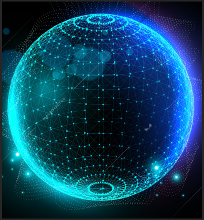
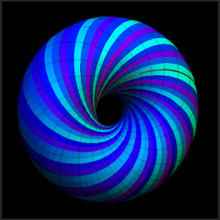
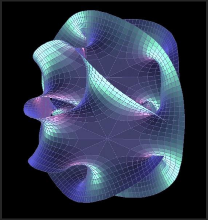
Strange Symbol Structures for a Sufficiently Supplemented Sunrise

arXiv:2209.03922 with **Andreas Forum**

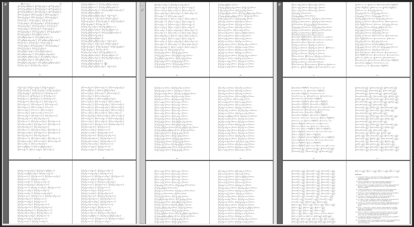
Matt von Hippel (Niels Bohr International Academy)

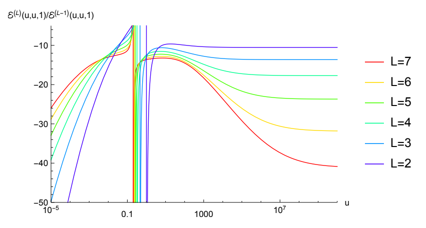
Our goal: Make everything as easy as polylogs

 $\Longrightarrow$  $\Longrightarrow$ 

Wouldn't it be nice to have a coaction?

A big part of what makes polylogs easy: coactions and symbols.





$$R_6^{(2)}(u_1,u_2,u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}. \qquad (3)$$

cluster algebras, Steinmann conditions, Landau equations, differential equations, bootstrap methods...

Too much to list, lots and lots of progress!

Wouldn't it be nice to have a coaction?

But wait a minute. . .

4.1 Periods and the motivic coaction

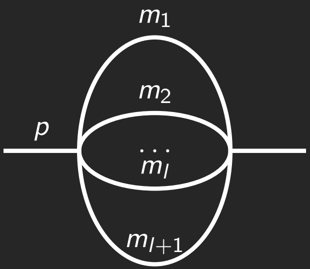
Our starting point for constructing a coaction on eMPLs is the *motivic coaction* of ref. [75]. Very loosely speaking, the motivic coaction provides a general framework to define a coaction on arbitrary periods, and it contains the coaction on MPLs in eq. (2.6) as a special case. The formula for the motivic coaction $\Delta^{\mathfrak{m}}$ can (schematically) be written as[4]

$$\Delta^{\mathfrak{m}}([\gamma,\omega]) = \sum_i [\gamma,\omega_i] \otimes [\omega_i,\omega]\,. \tag{4.1}$$

[Broedel, Duhr, Dulat, Penante, Tancredi: 1803.10256]

Why don't we just do *that*?

Sufficiently Supplementing the Sunrise



Depends on $l+1$ dimensionless variables,

$$\underline{z} = \{m_1^2/p^2, \dots m_{l+1}^2/p^2\}$$

First focus on equal-mass case, $z = m^2/p^2$

We supplement the sunrise with its master integrals in $D = 2 - 2\epsilon$ dimensions:

$$J_{l,0} = -\frac{\Gamma(1+\epsilon)}{\Gamma(1+l\epsilon)}$$

$$J_{l,1} = \frac{(-1)^{l+1}(m^2)^{1+l\epsilon}}{\Gamma(1+l\epsilon)} I_{\text{sunrise}}$$

$$J_{l,k} = (1+2\epsilon)\dots(1+k\epsilon)\partial_z^k J_{l,1}$$

Later we will take $\epsilon \to 0$

What are symbols supposed to do?

$$dF_n = \sum_i F^i_{n-1} \times \omega_i$$
$$\mathcal{S}(F_n) = \sum_i \mathcal{S}(F^i_{n-1}) \otimes \omega_i$$

For polylogs, $\omega_i = d \log \phi_i$. More generally, a one-form.

Essential properties:

- Homogeneous differential (terms in dF_n proportional to F_{n-1}'s)
- Recursion terminates (no infinite symbols!)

Unipotent and Semi-Simple Functions

Need functions satisfying unipotent differential equation:

$$d\underline{I} = A\underline{I}$$

where

$$A^n = 0$$

for some n.
Like factoring functions into polylog+rational part, factor general functions into unipotent and semi-simple:

$$f = \sum s \times u$$
$$\mathcal{S}(f) = \sum s \times \mathcal{S}(u)$$

[Broedel, Duhr, Dulat, Penante, Tancredi: 1803.10256]

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This is a choice! Many options! Some interesting! Some trivial!

