# Recent progress in intersection theory for Feynman integrals decomposition

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[2209.01997]

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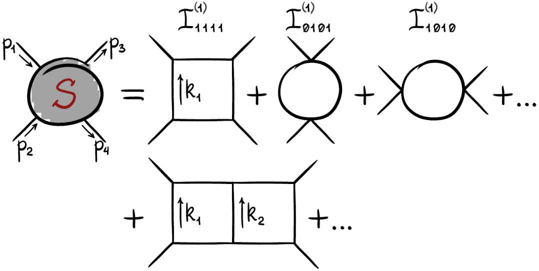
# Introduction

# Feynman integrals

$$\mathcal{I}^{(\ell)}_{a_1\dots a_n}(p_1,\dots,p_{E+1}) = \int \prod_{j=1}^{\ell} \mathrm{d}^d k_j \, \frac{1}{\mathcal{D}_1^{a_1}\dots\mathcal{D}_n^{a_n}}$$

1. $E$ independent external momenta
2. Exponents $a_i \in \mathbb{Z}$
3. Dimensional regularization: $d = 4 - 2\varepsilon$
4. $\mathcal{D}_i$ is either propagator or scalar product: e.g. $k^2 - m^2$ or $k \cdot p$
5. Number of factors $n = \ell(\ell+1)/2 + E\ell$



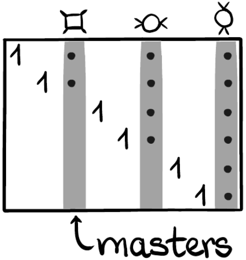
# Integration by Parts Identities (IBP)

Linear relations among Feynman integrals $\sim 40$ years and many applications:

[Tkachov] [Chetyrkin Tkachov] [Kotikov] [Remiddi] [Laporta] [Gehrmann Remiddi] [Henn] [Papadopoulos] ...

$$0 = \int \mathrm{d}^d k_j \, \frac{\partial}{\partial k_j^\mu} \left( v^\mu \, \frac{1}{\mathcal{D}_1^{a_1} \ldots \mathcal{D}_n^{a_n}} \right) , \quad \text{for some } v^\mu = v^\mu(p,k)$$

Decompose $\mathcal{I}$ in terms of a finite set of master integrals.
"Laporta algorithm": row reduction of a big linear system. [Laporta]




Some questions:

- What is the number of master integrals?
- How to compute the decomposition coefficients?
- Can we avoid row reductions of huge systems?

# Baikov representation

Parametric representation of Feynman integrals: [Baikov]

$$\mathcal{I} = \int \prod_{j=1}^{\ell} \mathrm{d}^d k_j \, \frac{1}{\mathcal{D}_1^{a_1} \dots \mathcal{D}_n^{a_n}} \Longrightarrow \int_{\mathcal{C}} \mathcal{B}(z)^{\gamma} \, \frac{\mathrm{d}^n z}{z_1^{a_1} \dots z_n^{a_n}}$$

Features:

1. $\mathrm{d}^n z := \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_n$
2. Baikov polynomial $\mathcal{B}(z) \in \mathbb{C}[z, p_i \cdot p_j]$
3. $\gamma = (d - \ell - E - 1)/2 \in \mathbb{C} \Longrightarrow \mathcal{B}^{\gamma}$ is multivalued!
4. Integration contour $\mathcal{C} \subset (\mathbb{C}^*)^n$ such that $\mathcal{B}(\partial\mathcal{C}) = 0$

# Intersection theory

# Basics of Intersection theory (from "physicist" POV)

$$\mathcal{I} = \int_{\mathcal{C}} u(z)\,\varphi(z)$$

Twisted cycle $\int_{\mathcal{C}} u(z)$: contour $\mathcal{C}$ & branch of the multivalued $u(z)$ [Cho, Matsumoto] [Matsumoto]
Twisted cocycle: holomorphic $n$-form $\varphi(z) = \hat{\varphi}(z)\,\mathrm{d}z_1 \wedge \ldots \mathrm{d}z_n$
Stokes theorem:

$$0 = \int_{\mathcal{C}} \mathrm{d}(u\,\xi) = \int_{\partial\mathcal{C}} u\,\xi = \int_{\mathcal{C}} u\,(\mathrm{d}\xi + \mathrm{d}\log u \wedge \xi)$$

