The Three-Loop Equal-Mass Banana Integral

ε-Factorization with Meromorphic Modular Forms

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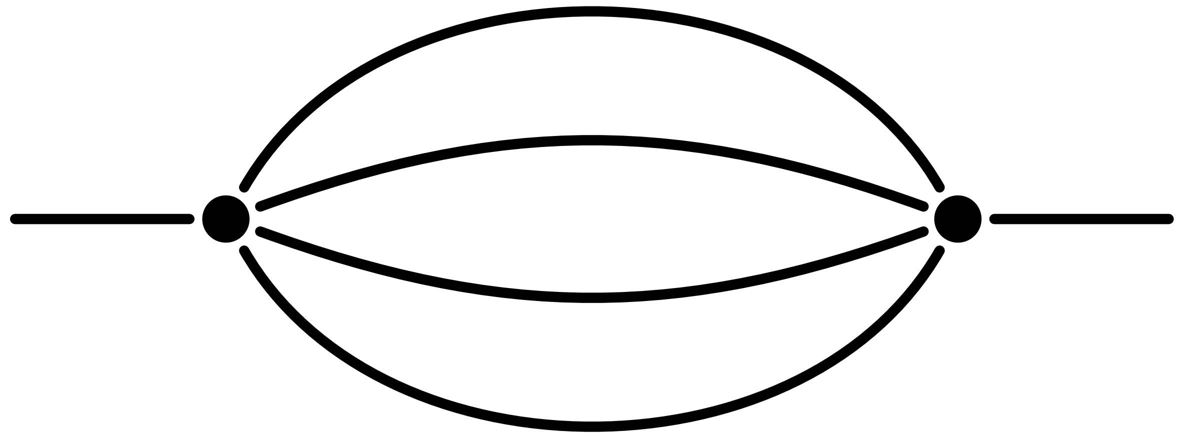


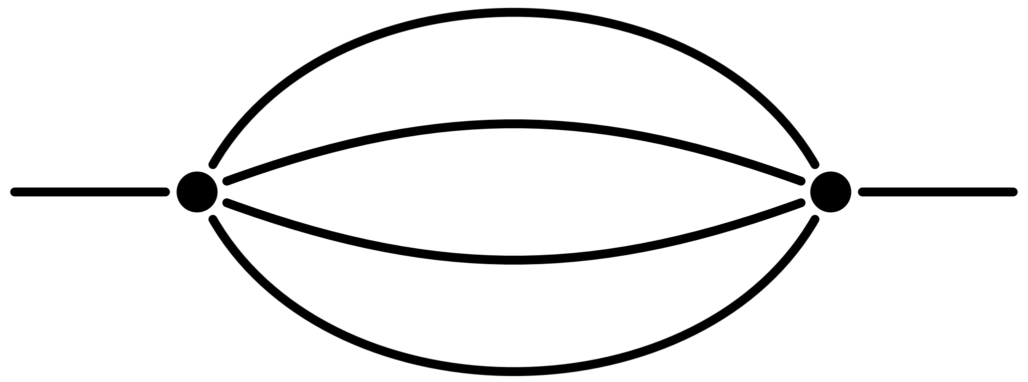
Work with Xing Wang and Stefan Weinzierl [2207.12893]

The Three-Loop Equal-Mass Banana Integral

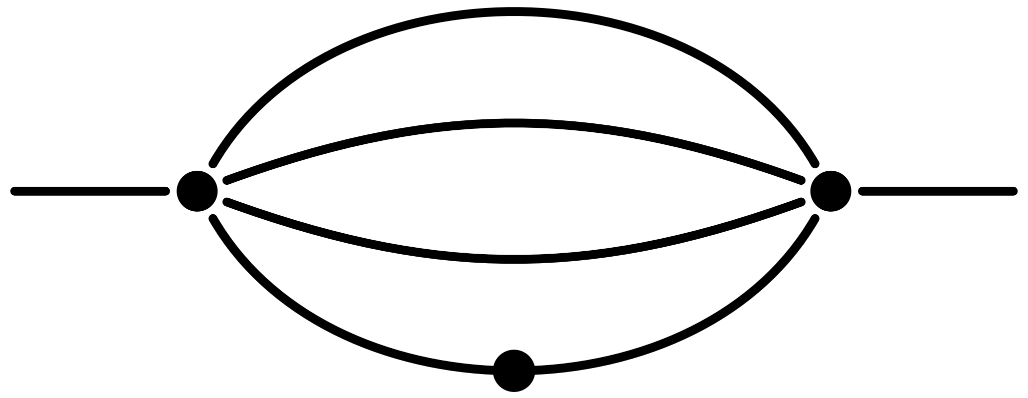
$$\boxed{D = 2 - 2\varepsilon}$$

$$I_{\nu_1\nu_2\nu_3\nu_4} = \quad e^{3\gamma_E\varepsilon}\left(m^2\right)^{\nu-\frac{3}{2}D}\int\left(\prod_{a=1}^{4}\frac{d^Dk_a}{i\pi^{\frac{D}{2}}}\right) i\pi^{\frac{D}{2}}\delta^D\left(p-\sum_{b=1}^{4}k_b\right)\left(\prod_{c=1}^{4}\frac{1}{(-k_c^2+m^2)^{\nu_c}}\right)$$

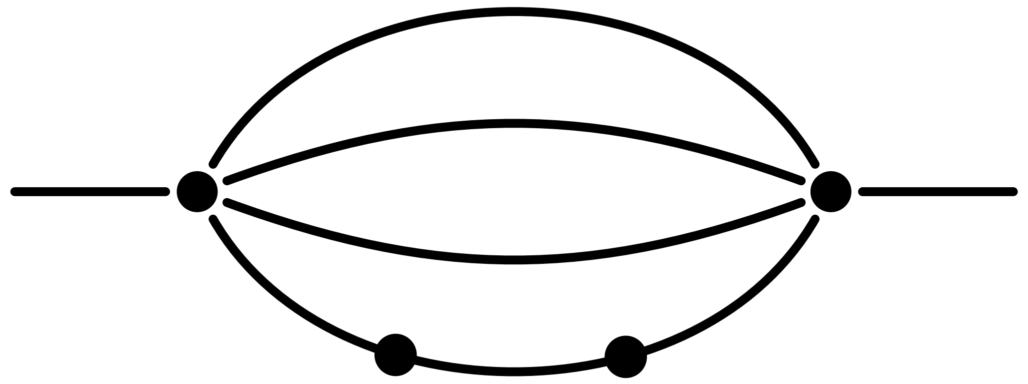
There are then four master integrals, *e.g.*



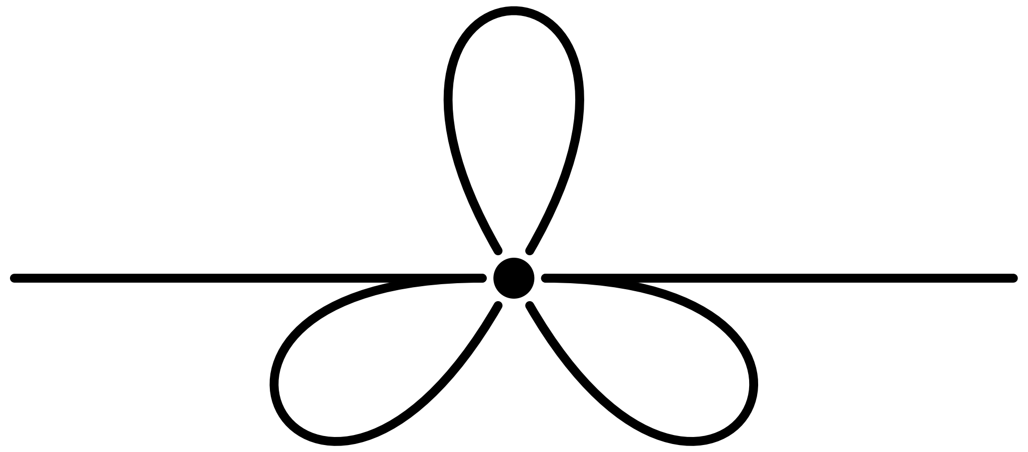
I_{1111}



I_{1112}



I_{1113}



I_{1110}

Why Study the Three-Loop Banana Integral?

