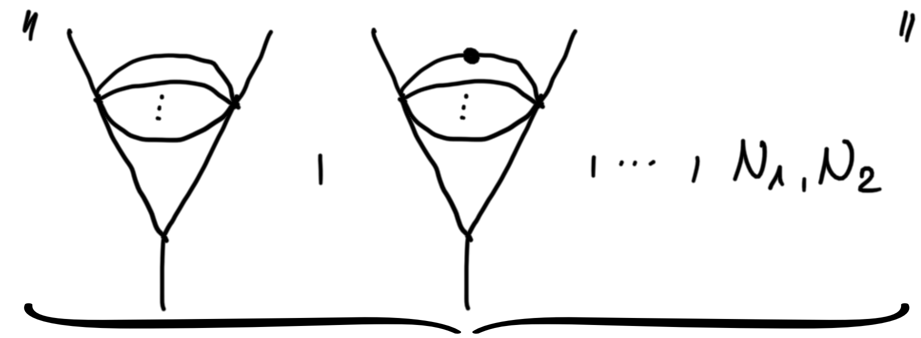

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- After a **simple rotation** in the space of non-trivial master integrals the GM connection for these integrals splits into three pieces:

$$\mathrm{d}\begin{pmatrix} I_1 \\ \vdots \\ I_{l-1} \end{pmatrix} = \mathrm{GM}_{ban}^{(l-1)}(-x)\begin{pmatrix} I_1 \\ \vdots \\ I_{l-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ \alpha(x) I_0 \end{pmatrix} ,$$

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$\alpha(x), \beta(x), \gamma(x), \delta(x)$
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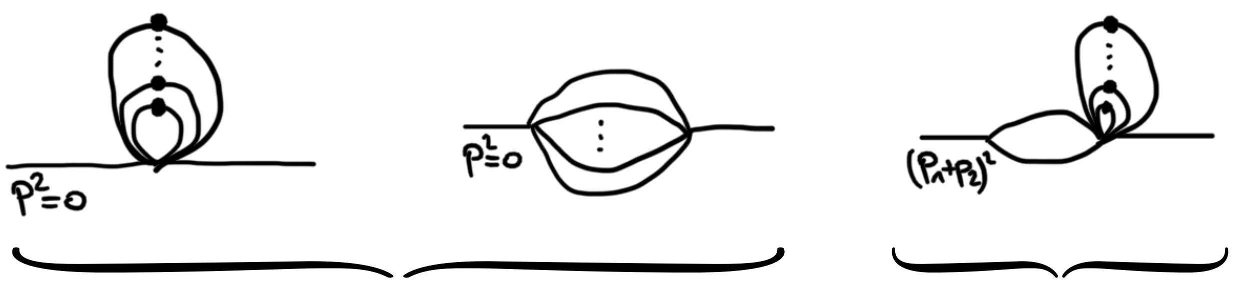

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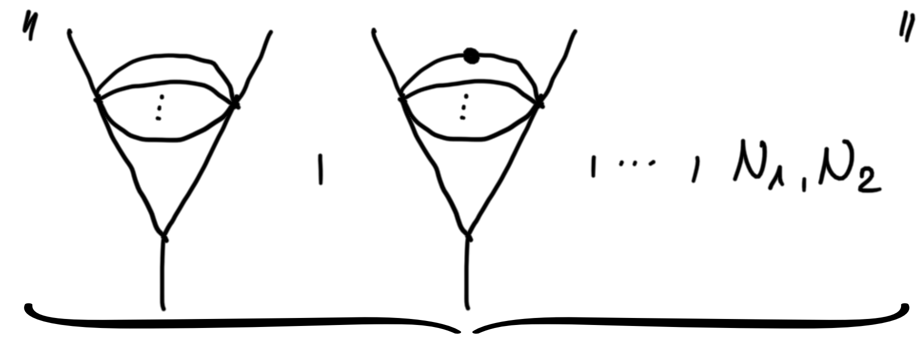