Covariant derivative: $\nabla_\omega := \mathrm{d} + \omega\wedge$ and

$$\omega := \mathrm{d}\log u$$

Linear relation between integrals:

$$\int_{\mathcal{C}} u\,\varphi \equiv \int_{\mathcal{C}} u\,(\varphi + \nabla_\omega \xi)$$

Dual integrals are similar: $\mathcal{I} \mapsto \mathcal{I}^\vee$, $u \mapsto u^{-1}$, $\nabla_\omega \mapsto \nabla_{-\omega}$

# A quick example

$$ {}_2F_1(\alpha - n, \beta, \gamma; x) \approx \int_0^1 \underbrace{z^{\alpha}(1-z)^{\beta}(x-z)^{\gamma}}_{\text{multivalued } u} \underbrace{\frac{\mathrm{d}z}{z^n}}_{\text{cocycle } \varphi} $$

# De Rham twisted cohomology

Equivalence class of forms: $\langle\varphi| : \varphi \sim \varphi + \nabla_\omega \xi$
Twisted cohomology groups: [Aomoto]

$$\langle\varphi| \in \mathbb{H}^n_\omega := \{n \text{ forms} \,|\, \nabla_\omega \varphi = 0\}/\{\nabla_\omega \xi\}$$

Equivalence class of dual forms: $|\psi\rangle : \psi \sim \psi + \nabla_{-\omega} \xi$
Dual twisted cohomology groups:

$$|\psi\rangle \in \mathbb{H}^n_{-\omega} := \{n \text{ forms} \,|\, \nabla_{-\omega} \psi = 0\}/\{\nabla_{-\omega} \xi\}$$

Also see nice reviews: [MathemAmplitudes'19] [Cacciatori Conti, Trevisan] [Abreu Britto, Duhr] [Mathieu's talk talk on Thursday?]

# Features of twisted cohomology

Relates various fields of mathematics to Feynman Integrals.

Many ways to count master integrals $m$:

0. The number is finite [Smirnov Petukhov]
1. Laporta algorithm [Laporta]
2. Number of critical points $\mathrm{d}\log u = \omega = 0$ [Baikov] [Lee Pomeransky]
3. Number of independent cycles [Bosma, Sogaard Zhang] [Primo Tancredi] [Hjalte's talk on Thursday]
4. Number of independent cocyles $= \dim(\mathbb{H}^n_{\pm\omega})$ [Mastrolia Mizera] [FGLMMMM]
5. Holonomic rank of GKZ system (volumes of polytopes) [de la Cruz] [Klausen] [2204.12983]

# A quick example

$$\int_{\mathcal{C}} u\,\varphi := \int_0^1 z^{\alpha}(1-z)^{\beta}(x-z)^{\gamma}\,\frac{\mathrm{d}z}{z^n} \approx {}_2F_1(\alpha - n, \beta, \gamma; x)$$

The number of master integrals

$$\omega = \mathrm{d}\log u = \left(\frac{\alpha}{z} + \frac{\beta}{1-z} + \frac{\gamma}{x-z}\right)\mathrm{d}z = 0 \implies \text{2 solutions}$$

Now let's turn to the decomposition coefficients.

# Cohomology intersection numbers

Fix contour $\mathcal{C} \implies$ finite dimensional vector space of integrals: [FGMMMM] [Agostini, Fevola Sattelberger, Telen]

$$\mathcal{I} = \langle \varphi | \mathcal{C}] = \int_{\mathcal{C}} u\, \varphi\ , \quad \mathcal{I}^{\vee} = [\mathcal{C} | \psi \rangle = \int_{\mathcal{C}} u^{-1}\, \psi$$

Master integrals form basis: $\langle e_\lambda |$ for $\lambda \in \{1, \ldots, m\}$

Cohomology intersection number $=$ scalar product of integrals:

$$\langle \varphi | \psi \rangle = \frac{1}{(2\pi \mathrm{i})^n} \int_X \iota(\varphi) \wedge \psi \in \mathbb{C}(\varepsilon, p_i \cdot p_j)$$

is a rational function of kinematics and $\varepsilon$. Think of $X$ as $\mathbb{C}\mathrm{P}^n \setminus \{\text{singularities}\}$

How to use it?