Simplest example of Feynman integral beyond elliptic: K3 surface

Already well studied in the past:

Leading term in ε [Bloch, Kerr, Vanhove, 1406.2664]

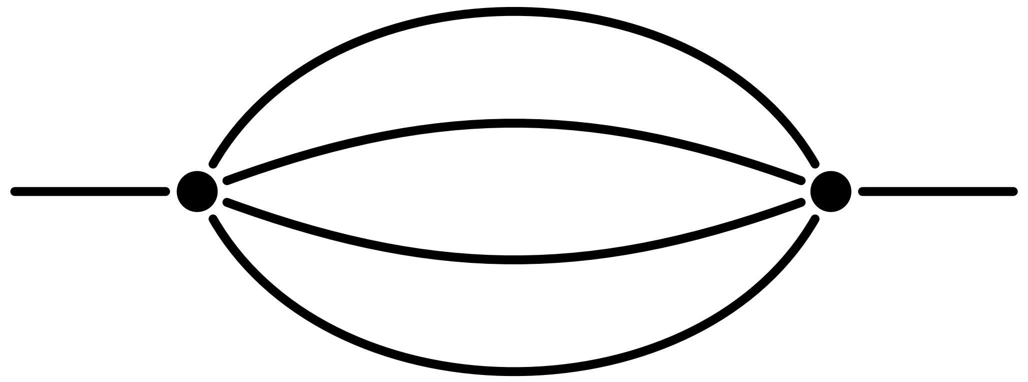
ε-factorized form [Primo, Tancredi, 1704.05465]

Master integrals in D=2 in terms of eMPLs $\tilde{\Gamma}$ [Broedel, Duhr, Dulat, Marzucca, Penante, 1907.03787]

DEQ with meromorphic modular forms [Broedel, Duhr, Matthes, 2109.15251]

Part of larger l-Loop banana program [Bönisch, Duhr, Klemm, Nega, Safari; Kreimer; Forum, von Hippel]

See also talks by Matt, Christoph, Mathieu



Equal-mass case closely connected to sunrise integral

The Banana: A Symmetric Square

Picard-Fuchs operator of Banana

$$L_3 = \frac{d^3}{dx^3} + \left[\frac{3}{x} + \frac{3}{2\,(x-4)} + \frac{3}{2\,(x-16)}\right]\frac{d^2}{dx^2} + \frac{7x^2-68x+64}{x^2\,(x-4)\,(x-16)}\frac{d}{dx} + \frac{1}{x^2\,(x-16)}.$$

With solutions $L_3\,\omega_i = 0$ where $\omega_i = \mathrm{MaxCut}(I_{1111})|_{\gamma_i}$ on three independent contours γ_i

L_3 is a symmetric square [Verrill, 96'; Joyce, 72']

There exists an operator

$$L_2 = \frac{d^2}{dx^2} + \left[\frac{1}{x} + \frac{1}{2\,(x-4)} + \frac{1}{2\,(x-16)}\right]\frac{d}{dx} + \frac{(x-8)}{4x\,(x-4)\,(x-16)}$$

with solutions ψ_1, ψ_2, $L_2\,\psi_i = 0$ such that

$$\boxed{\omega_i \in \langle \psi_1^2,\, \psi_1\psi_2,\, \psi_2^2 \rangle}$$

Sunrise in Disguise

$$\underbrace{\left[\frac{d^2}{dx^2}+\left(\frac{1}{x}+\frac{1}{2(x-4)}+\frac{1}{2(x-16)}\right)\frac{d}{dx}+\frac{(x-8)}{4x(x-4)(x-16)}\right]}_{L_2}\psi_j=0$$

$x = -\frac{(y-1)(y-9)}{y}$

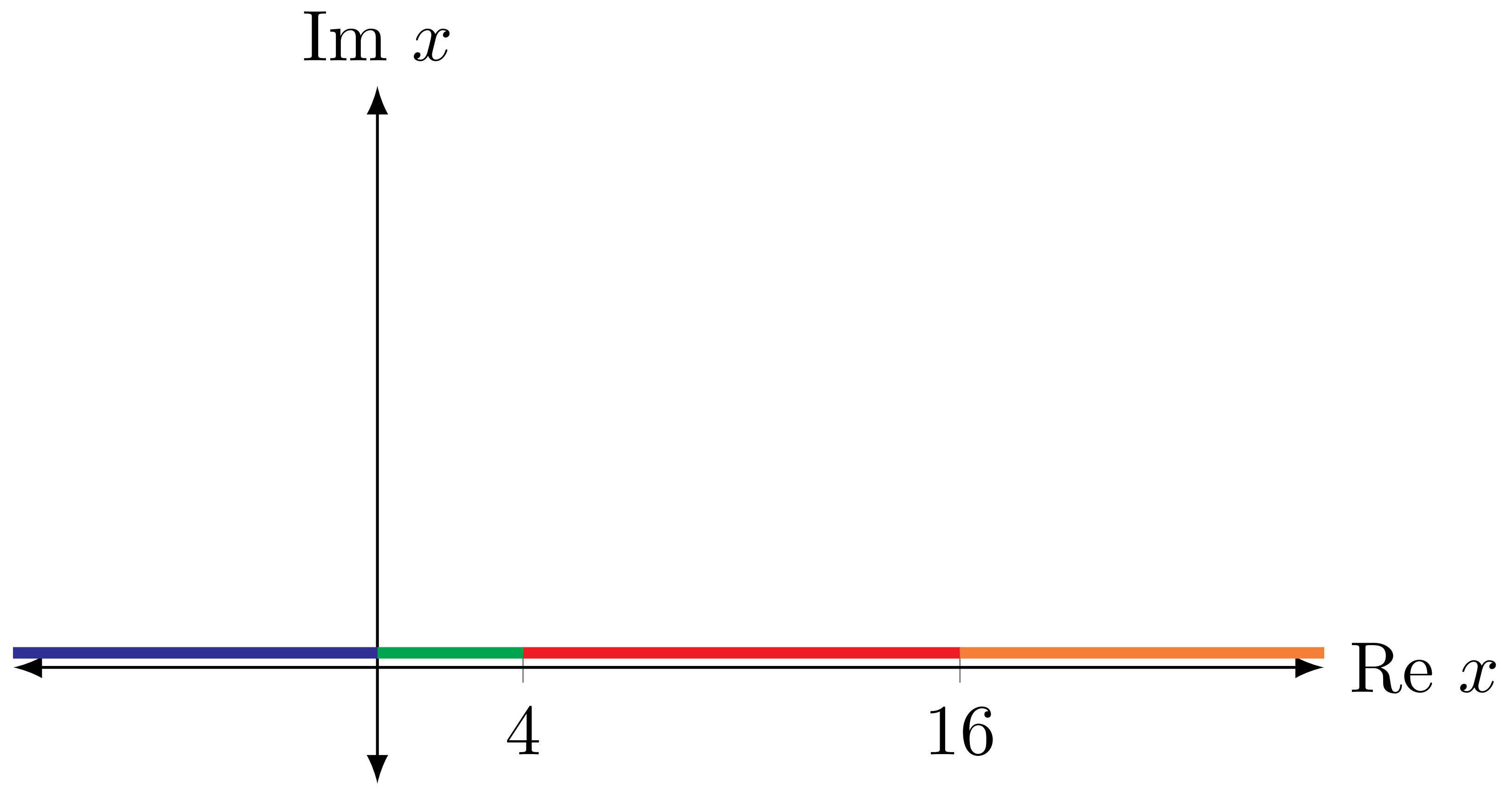
$$\left[\frac{d^2}{dy^2}+\left(\frac{1}{y-1}+\frac{1}{y-9}\right)\frac{d}{dy}+\frac{y^2-2y+9}{4y^2(y-1)(y-9)}\right]\psi_j=0$$

$\psi_j = \sqrt{y}\,\psi_j^{\mathrm{sun}}$

$$\underbrace{\left[\frac{d^2}{dy^2}+\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-9}\right)\frac{d}{dy}+\frac{y-3}{y(y-1)(y-9)}\right]}_{\text{Picard-Fuchs operator of sunrise integral } L_2^{\mathrm{sun}}}\psi_j^{\mathrm{sun}}=0$$