[Broedel, Duhr, Dulat, Penante, Tancredi: 1803.10256]

Cocktail Recipe for a Coaction

We need two ingredients:

Motivic Coaction

Symbol on de Rham periods

[Brown: 1512.06410]
[Broedel, Duhr, Dulat, Penante, Tancredi: 1803.10256]

The Motivic Coaction

$$\boxed{\Delta^{\mathfrak{m}}([\gamma,\omega]) = \sum_i [\gamma,\omega_i] \otimes [\omega_i,\omega]}$$

- γ a cycle, ω a form
- $[\gamma,\omega]$ is a motivic period
- $[\omega_i,\omega]$ pairs forms: de Rham period
- Next must make sense of it: make a symbol!

[Brown: 1512.06410]

Specifying our Symbology

These forms come from our basis of integrals:

$$\underline{I} = \{\int \omega_1, \int \omega_2, \ldots\}$$

Using A from the differential equation, make matrix of words

$$T_A = 1 + [A]^R + [A|A]^R + [A|A|A]^R + \ldots$$

(Unipotence: guaranteed to terminate!)
Symbol then defined:

$$\boxed{\mathcal{S}([\omega_i, \omega_j]) \equiv (T_A)_{ji}}$$

[Brown: 1512.06410]

All Together Now

Stick the symbol in the coaction to make sense of the de Rham periods:

$$\Delta \equiv (\mathrm{id} \otimes \mathcal{S})\Delta^{\mathfrak{m}}$$

Get reasonable action on our functions:

$$\boxed{\Delta(I_a) = \sum_c I_c \otimes \mathcal{S}([\omega_c, \omega_a])}$$

[Broedel, Duhr, Dulat, Penante, Tancredi: 1803.10256]

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We've got the recipe...now to apply it!

[Broedel, Duhr, Dulat, Penante, Tancredi: 1803.10256]

Basis Cleaning

We're going to "wash" our basis a few times, to get from the $J_{l,k}$ to a basis with nice properties.

Sorting Laundry

Separate out the top sector (i.e., throw out tadpoles).
Gives inhomogeneous system:

$$d\underline{J}(z,\epsilon) = B(z,\epsilon)\underline{J}(z,\epsilon) + \underline{N}(z,\epsilon)$$

Pre-Wash

Maximal cuts satisfy homogeneous version.
Write $D = 2$ solutions with **Wronskian**:

$$dW(z) = B(z,0)W(z)$$

First "wash": use to transform to new basis:

$$\underline{L}(z,\epsilon) = W^{-1}(z)\underline{J}(z,\epsilon)$$

Taking $\epsilon \to 0$, this obeys:

$$\boxed{d\underline{L}(z,0) = W^{-1}(z)\underline{N}(z,0)}$$

[Bönisch, Duhr, Fischbach, Klemm, Nega: 2108.05310]

Griffiths Cycle

$W^{-1}(z)$ in principle quite complicated.
Simplified due to **Griffiths Transversality**

$$\underline{\Pi}(z)^T \Sigma_l \partial_z^k \underline{\Pi}(z) = \begin{cases} 0 & \text{for } k < l-1 \\ C_k(z) & \text{for } k = l-1 \end{cases}$$

Quadratic relations of **Frobenius basis**

$$\underline{\Pi}(z) = \{\varpi_{l,0}(z), \ldots, \varpi_{l,l-1}(z)\}$$

Gives,

$$W_{k,l}^{-1}(z) = \frac{(-1)^{l+k} \varpi_{l,l-k}(z)}{C_l(z)}$$

[Bönisch, Duhr, Fischbach, Klemm, Nega: 2108.05310]

Add Detergent

$$dL_{l,k} = (-1)^{k+1}\frac{(l+1)!}{z^2}\varpi_{l,k-1}(z)dz$$

To get a unipotent system, we need to add some functions back in.

For a finer wash, add ratios of periods:

$$\underline{T}_l = (L_{l,1}, \ldots, L_{l,l}, \tau_0, \ldots, \tau_{l-2}, 1)$$

with:

$$\tau_j(z) = \frac{\varpi_{l,j}(z)}{\varpi_{l,l-1}(z)}$$

[Us! (MvH and Andreas Forum)]

On Potency

$$\tau_j(z) = \frac{\varpi_{l,j}(z)}{\varpi_{l,l-1}(z)}$$

An implication here: while $\tau_j(z)$ are now *unipotent* functions, we've now made $\varpi_{l,l-1}(z)$ *semi-simple*.

Usual elliptic symbol does this for one of the periods. Also what happens with $i\pi$ in polylog symbol.