# Master decomposition formula

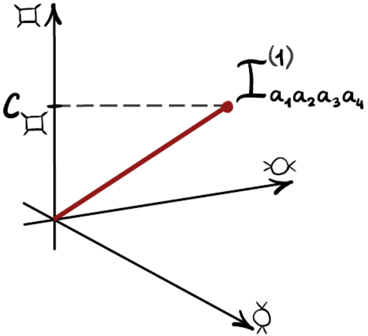
Allows to directly decompose $\mathcal{I} = \sum_{\lambda=1}^{m} c_\lambda \, \mathcal{J}_\lambda$ in a basis of master integrals $\mathcal{J}_\lambda = \langle e_\lambda | \mathcal{C}]$

Master decomposition formula: [Mastrolia Mizera] [FGLMMMM]

$$\langle \varphi | = \sum_{\lambda=1}^{m} c_\lambda \, \langle e_\lambda | \; , \quad c_\lambda \in \mathbb{C}(\varepsilon, p_i \cdot p_j)$$

The coefficients (independent of $|h_\mu\rangle$):

$$c_\lambda = \sum_{\mu=1}^{m} \langle \varphi | h_\mu \rangle \left(C^{-1}\right)_{\mu\lambda} \quad \text{with} \quad C_{\lambda\mu} = \langle e_\lambda | h_\mu \rangle$$




$\Longrightarrow$ learn how to compute $\langle \varphi | \psi \rangle$ first!

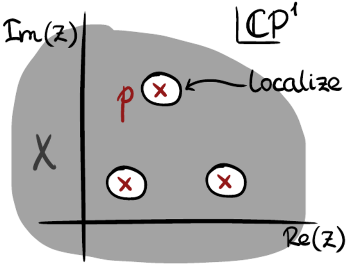
Another important connection: diagrammatic coaction [Abreu, Britto Duhr, Gardi] [Abreu, Britto Duhr, Gardi] [Abreu, Britto, Duhr Gardi, Matthew]

# Iterative method

# Univariate intersection numbers $n = 1$

Localize on poles of $\omega$ ($=$ zeros of $u$): [Cho, Matsumoto] [Matsumoto] [Mastrolia Mizera] [FGLMMMM]

$$\begin{aligned}
\langle\varphi|\psi\rangle &= \frac{1}{2\pi\mathrm{i}}\int_X \iota(\varphi)\wedge\psi \\
&\equiv \frac{1}{2\pi\mathrm{i}}\int_X \Big(\varphi - \sum_{p\in\mathcal{P}_\omega}\nabla_\omega\big(\theta(|z-p|)f_p\big)\Big)\wedge\psi \\
&= \sum_{p\in\mathcal{P}_\omega}\mathrm{Res}_{z=p}\left(f_p\,\psi\right)
\end{aligned}$$




- Integrate over $X = \mathbb{C}\mathrm{P}^1 \setminus \{\text{singularities}\}$
- $\mathcal{P}_\omega := \{\text{poles of } \omega\}$, including $\infty$
- Regulate using local potential $\nabla_\omega f_p \equiv (\mathrm{d} + \omega\wedge) f_p = \varphi$ around $p$

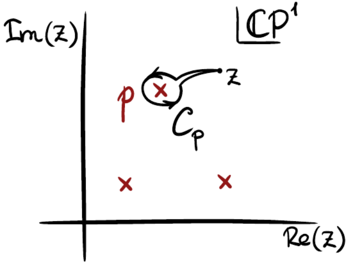
# Solving strategies: ansatz vs integral formula

$$\langle\varphi|\psi\rangle = \sum_{p\in\mathcal{P}_\omega} \mathrm{Res}_{z=p}\left(f_p\,\psi\right)\ , \quad \nabla_\omega f_p = \varphi \quad \text{around } p$$