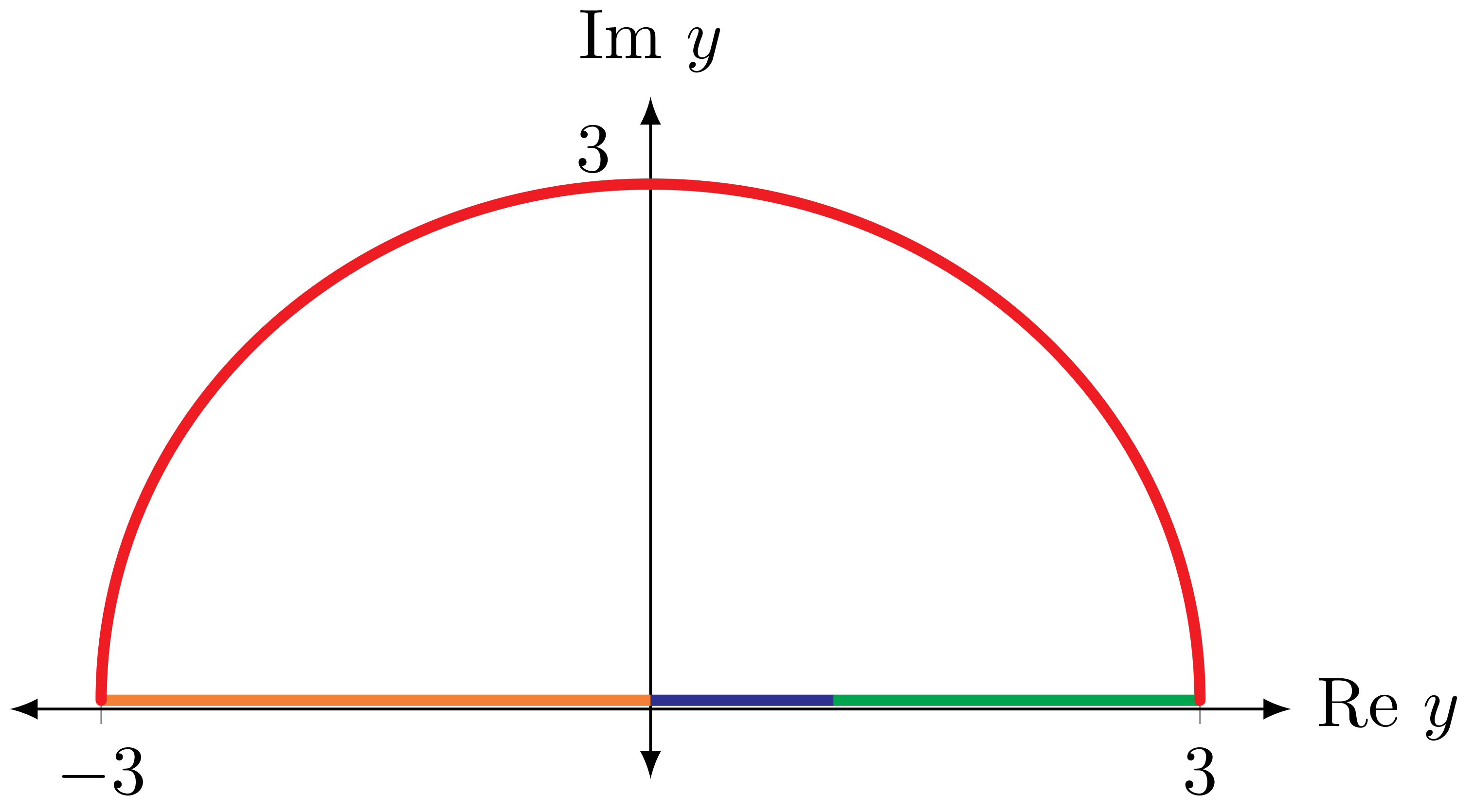
Transformation to Sunrise

We are interested in $x \in \mathbb{R} + i\delta^{+}$



$$y = \frac{1}{2}\left[10 - x - \sqrt{4-x}\sqrt{16-x}\right]$$

$$x = -\frac{(y-1)\,(y-9)}{y}$$



(Pseudo-) threshold $x = 4, 16$ mapped to $y = \pm\, 3$

The Elliptic Curve

$$L_2^{\rm sun} = \frac{d^2}{dy^2} + \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-9}\right)\frac{d}{dy} + \frac{y-3}{y\,(y-1)\,(y-9)}$$

defines elliptic curve

$$v^2 = (u-u_1)\,(u-u_2)\,(u-u_3)\,(u-u_4)$$

Roots: $u_1 = -4$, $u_3 = -(1-\sqrt{y})^2$, $u_4 = 0$, $u_2 = -(1+\sqrt{y})^2$

Periods: $\psi_1^{\rm sun} \sim K(k)$, $\psi_2^{\rm sun} \sim iK(\overline{k})$

where $k^2 = \frac{(u_3-u_2)\,(u_4-u_1)}{(u_3-u_1)\,(u_4-u_2)}$, $\overline{k}^2 = 1-k^2$

Associated to congruence subgroup $\Gamma_1(6)$

→ Sunrise in terms of modular forms of $\Gamma_1(6)$

For Banana, elliptic curve not degenerate at (pseudo-) threshold $y = \pm 3$!

→ Might expect meromorphic modular forms of $\Gamma_1(6)$, with poles at $y = \pm 3$

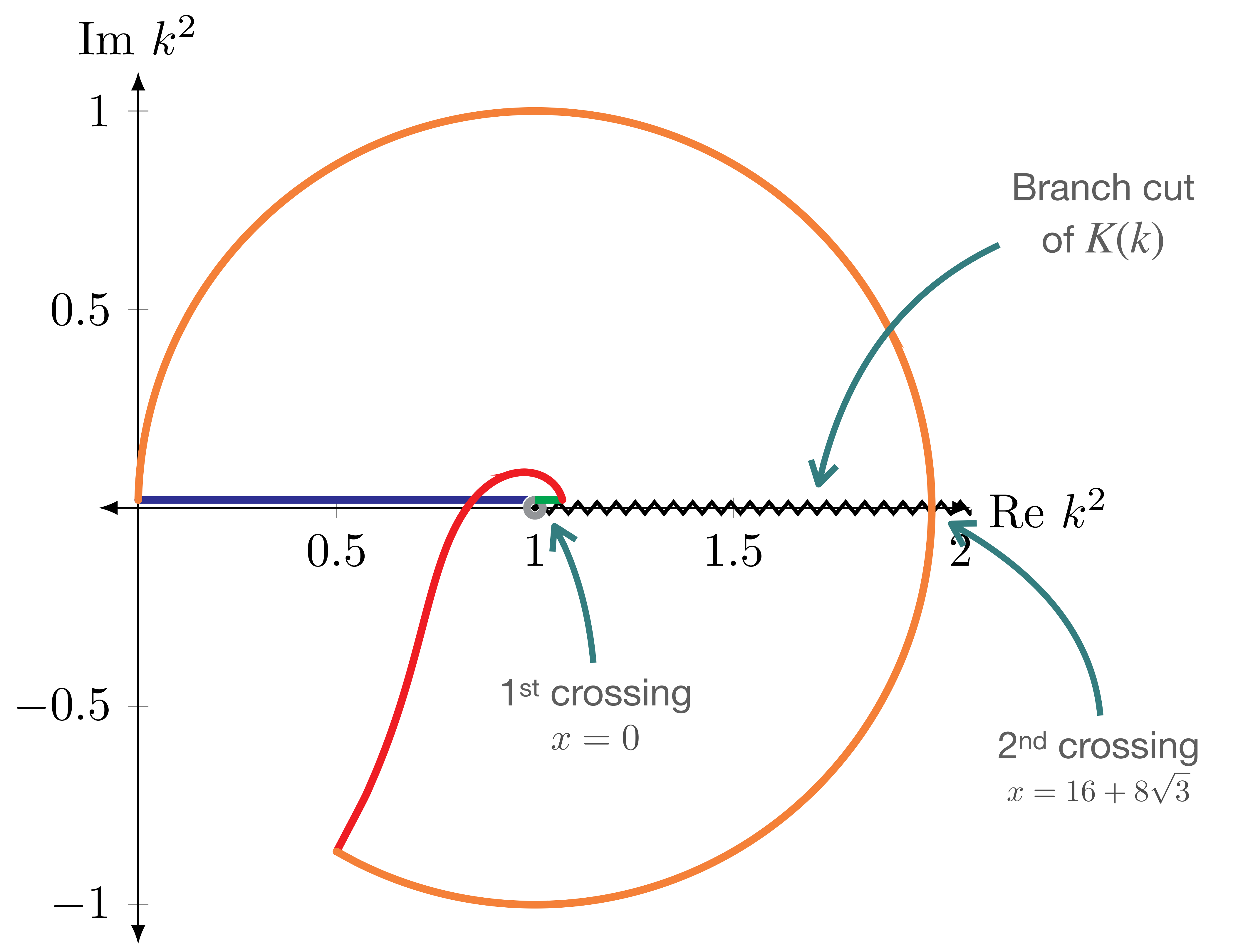
Path $x \in \mathbb{R} + i\delta^+$ crosses branch cut of elliptic integral twice

Take discontinuities into account:

$$\begin{pmatrix} \psi_2^{\text{sun}} \\ \psi_1^{\text{sun}} \end{pmatrix} = \frac{4}{U_3^{\frac{1}{2}}} \gamma \begin{pmatrix} iK\left(\bar{k}\right) \\ K\left(k\right) \end{pmatrix}$$