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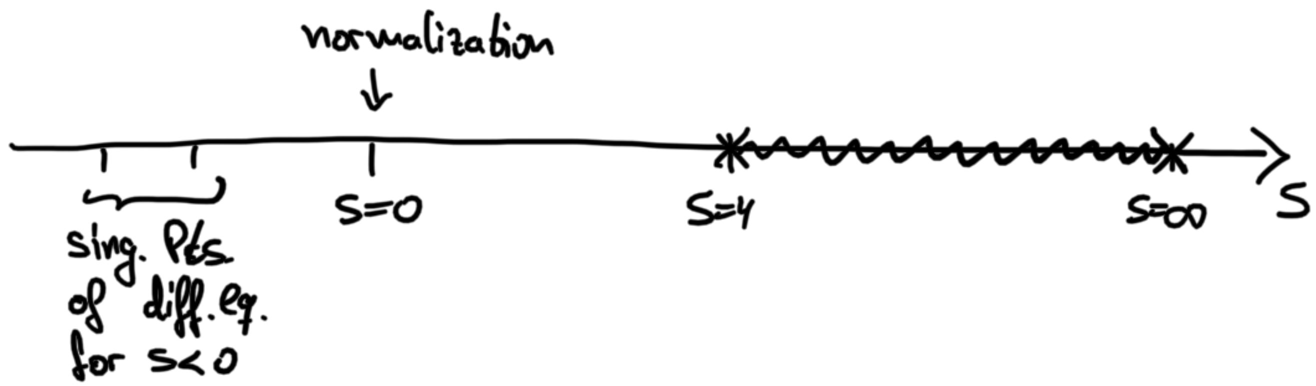
→ The function space of ice cone integrals is given by one-fold iterated CY period integrals.

Linear Combination and Monodromy

- We still have to combine these function to obtain the ice cone integral.
- For $s = 0$ the ice cone integrals get proportional to the banana integral:

$$I^{(l)}_{\text{ice}}(0) = -\frac{1}{l+1} I^{(l)}_{\text{ban}}(0) \qquad \text{normalization}$$

- The ice cone integrals have a branch cut from $s = 4$ until infinity:



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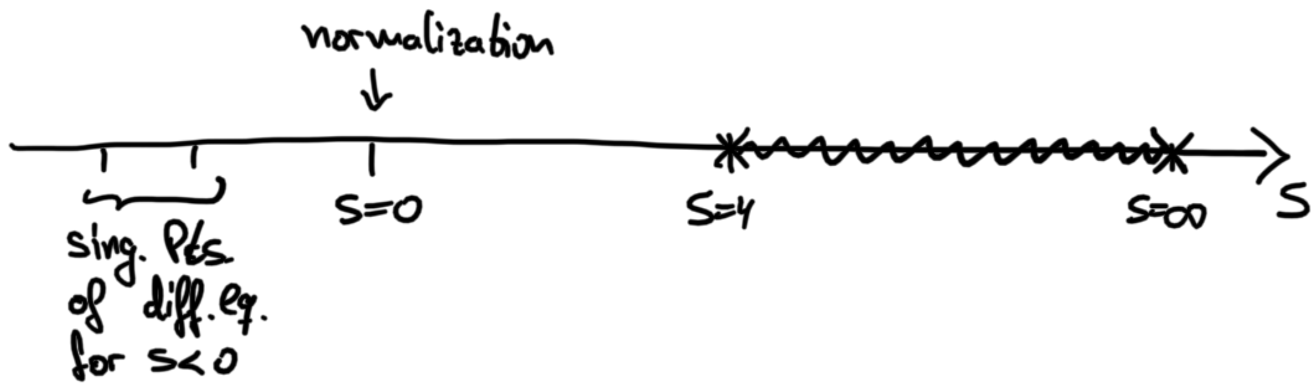