Ansatz $f_p = (z-p)^{\min} f_{p,\min} + \ldots + (z-p)^{\max} f_{p,\max} \implies$ solve linear system for $f_{p,n}$

Alternatively, series expand the integral formula:

$$f_p(z) = \tfrac{1}{(1-\lambda_p)} u(z)^{-1} \int_{\mathcal{C}_p} u\,\varphi = u^{-1}\int_p^z u\,\varphi$$




Plugging back:

$$\langle\varphi|\psi\rangle = \sum_{p\in\mathcal{P}_\omega} \mathrm{Res}_{z=p}\left(\psi\, u^{-1}\int_p^z u\,\varphi\right)$$

# A quick example

$$\int_{\mathcal{C}} u\,\varphi := \int_0^1 z^\alpha (1-z)^\beta (x-z)^\gamma \,\frac{\mathrm{d}z}{z^n} \approx {}_2F_1(\alpha-n,\beta,\gamma;x)$$

Singularities are $\mathcal{P}_\omega = \{0,1,x,\infty\}$

$$\left\langle \mathrm{d}z/z \,\middle|\, \mathrm{d}z/z^2 \right\rangle = \mathrm{Res}_{z=0}\left(\mathrm{d}z/z^2\, u^{-1}\int_0^z u(t)\,\mathrm{d}t/t\right) + 0 + 0 + 0 = -\frac{\gamma+\beta x}{\alpha(\alpha+1)x}$$

How? Use series expansion

$$u(z) = z^\alpha(x^\gamma - x^{\gamma-1}(\gamma+\beta x)z + \mathcal{O}(z^2))$$

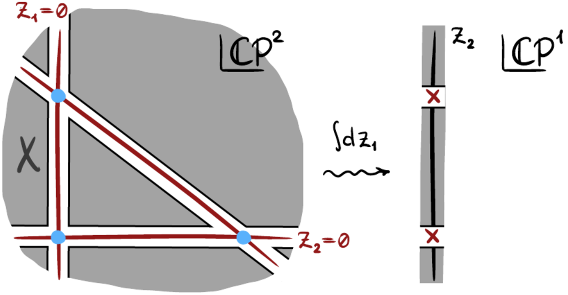
integrate $\int_0^z t^\alpha\,\mathrm{d}t = z^{\alpha+1}/(\alpha+1)$, and collect the residue terms $\sim z^{-1}$

# Multivariate intersection numbers $n > 1$

Recursive method: apply the univariate procedure one variable at a time.

[Ohara] [Mizera] [Mastrolia Mizera] [FGLMMMM] [Weinzierl]

So given $\varphi(z_1, z_2)$, $\psi(z_1, z_2)$ to compute $\langle\varphi|\psi\rangle \rightsquigarrow$ integrate out $z_1$ first, then $z_2$



# Multivariate intersection numbers $n > 1$

Pick $\langle e_\lambda|$ and $|h_\mu\rangle$ bases for $z_1$-intersections & project $\varphi$, $\psi$ onto them:

$$\langle\varphi| = \langle e_\lambda| \wedge \langle\varphi_\lambda| \ , \quad \langle\varphi_\lambda| = \langle\varphi|h_\mu\rangle\big(C^{-1}\big)_{\mu\lambda}$$
$$|\psi\rangle = |h_\mu\rangle \wedge |\psi_\mu\rangle \ , \quad |\psi_\mu\rangle = \big(C^{-1}\big)_{\mu\lambda}\langle e_\lambda|\psi\rangle$$

Integration over $z_1$ can be seen as insertion of identity operator $\mathbb{I}_\mathrm{c}$:

$$\langle\varphi|\psi\rangle = \langle\varphi\,\underbrace{|h_\mu\rangle\big(C^{-1}\big)_{\mu\lambda}\langle e_\lambda|}_{\mathbb{I}_\mathrm{c}}\,\psi\rangle = \sum_{p\in\mathcal{P}_\Omega} \mathrm{Res}_{z_2=p}\left(f_{p,\lambda}\, C_{\lambda\mu}\, \psi_\mu\right)$$