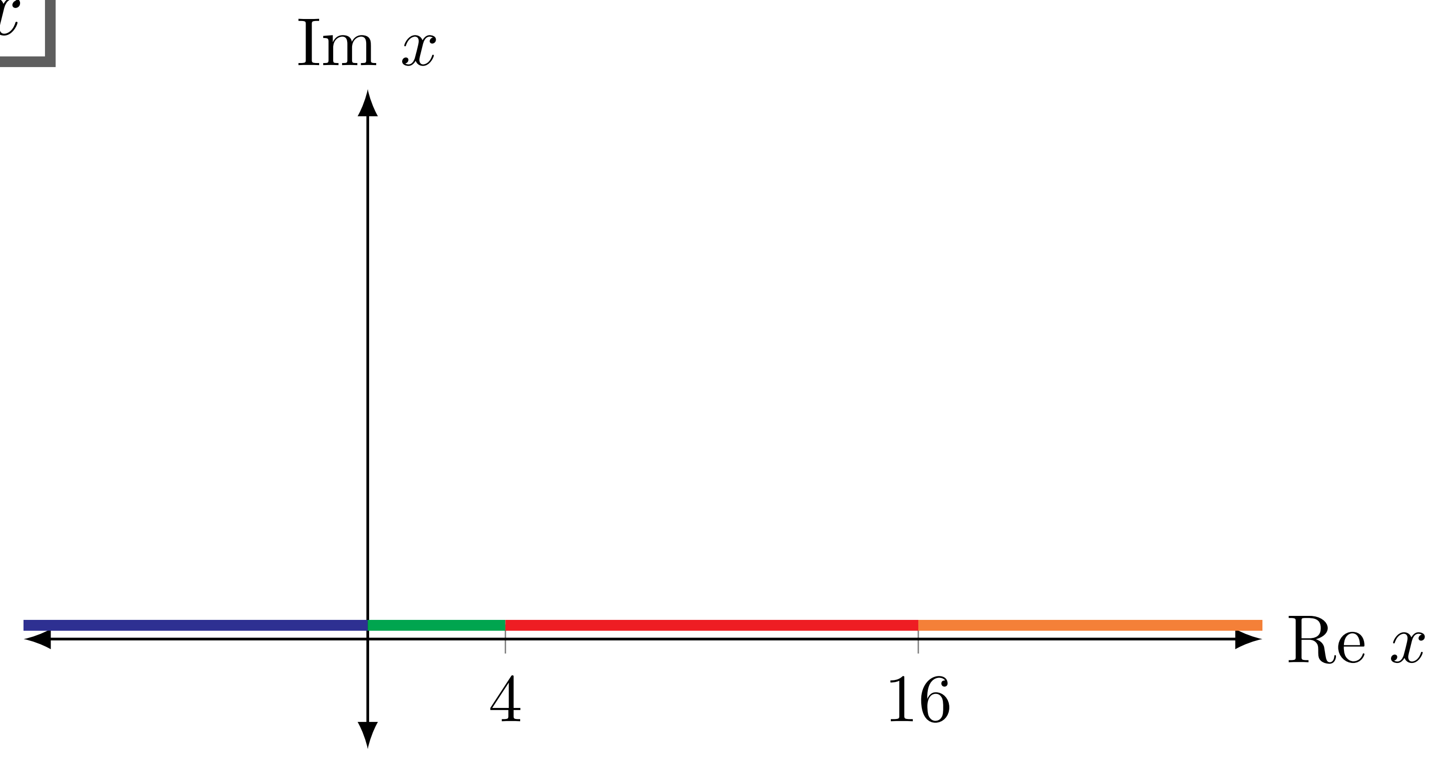
$$\gamma = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if} \quad x < 0 \quad \text{or} \quad 16 + 8\sqrt{3} < x, \\ \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} & \text{if} \quad 0 < x \leq 16 + 8\sqrt{3}. \end{cases}$$