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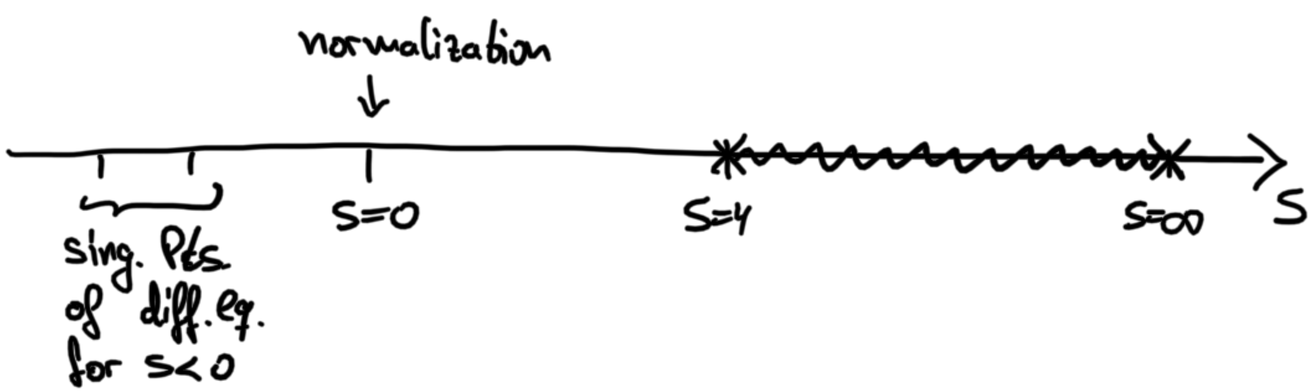


> Using the **monodromy properties** of the iterated CY periods and the **normalization** we could fix the linear combination.

- We could even construct a basis of solutions such that **all monodromies** are **integral**. The ice cone integral itself was a member of this basis. The intersection form in this basis was the one related to the banana integrals inside the ice cone.

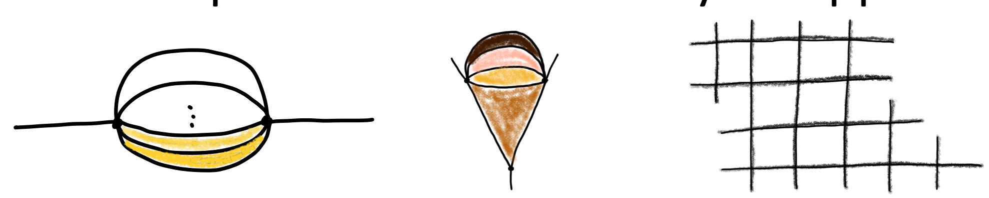
→ What is the meaning/interpretation of such an **integral basis**?

Conclusions

- Unterstanding CY geometries is essential for understanding higher loop Feynman integrals.
- Many concepts from the world of CYs have a direct interpretation and profit for (some) Feynman integrals:

(Relative) CY period integrals of varieties M	$\longleftrightarrow$	Feynman integral
Complex moduli of M	$\longleftrightarrow$	Physical parameters
Cohomology group of M, GM connection	$\longleftrightarrow$	Master integrals, differential relations between them
Monodromy relations	$\longleftrightarrow$	Analytic properties
	$\vdots$	

- CY geometries give a function space and simultaneously boundary conditions for Feynman integrals:
  - $\rightarrow$ Iterated CY period integrals
  - $\rightarrow$ Monodromy properties
- So far CY techniques could successfully be applied on three different families of Feynman graphs:



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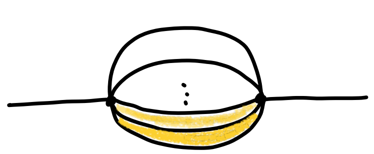

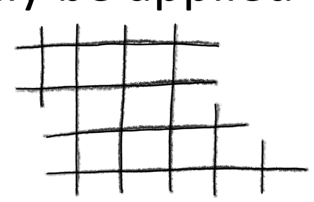

Further Questions:

- How useful is the function space of iterated CY period integrals for Feynman integrals? What is a proper definition of it?
- Which other graphs can be solved using CY techniques? What is the best starting point?

Thank you for your attention