Requires a local vector potential $\vec{f}_p \equiv f_{p,\lambda}$:

$$\partial_{z_2}\vec{f}_p + \Omega\vec{f}_p = \vec{\varphi} \quad \text{around pole } p \ , \quad \Omega_{\lambda\nu} := \langle(\partial_{z_2}+\omega_2)e_\lambda|h_\mu\rangle\big(C^{-1}\big)_{\mu\nu}$$

# Solving via ansatz: what to row reduce?

Solve for $\rho$:

$$\begin{cases} [\partial_z + \Omega]\vec{f} = \vec{\varphi} \\ \rho = \mathrm{Res}_{z=0}\left(\vec{\psi} \cdot \vec{f}\right) \end{cases}$$

Given Laurent expansions:

$$\Omega = z^{-1}\Omega_{-1} + \Omega_0 + \dots$$

$$\vec{\varphi} = z^{-n}\vec{\varphi}_{-n} + \dots\ , \quad \vec{\psi} = z^{-k}\vec{\varphi}_{-k} + \dots$$

Row reduce to find this element

$$\begin{array}{c|ccccc|c}
1 & \vec{\psi}_1^{\mathsf{T}} & \vec{\psi}_0^{\mathsf{T}} & \vec{\psi}_{-1}^{\mathsf{T}} & \vec{\psi}_{-2}^{\mathsf{T}} & \vec{\psi}_{-3}^{\mathsf{T}} & \boxed{\cdot} \\
\cdot & \Omega_{-1}-2 & \cdot & \cdot & \cdot & \cdot & \vec{\varphi}_{-3} \\
\cdot & \Omega_0 & \Omega_{-1}-1 & \cdot & \cdot & \cdot & \vec{\varphi}_{-2} \\
\cdot & \Omega_1 & \Omega_0 & \Omega_{-1} & \cdot & \cdot & \vec{\varphi}_{-1} \\
\cdot & \Omega_2 & \Omega_1 & \Omega_0 & \Omega_{-1}+1 & \cdot & \vec{\varphi}_0 \\
\cdot & \Omega_3 & \Omega_2 & \Omega_1 & \Omega_0 & \Omega_{-1}+2 & \vec{\varphi}_1
\end{array}$$

# Nuances and remarks

- The system is triangular and is *much* smaller than in the Laporta algorithm
- If $\Omega$ has only simple poles $\Longrightarrow$ apply the global residue theorem [Weinzierl]
- Spectral issues: $\Omega_{-1}$ may have integer eigenvalues $\Longrightarrow$ unfixed coefficients
- Higher order poles of $\Omega$ (luckily, so far are always Moser-reducible)

Open questions:

- Make use of the relative dual basis [Matsumoto] [Caron-Huot Pokraka] [Caron-Huot Pokraka]
- $\Omega$ has simple pole and $\Omega_{-1}$ has "good spectrum" $\Longrightarrow$ integral formula for $\vec{f}$

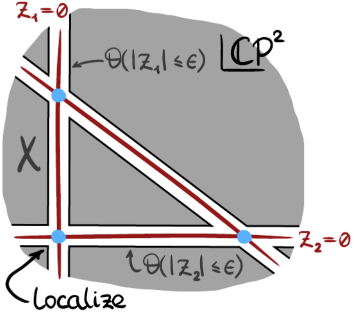
# $n$PDE method

# From hyperplane arrangement . . .

Consider $u(z_1, z_2) = (z_1 z_2 (z_1 + z_2 - 1))^\gamma$ with singularities along hyperplanes. [Matsumoto]
Define three potentials:

$$\begin{aligned} \nabla(\hat{f}_1\, \mathrm{d}z_2) &= \varphi && \text{near } z_1 = 0 \\ \nabla(\hat{f}_2\, \mathrm{d}z_1) &= \varphi && \text{near } z_2 = 0 \\ \nabla f_{12} = f_1 - f_2 & && \text{near } z_1 = z_2 = 0 \end{aligned}$$

$$\langle \varphi | \psi \rangle = \sum_{p \in \mathcal{P}_\omega} \mathrm{Res}_{z_1 = p_1} \mathrm{Res}_{z_2 = p_2} (f_{12}\, \psi)$$




Localize on "poles of $\omega$" (= zeros of $u$): $\mathcal{P}_\omega := \{\text{intersection points of singularities}\}$
Can we do better than hyperplanes?