Natural variable: $\tau = \dfrac{\psi_2^{\text{sun}}}{\psi_1^{\text{sun}}}$

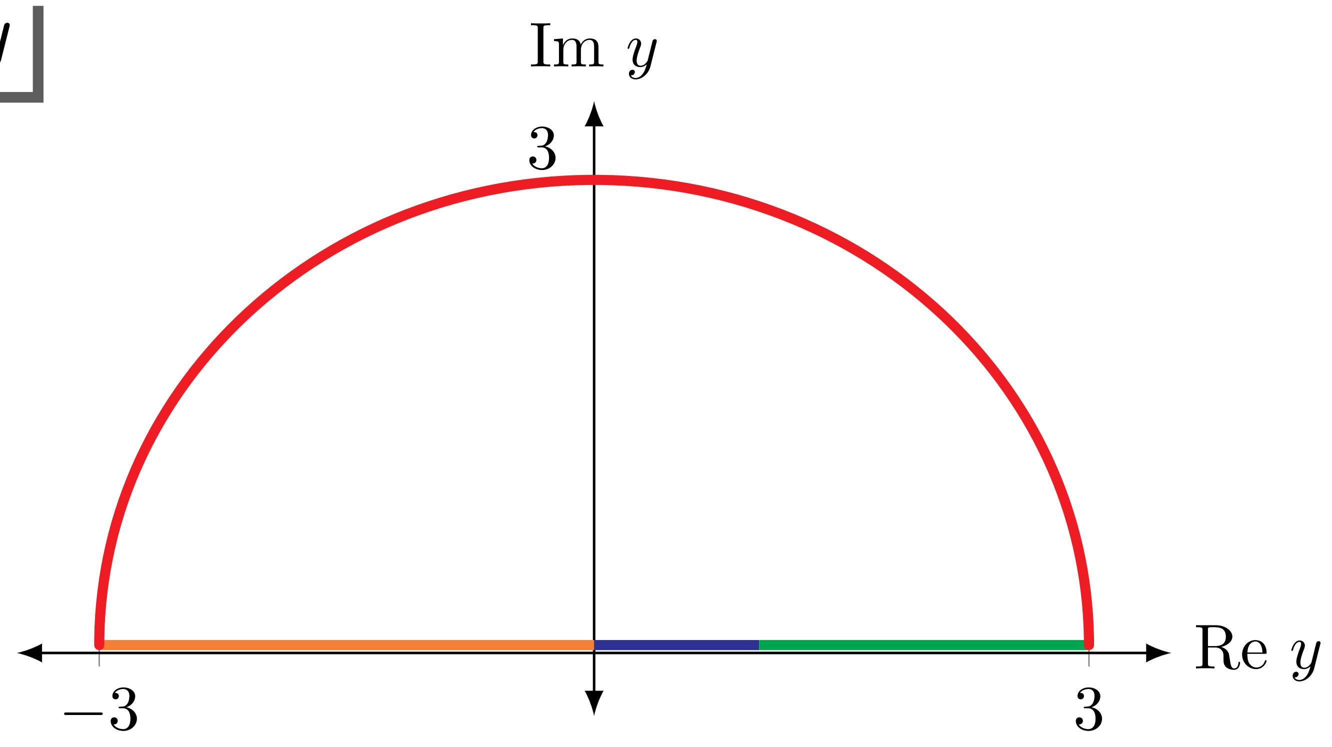


Kinematic Maps

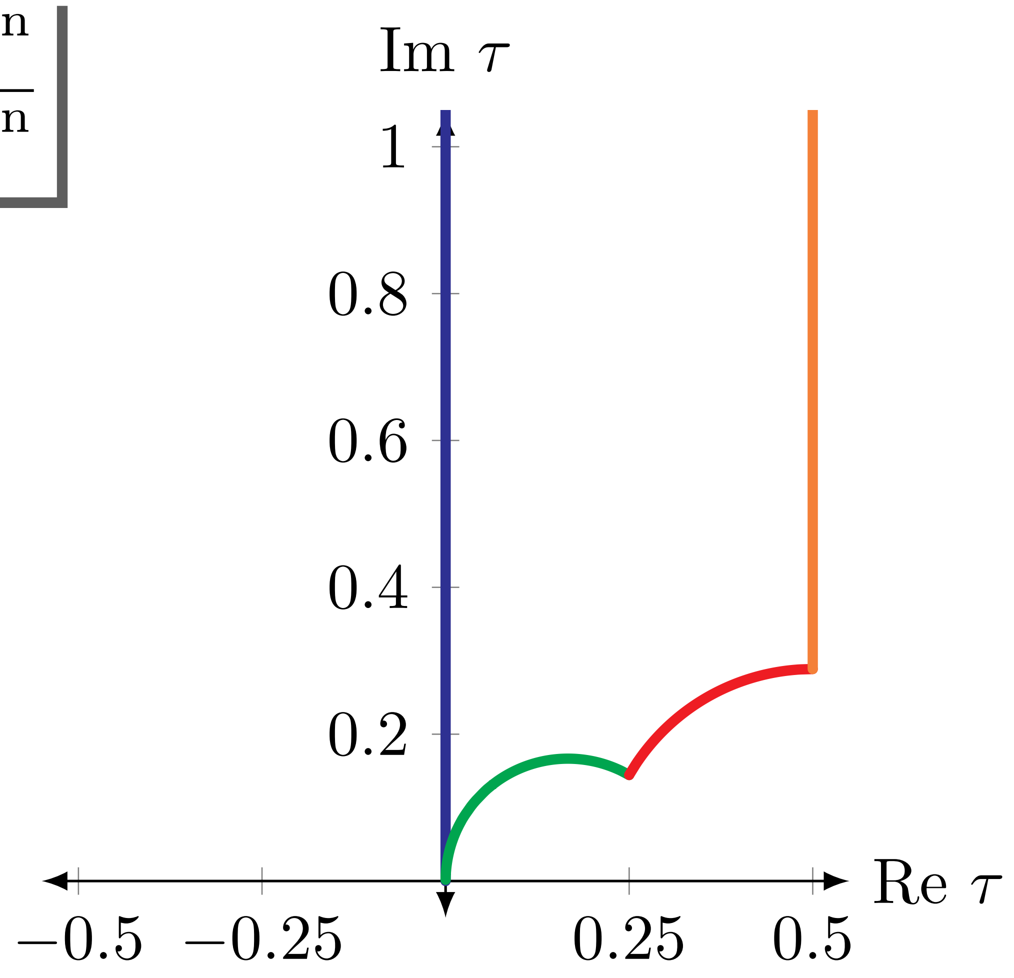
x



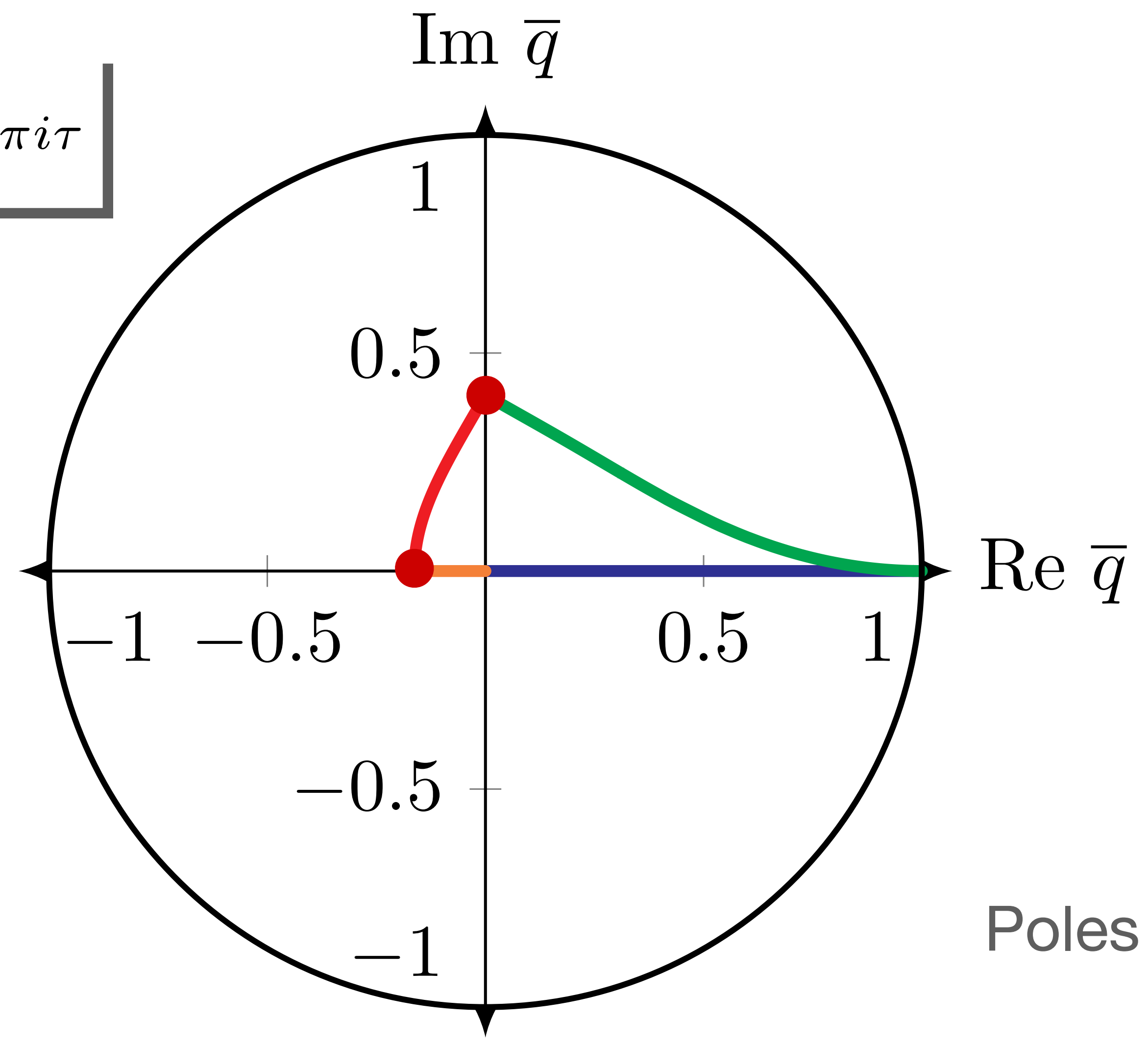
y



$\tau = \frac{\psi_2^{\text{sun}}}{\psi_1^{\text{sun}}}$



$\bar{q} = e^{2\pi i \tau}$



Mapping of special points

x	$-\infty$	0	4	16	∞
y	0	1	3	-3	0
τ	$i\infty$	0	$\frac{1}{4} + \frac{i\sqrt{3}}{12}$	$\frac{1}{2} + \frac{i\sqrt{3}}{6}$	$i\infty$
$\bar{q}$	0	1	$ie^{-\frac{\pi}{6}\sqrt{3}}$	$-e^{-\frac{\pi}{3}\sqrt{3}}$	0

Poles ● limit $\bar{q}$ expansion later on

Obtain $dI = \varepsilon A I$

Wishlist:

- A independent of ε
- A consisting of (meromorphic) modular forms

Again, the look to the Sunrise

Making the Ansatz

$$I_1^{\text{sun}} = \varepsilon^2 I_{110},$$
$$I_2^{\text{sun}} = \varepsilon^2 \frac{\pi}{\psi_1^{\text{sun}}} I_{111},$$
$$I_3^{\text{sun}} = \frac{1}{2\pi i \varepsilon} \frac{d}{d\tau} I_2^{\text{sun}} + F_{32}^{\text{sun}} I_2^{\text{sun}},$$

leads to

$$\mathrm{d}I^{\text{sun}} = \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta_2 & \eta_0 \\ \eta_3 & \eta_4 & \eta_2 \end{pmatrix} I^{\text{sun}}$$

A organised by modular weight

η_k : Modular forms of $\Gamma_1(6)$ of weight k, independent of ε

✓ A independent of ε

✓ A consists of modular forms

Banana Ansatz

Analogous Ansatz for three-loop banana:

$$
\begin{aligned}
I_1 &= \varepsilon^3 I_{1110}, \\
I_2 &= \varepsilon^3 \frac{\pi^2}{\omega} I_{1111}, \\
I_3 &= \frac{1}{2\pi i \varepsilon}\frac{d}{d\tau} I_2 + F_{32} I_2, \\
I_4 &= \frac{1}{2\pi i \varepsilon}\frac{d}{d\tau} I_3 + F_{42} I_2 + F_{43} I_3.
\end{aligned}
$$

Making no assumptions for ω and τ

Requiring $dI = \varepsilon A I$ constrains form of ω, J, F_{32}, F_{42}, F_{43}

$$\frac{\mathrm{d}x}{\mathrm{d}\tau}$$

Eliminating Non-Factorized Pieces

$$dI = \varepsilon \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{pmatrix} I$$

Eliminating Non-Factorized Pieces

$$dI = \varepsilon \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & \color{red}{A_{4,2}} & A_{4,3} & A_{4,4} \end{pmatrix} I$$

Remove ε^2 from $\color{red}{A_{4,2}}$:

$$L_3\,\omega = 0$$

Eliminating Non-Factorized Pieces

$$dI = \varepsilon \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & \color{red}{A_{4,4}} \end{pmatrix} I$$

Remove ε^0 from $\color{red}{A_{4,4}}$:

$$\frac{d \ln J}{dx} = \frac{d \ln \omega}{dx} + \frac{2\left(x^2 - 15x + 32\right)}{x\,(x-4)\,(x-16)}$$

Eliminating Non-Factorized Pieces

$$dI = \varepsilon \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{pmatrix} I$$

Remove ε^{-1} from $A_{4,3}$ (plus previous):

$$\frac{1}{\omega}\frac{d^2\omega}{dx^2} - \frac{1}{2}\left(\frac{1}{\omega}\frac{d\omega}{dx}\right)^2 + \frac{2\left(x^2 - 15x + 32\right)}{x\left(x-4\right)\left(x-16\right)}\frac{1}{\omega}\frac{d\omega}{dx} + \frac{\left(x-8\right)}{2x\left(x-4\right)\left(x-16\right)} = 0$$

Eliminating Non-Factorized Pieces

$$dI = \varepsilon \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & \color{red}{A_{4,2}} & A_{4,3} & A_{4,4} \end{pmatrix} I$$

Remove ε^{-1} from $\color{red}{A_{4,2}}$:

$$\frac{d^2 F_{32}}{dx^2} + \left[\frac{d\ln\omega}{dx} + \frac{2\left(x^2 - 15x + 32\right)}{x\,(x-4)\,(x-16)}\right]\frac{dF_{32}}{dx} + \frac{3J}{2\pi i}\left[-\frac{(x-10)}{(x-4)\,(x-16)}\left(\frac{d\ln\omega}{dx}\right)^2 \right.$$
$$\left. - \frac{2\left(x^3 - 30x^2 + 228x - 640\right)}{x\,(x-4)^2\,(x-16)^2}\frac{d\ln\omega}{dx} - \frac{\left(x^3 - 28x^2 + 168x - 384\right)}{x^2\,(x-4)^2\,(x-16)^2}\right] = 0$$

Eliminating Non-Factorized Pieces

$$dI = \varepsilon \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{pmatrix} I$$

Remove ε^0 from $A_{4,2}$:

$$\frac{dF_{42}}{dx} - 3F_{32}\frac{dF_{32}}{dx} + \frac{3J}{2\pi i}\frac{2\,(x-10)}{(x-4)\,(x-16)}\frac{dF_{32}}{dx}$$
$$+\frac{3J}{2\pi i}\left[\frac{2\,(x-10)}{(x-4)\,(x-16)}\frac{d\ln\omega}{dx} + \frac{2\left(x^3-30x^2+228x-640\right)}{x\,(x-4)^2\,(x-16)^2}\right]F_{32}$$
$$+\frac{J^2}{(2\pi i)^2}\left[-\frac{(11x+16)}{x^2\,(x-16)}\frac{d\ln\omega}{dx} - \frac{(11x-14)}{x^2\,(x-4)\,(x-16)}\right] = 0$$

Eliminating Non-Factorized Pieces

$$dI = \varepsilon \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{pmatrix} I$$

Remove ε^0 from $A_{4,3}$:

$$\frac{dF_{43}}{dx} + 2\frac{dF_{32}}{dx} + \frac{3J}{2\pi i}\left[-\frac{2\,(x-10)}{(x-4)\,(x-16)}\frac{d\ln\omega}{dx} - \frac{2\left(x^3-30x^2+228x-640\right)}{x\,(x-4)^2\,(x-16)^2}\right] = 0$$

Solution for Normalisation ω

First constraint is just Picard-Fuchs operator

$$L_3\,\omega = 0$$

Symmetric square

$$\omega_i \in \langle \psi_1^2,\, \psi_1\psi_2,\, \psi_2^2 \rangle$$

Second constraint is non-linear

$$\frac{1}{\omega}\frac{d^2\omega}{dx^2} - \frac{1}{2}\left(\frac{1}{\omega}\frac{d\omega}{dx}\right)^2 + \frac{2\left(x^2-15x+32\right)}{x\left(x-4\right)\left(x-16\right)}\frac{1}{\omega}\frac{d\omega}{dx} + \frac{\left(x-8\right)}{2x\left(x-4\right)\left(x-16\right)} = 0$$

$$\omega_i \in \langle \psi_1^2,\, \cancel{\psi_1\psi_2},\, \psi_2^2 \rangle$$

$$\boxed{\text{We choose: } \omega = \psi_1^2}$$

Next, Fix Kinematic Variable τ

With $\omega = \psi_1^2$ the constraint for τ is

$$\frac{d\ln J}{dx} = \frac{d\ln\omega}{dx} + \frac{2\left(x^2 - 15x + 32\right)}{x\left(x-4\right)\left(x-16\right)}$$

As hoped, satisfied by $\tau = \frac{\psi_2}{\psi_1} = \frac{\psi_2^{\text{sun}}}{\psi_1^{\text{sun}}} \qquad J = \frac{\psi_1^2}{W}$

Wronskian $W = \psi_1 \frac{d}{dx}\psi_2 - \psi_2 \frac{d}{dx}\psi_1$

Constraints for F_{32}, F_{42}, F_{43}

Remaining differential equations are fulfilled for

$$F_{32} = \boxed{F_2} - \frac{\pi i\,(x-10)}{(x-4)\,(x-16)\,W}\left(\frac{\psi_1}{\pi}\right)^2$$

$$F_{42} = \frac{3}{2}\boxed{F_2^2} + \frac{\pi^2\,(x+8)^2\,(x^2-8x+64)}{8x^2\,(x-4)^2\,(x-16)^2\,W^2}\left(\frac{\psi_1}{\pi}\right)^4$$

$$F_{43} = -2\boxed{F_2} - \frac{\pi i\,(x-10)}{(x-4)\,(x-16)\,W}\left(\frac{\psi_1}{\pi}\right)^2$$

All depend on one additional function F_2

F_2 has to satisfy

$$\frac{d^2F_2}{dx^2}+\left[\frac{2\left(x^2-15x+32\right)}{x\left(x-4\right)\left(x-16\right)}+2\left(\frac{d\ln\psi_1}{dx}\right)\right]\frac{dF_2}{dx}=\frac{\pi i\left(x-8\right)\left(x+8\right)^3}{x^2\left(x-4\right)^3\left(x-16\right)^3W}\left(\frac{\psi_1}{\pi}\right)^2$$

Solution: Iterated integral of meromorphic modular form of weight 6!

$$F_2=\left(2\pi i\right)^2\int_{i\infty}^{\tau}d\tau_1\int_{i\infty}^{\tau_1}d\tau_2\,\underbrace{\frac{x\left(x-8\right)\left(x+8\right)^3}{864\left(4-x\right)^{\frac{3}{2}}\left(16-x\right)^{\frac{3}{2}}}\left(\frac{\psi_1}{\pi}\right)^6}_{g_6}$$

Properties:

- $\overline{q}$ expansion of g_6 has only integer coefficients
- $\overline{q}^n$ coefficient of g_6 divisible by n^2
- Carrying out integration, F_2 has simple poles at $x=4,\ 16$

$$dI = \varepsilon A I$$

with

$$A = 2\pi i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -f_{2,a} - f_{2,b} & 1 & 0 \\ 0 & f_{4,b} & -f_{2,a} + 2f_{2,b} & 1 \\ f_{4,a} & f_{6} & f_{4,b} & -f_{2,a} - f_{2,b} \end{pmatrix} d\tau.$$

$$\underbrace{\qquad}_{\text{Holomorphic}} \qquad \underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{Meromorphic}}$$

Alphabet: $\mathcal{A} = \{1, f_{2,a}, f_{2,b}, f_{4,a}, f_{4,b}, f_{6}\}\,.$

Letters are all (meromorphic) modular forms, except $f_{2,b}$!

Just F_2

Basis of Modular Forms

Two classes

Holomorphic: $b_0 = \frac{\psi_1^{\text{sun}}}{\pi}$ $\qquad b_1 = y\frac{\psi_1^{\text{sun}}}{\pi}$

Meromorphic: $b_3 = \frac{1}{(y-3)}\frac{\psi_1^{\text{sun}}}{\pi}$ $\qquad b_{-3} = \frac{1}{(y+3)}\frac{\psi_1^{\text{sun}}}{\pi}$

Poles corresponding to (pseudo-)threshold

Can use these to express all modular forms appearing

Example: $$f_{2,a} = \left(\frac{1}{x-4} + \frac{1}{x-16}\right)\frac{\psi_1^2}{2\pi i W}$$
$$= \left[\frac{1}{6}y^2 - \frac{5}{3}y + \frac{9}{2} - \frac{6}{y-3} - \frac{24}{y+3}\right]\left(\frac{\psi_1^{\text{sun}}}{\pi}\right)^2$$
$$= \frac{1}{6}b_1^2 - \frac{5}{3}b_0 b_1 + \frac{9}{2}b_0^2 - 6b_0 b_3 - 24 b_0 b_{-3}.$$

Letter $f_{2,b}$ is not a modular form, but iterated integral of one: non-trivial transformation under $\Gamma_1(6)$

Path decomposition gives us

$$
\begin{aligned}
(f_{2,b}|_2\gamma)(\tau) = f_{2,b}(\tau) \\
&-6\frac{c}{c\tau+d}\frac{1}{2\pi i}I\left(1,1,g_6;\tau\right)+18\left(\frac{c}{c\tau+d}\right)^2\frac{1}{(2\pi i)^2}I\left(1,1,1,g_6;\tau\right) \\
&-24\left(\frac{c}{c\tau+d}\right)^3\frac{1}{(2\pi i)^3}I\left(1,1,1,1,g_6;\tau\right) \\
&+\frac{C_{1,6}}{(c\tau+d)^2}-\frac{2\pi i C_6}{c\left(c\tau+d\right)^3}
\end{aligned}
$$

Constants:

$$
\begin{aligned}
C_{1,6} &= I\left(1,g_6;i\infty,\frac{a}{c}\right) \\
C_6 &= I\left(g_6;i\infty,\frac{a}{c}\right)
\end{aligned}
$$