# . . . to multivariate residues and $n$-th order PDE

$$\langle\varphi|\psi\rangle = \int_X \iota(\varphi) \wedge \hat{\psi}\, \mathrm{d}z_1 \wedge \mathrm{d}z_2$$

where $X = \mathbb{C}\mathrm{P}^2 \setminus \{\text{singularities}\}$ and $\iota$-regulator is: [Matsumoto]

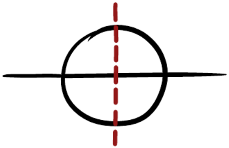
$$\begin{aligned}\iota(\varphi) &= \varphi - \nabla\big(\theta_1 f_1 + \theta_2(1-\theta_1) f_2 + \theta_2 \mathrm{d}\theta_1 f_{12}\big) \\ &= \ldots - \underbrace{\mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2}_{\text{gives Res Res}}\ f_{12}\end{aligned}$$

Define partial derivative $\nabla_i := \mathrm{d}z_i\, \partial_i + \hat{\omega}_i\, \mathrm{d}z_i \wedge \rightsquigarrow n$PDE:

$$\boxed{\nabla_1 \nabla_2\, f_{12} = \varphi} \Longleftarrow \text{[2209.01997]}$$

Use $f_{12}$ to localize $\langle\varphi|\psi\rangle$ on intersection points of general singular hypersurfaces.

# Sanity check: massless sunrise

 $\rightsquigarrow u(z_1, z_2) = \big(z_1 z_2 (z_1 + z_2 - s)\big)^\gamma$

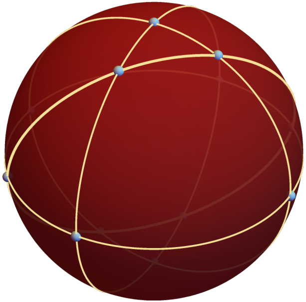
Singular hyperplanes:

$$\{z_1 = 0,\ z_2 = 0,\ z_1 + z_2 - s = 0\}$$

$\Longrightarrow$ 6 intersection points in $\mathbb{CP}^2$

Consider $\varphi = \psi = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{z_1 z_2} \rightsquigarrow$ residues:

$$\langle \varphi | \psi \rangle = \frac{-1}{3\gamma^2} + \frac{-1}{3\gamma^2} + \frac{1}{\gamma^2} + 0 + 0 + 0 = \frac{1}{3\gamma^2}$$



Reproduce results of the iterative method.

# Solving via an ansatz

Consider $p = 0$ contribution to

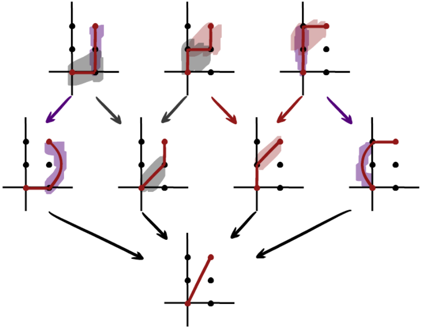
$$\langle \varphi | \psi \rangle = \sum_p \mathrm{Res}_{z=p}(f_p \psi) \ , \quad \nabla_n \ldots \nabla_1 \, f_p = \varphi$$

Expansions:

$$\varphi = \sum_{i \geq \mu_\varphi} \varphi_i \, z^i \ , \quad \psi = \sum_{i \geq \mu_\psi} \psi_i \, z^i$$

$$w := \log(u) = \sum_{i \geq \mu_w} w_i \, z^i + \sum_{j=1}^{n} \gamma_j \log(z_j)$$

$\implies$ explicit rational kernel $Y(h, \tau, \sigma) \in \mathbb{C}(w_i, \gamma_i)$ in




$$\boxed{\mathrm{Res}(f_p \, \psi) = \sum_{\tau=0}^{n\mu_w - \mu_L - \mu_R - 2} \ \sum_{\sigma \in \mathrm{VC}(\tau)} \ \sum_{h=\mu_L - n\mu_w + 1}^{-\mu_R - \tau - 1} Y(h, \tau, \sigma) \, \varphi_{h + n\mu_w - 1} \, \psi_{-h - \tau - 1}}$$

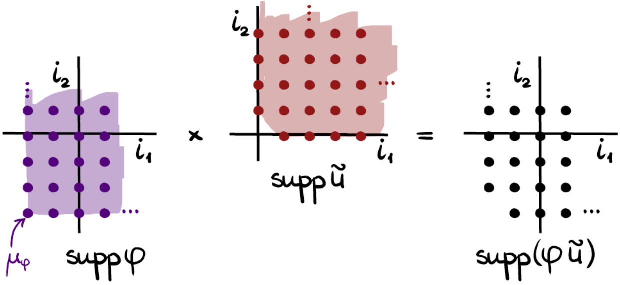
# Solving via integral formula

Consider $p = 0$ contribution to

$$\langle\varphi|\psi\rangle = \sum_p \mathrm{Res}_{z=p}(\psi\, u^{-1} \int_p^z u\, \varphi)$$

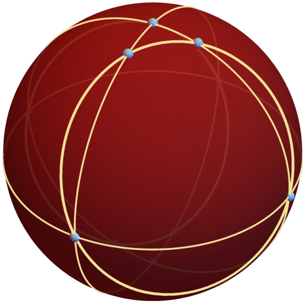
$$\varphi = \sum_{i \geq \mu_\varphi} \varphi_i\, z^i\ , \quad \psi = \sum_{i \geq \mu_\psi} \psi_i\, z^i$$

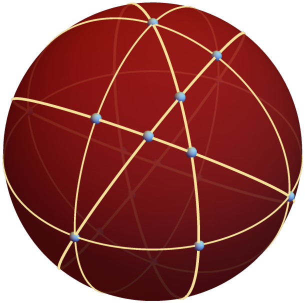
$$u = z^\gamma (1 + \sum_{i \geq 1} u_i\, z^i)$$

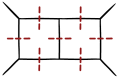



1. Geometric series for $u^{-1}$
2. Integrate termwise $\int_0^z t^\alpha \mathrm{d}t = z^{\alpha+1}/(\alpha+1)$
3. Multiply series expansions
4. Take the residue

# Other examples: degenerate intersection points





Double box 

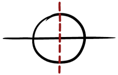
$$u = \big(z_1 z_2 (st + z_1 z_2 + s(z_1 + z_2))\big)^{\gamma}$$

${}_3F_2$ hypergeometric function

$$u = z_1^{\gamma_1} z_2^{\gamma_2} (1 - z_1)^{\gamma_3} (s - z_2)^{\gamma_4} (z_1 - z_2)^{\gamma_5}$$

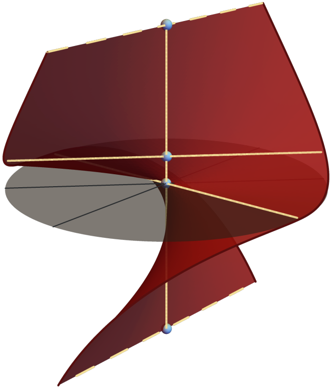
Points with $3$ curves going through $\Longrightarrow$ resolution of singularities.