Singularities obstruct simple evaluation

E.g. $a/c=1/6$: $C_{1,6}=5$, $C_6=\frac{1620\zeta_3}{\pi^4}-i\frac{42}{\pi}$

Defining "Quasi-Eichler" of weight k, depth p:

$$(f|_k\gamma)(\tau)=f(\tau)+\sum_{j=1}^{p}\left(\frac{c}{c\tau+d}\right)^j f_j(\tau)+\frac{P_\gamma(\tau)}{(c\tau+d)^p}$$

$f_{2,b}$ transforms "Quasi-Eichler" of modular weight 2 and depth 3

Solution for Master Integrals

Initial condition of I_{1111} in limit $1/x \to 0$ from Mellin-Barnes representation

Master integrals to arbitrary power in ε as iterated integrals over $\{1, f_{2,a}, f_{2,b}, f_{4,a}, f_{4,b}, f_6\}$

$$I\left(f_1,...,f_n;\tau\right) = (2\pi i)^n \int_{i\infty}^{\tau} d\tau_1 ... \int_{i\infty}^{\tau_{n-1}} d\tau_n \; f_1\left(\tau_1\right)...f_n\left(\tau_n\right)$$

e.g., with $I_2 = \varepsilon^3 \dfrac{\pi^2}{\psi_1^2} I_{1111} = \varepsilon^3 I_2^{(3)} + \varepsilon^4 I_2^{(4)} + \mathcal{O}(\varepsilon^5)$

$$I_2^{(3)} = \frac{4}{3}\zeta_3 + I\left(1,1,f_{4,a};\tau\right)$$

Holomorphic, agrees with [Bloch, Kerr, Vanhove]

$$\begin{aligned} I_2^{(4)} = {} & 2\zeta_4 + \frac{4}{3}\zeta_3\left[\frac{11}{2}\ln\left(\bar{q}\right) - I\left(f_{2,a};\tau\right) - I\left(f_{2,b};\tau\right)\right] + \zeta_2 \ln^2\left(\bar{q}\right) - I\left(1,1,f_{2,a},f_{4,a};\tau\right) \\ & - I\left(1,f_{2,a},1,f_{4,a};\tau\right) - I\left(f_{2,a},1,1,f_{4,a};\tau\right) - I\left(1,1,f_{2,b},f_{4,a};\tau\right) \\ & + 2I\left(1,f_{2,b},1,f_{4,a};\tau\right) - I\left(f_{2,b},1,1,f_{4,a};\tau\right) \end{aligned}$$

Obtained explicit expressions for all master integrals up to ε^6

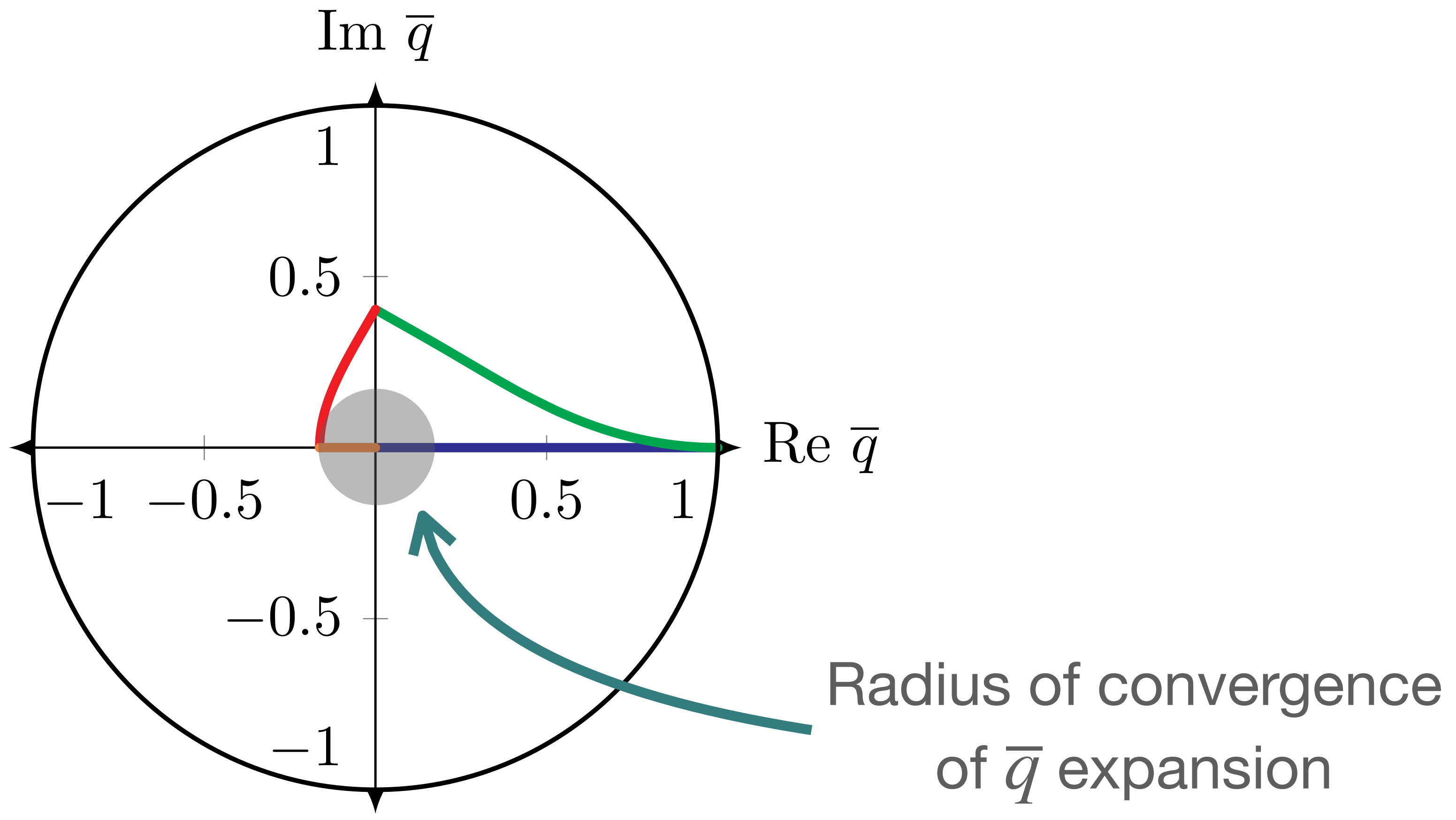
All integrals have uniform length

Numeric Verification

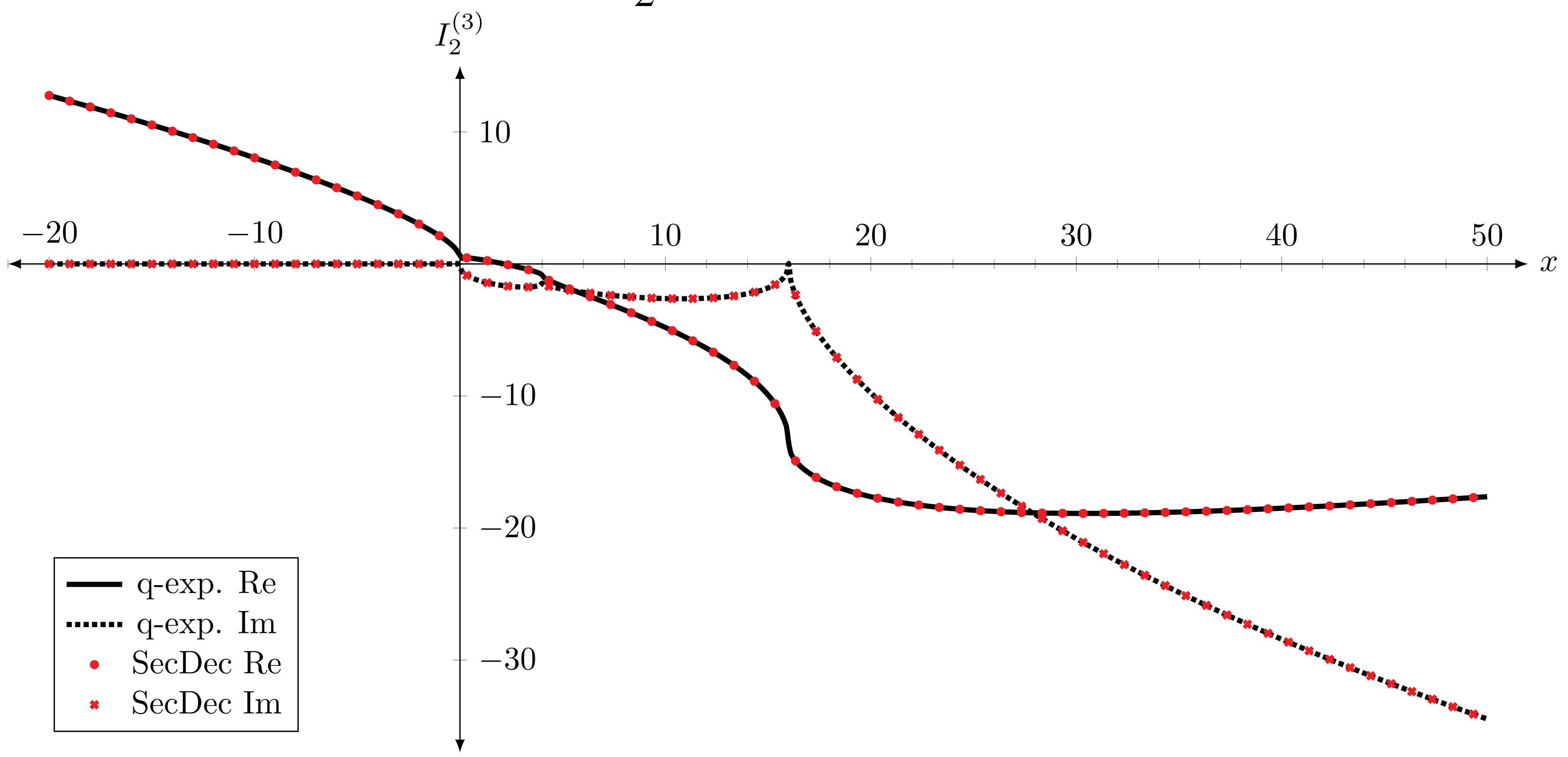
Numeric evaluation via $\overline{q}$-expansion

Singularities limit radius of convergence

Comparison against `SecDec`

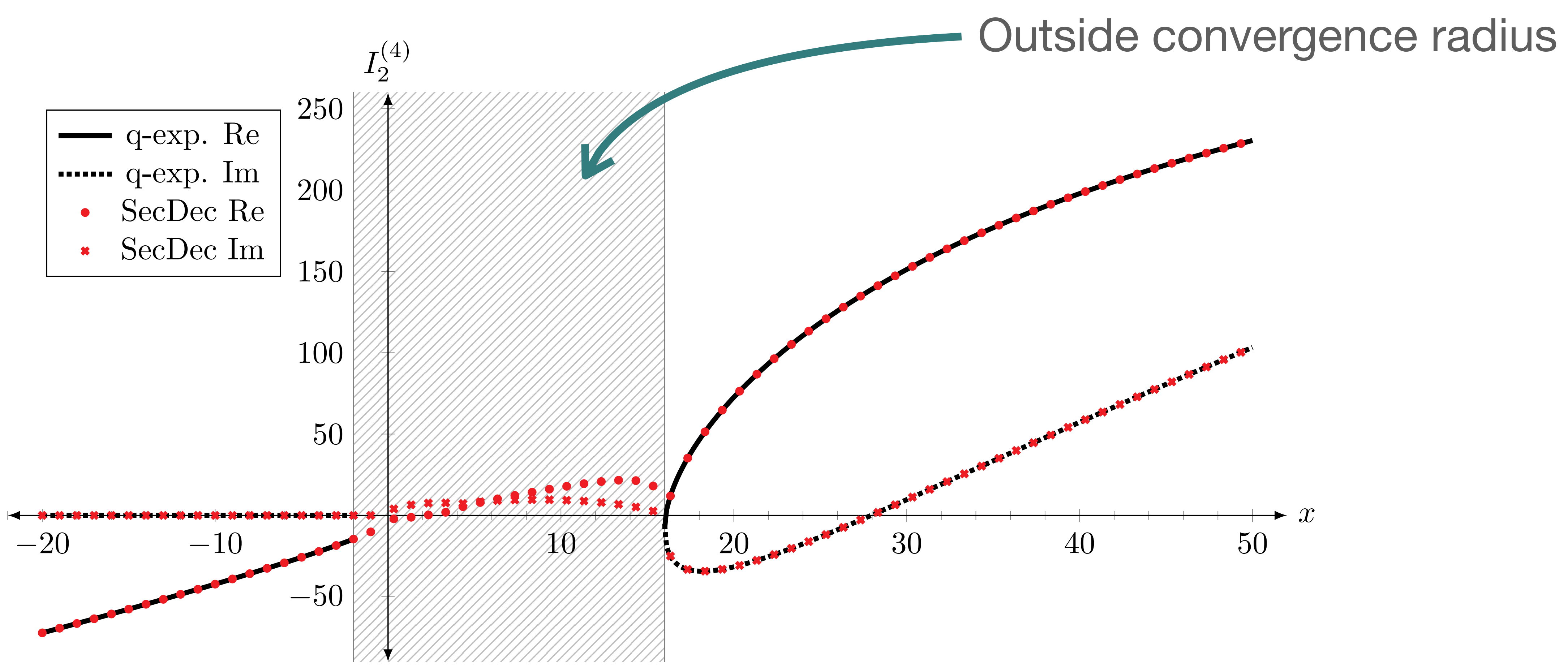


I_2, order ε^3



Only $f_{4,a}$, therefore holomorphic

I_2, order ε^4



Also meromorphic $f_{2,a}, f_{2,b}$

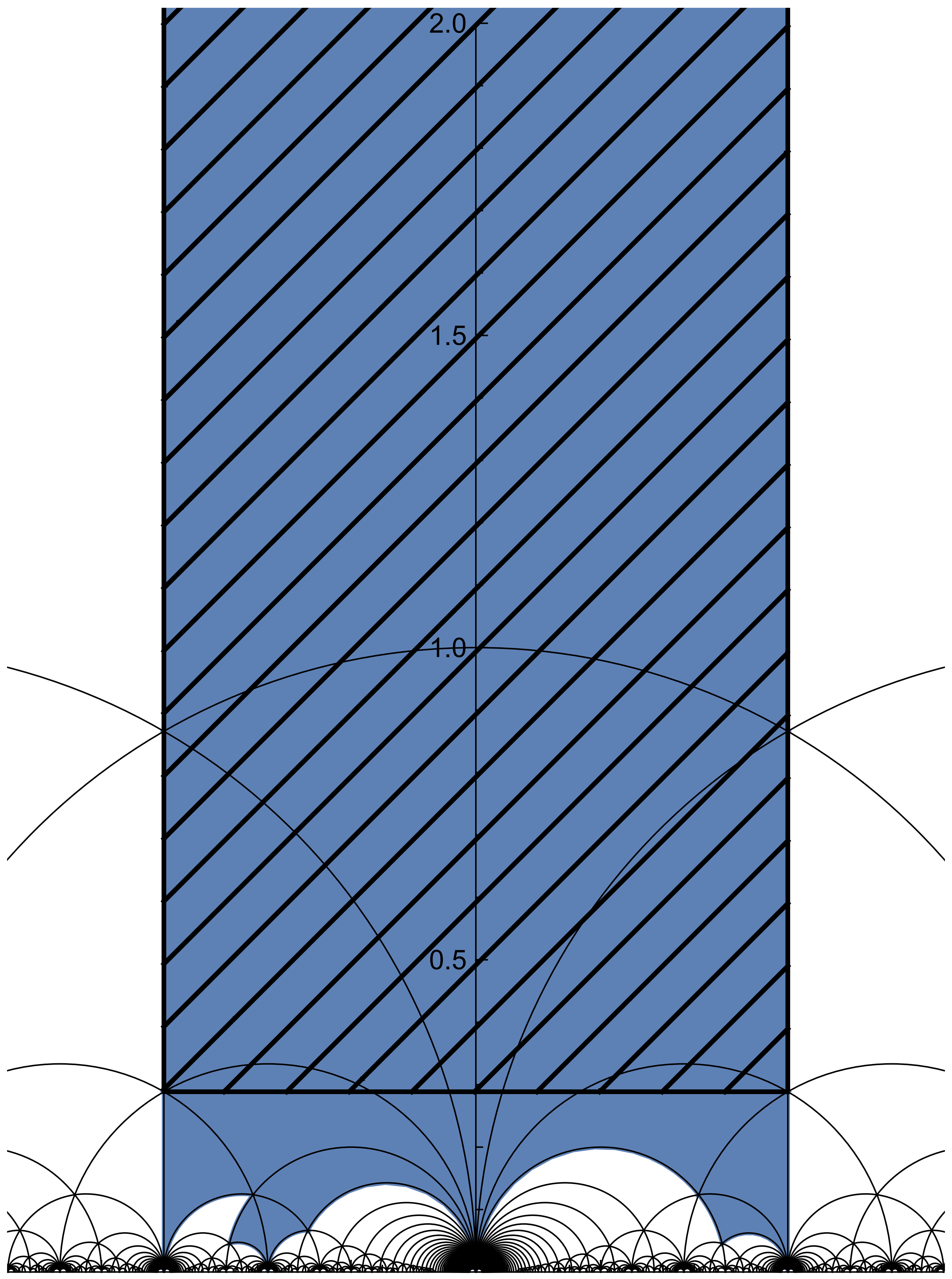
Summary

✓ Differential equation in ε-factorised form

✓ Meromorphic modular form…

? …except for F_2

? What about four loop?


2.0
1.5
1.0
0.5