# Resolution of singularities



$$\Big|_{\substack{z_1 \to z_1 + s \\ z_2 \to -z_2}} = \big((z_1 + s) z_2 (z_1 - z_2)\big)^{\gamma}$$

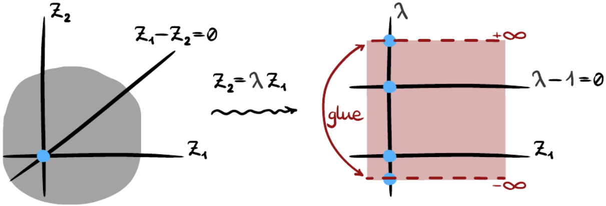
focus on $z = 0$



Problem: expansion depends on the order

$$1/(z_1 - z_2) = \begin{cases} -z_2^{-1} - z_1 z_2^{-2} - \dots & \text{first } z_1\text{, then } z_2 \\ z_1^{-1} + z_1^{-2} z_2 + \dots & \text{first } z_2\text{, then } z_1 \end{cases}$$

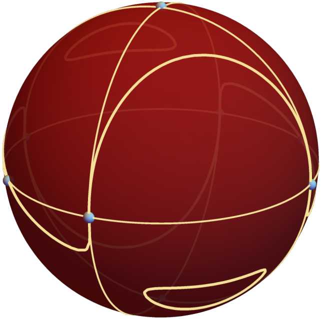
New variable: $z_2 = \lambda z_1 \implies$ new intersection points!




Closely related to sector decomposition [pySecDec] [Bogner Weinzierl]

# More examples

Cubic with self-intersection
$u = \left(z_1 z_2 (z_1^2 - (1 + z_2) z_2^2)\right)^\gamma$

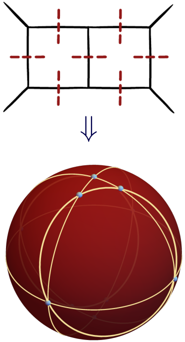


Equal mass sunrise
$u = \left(z_1 z_2 (s^2 - z_1 z_2 (4 + s + z_1 + z_2))\right)^\gamma$

# Conclusions

# Conclusions

1. Twisted cohomology unites areas of math related to Feynman integrals
2. Intersection number $=$ scalar product $\rightsquigarrow$ direct projection onto basis of master integrals
3. $n$PDE method:
   - Localizes on intersection points of singular surfaces
   - At the moment requires resolution of singularities (sector decomposition)
   - No ansatz is needed: explicit formulas for residues are available



Outlook:

- Avoid resolution of singularities somehow?
- Higher dimensions $n \geq 3$ for $n$PDE method
- Incorporate relative twisted cohomology
- Canonical bases, less cuts, more loops and legs, coaction, twisted period relations . . .

# Bonus slides

# Riemann twisted period relations (a slide for David)

Quadratic relations between Feynman Integrals. [Broadhurst Roberts] [Zhou] [Broadhurst Mellit] [Lee]
Two identity operators: cohomological [Cho, Matsumoto] [Mastrolia Mizera] [FGLMMMM]

$$\mathbb{I}_\mathrm{c} = \sum_{\lambda,\mu} |h_\mu\rangle \left(C^{-1}\right)_{\mu\lambda} \langle e_\lambda| \ , \quad C_{\lambda\mu} = \langle e_\lambda | h_\mu \rangle \ ,$$

($C_{\lambda\mu}$ from, say, intersections or GKZ systems) and homological [2204.12983]

$$\mathbb{I}_\mathrm{h} = \sum_{\lambda,\mu} |\mathcal{C}_\mu] \left(H^{-1}\right)_{\mu\lambda} [\mathcal{D}_\lambda| \ , \quad H_{\lambda\mu} = [\mathcal{D}_\lambda | \mathcal{C}_\mu] \ .$$

Twisted Riemann Period Relations:

$$\langle \varphi | \psi \rangle = \langle \varphi | \mathbb{I}_\mathrm{h} | \psi \rangle = \sum_{\lambda,\mu} \underbrace{\langle \varphi | \mathcal{C}_\mu]}_{\text{Feyn. Int.}} \left(H^{-1}\right)_{\mu\lambda} \underbrace{[\mathcal{D}_\lambda | \psi \rangle}_{\text{Feyn. Int.}} \ .$$

Can continue inserting $\mathbb{I}_\mathrm{h}$ and $\mathbb{I}_\mathrm{c}$ $\rightsquigarrow$ higher order identities, coaction? [Abreau, Britto Duhr, Gradi]