Coaction acts as:

$$\Delta(\varpi_{l,l-1}(z)) = \varpi_{l,l-1}(z) \otimes 1$$

The Unipotent Differential Equation

Now unipotent:

$$d\underline{T}_l = M_l(z)\underline{T}_l\,, \quad \text{with } M_l(z)^3 = 0 \text{ to all loops!}$$

$$M_l(z)_{i,j} = \begin{cases} (-1)^{j+1}\frac{(l+1)!}{z^2}\varpi_{l,l-1}dz & i+j = 2l+1, 1 \le j \le l, l < i \le 2l \\ d\tau_{j-l-1} & i = 2l, l < j < 2l \\ 0 & \text{otherwise} \end{cases}$$

$l = 4$ Example

$$M_4(z) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5!}{z^2}\varpi_{4,3}dz \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{5!}{z^2}\varpi_{4,3}dz & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{5!}{z^2}\varpi_{4,3}dz & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{5!}{z^2}\varpi_{4,3}dz & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d\tau_0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d\tau_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d\tau_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Symbol Results

Recall that

$$\mathcal{S}([\omega_i, \omega_j]) \equiv (T_A)_{ji} \quad \text{with} \quad T_A = 1 + [A]^R + [A|A]^R + \ldots$$

This gives us

$$\mathcal{S}([\omega_i, \omega_j]) = \begin{cases} 1 & i = j \\ (-1)^{j+1} \left[\frac{(l+1)!}{z^2} \varpi_{l,l-1} dz\right] & i + j = 2l + 1, \\ & 1 \leq j \leq l,\ l < i \leq 2l \\ (-1)^{j+1} \left[d\tau_j | \frac{(l+1)!}{z^2} \varpi_{l,l-j}(z) dz\right] & i = 2l,\ 2 \leq j \leq l \\ [d\tau_{j-l-1}] & i = 2l,\ l < j < 2l \\ 0 & \text{otherwise} \end{cases}$$

Symbol Results, $l = 4$ example

$$\mathcal{S}([\omega_i, \omega_i]) = 1$$
$$\mathcal{S}([\tau_0, L_{4,4}]) = \left[\frac{5!}{z^2}\varpi_{4,3}dz\right]$$
$$\mathcal{S}([\tau_1, L_{4,3}]) = \left[-\frac{5!}{z^2}\varpi_{4,3}dz\right]$$
$$\mathcal{S}([\tau_2, L_{4,2}]) = \left[\frac{5!}{z^2}\varpi_{4,3}dz\right]$$
$$\mathcal{S}([1, L_{4,1}]) = \left[-\frac{5!}{z^2}\varpi_{4,3}dz\right]$$

$$\mathcal{S}([1, L_{4,2}]) = \left[d\tau_2|\frac{5!}{z^2}\varpi_{4,3}dz\right]$$
$$\mathcal{S}([1, L_{4,3}]) = \left[d\tau_1|-\frac{5!}{z^2}\varpi_{4,3}dz\right]$$
$$\mathcal{S}([1, L_{4,4}]) = \left[d\tau_0|\frac{5!}{z^2}\varpi_{4,3}dz\right]$$
$$\mathcal{S}([1, \tau_0]) = [d\tau_0]$$
$$\mathcal{S}([1, \tau_1]) = [d\tau_1]$$
$$\mathcal{S}([1, \tau_2]) = [d\tau_2]$$

Note: symbols still of length two!

Coaction Results, $l = 4$ example

Using the symbols, we build coactions:

$$\Delta(\tau_k) = \tau_k \otimes 1 + 1 \otimes [d\tau_k]$$

$$\Delta(L_{4,1}) = L_{4,1} \otimes 1 - 1 \otimes \left[\frac{5!}{z^2}\varpi_{4,3}dz\right]$$

$$\Delta(L_{4,2}) = L_{4,2} \otimes 1 + 1 \otimes \left[d\tau_2|\frac{5!}{z^2}\varpi_{4,3}dz\right] + \tau_2 \otimes \left[\frac{5!}{z^2}\varpi_{4,3}dz\right]$$

$$\Delta(L_{4,3}) = L_{4,3} \otimes 1 - 1 \otimes \left[d\tau_1|\frac{5!}{z^2}\varpi_{4,3}dz\right] - \tau_1 \otimes \left[\frac{5!}{z^2}\varpi_{4,3}dz\right]$$

$$\Delta(L_{4,4}) = L_{4,4} \otimes 1 + 1 \otimes \left[d\tau_0|\frac{5!}{z^2}\varpi_{4,3}dz\right] + \tau_0 \otimes \left[\frac{5!}{z^2}\varpi_{4,3}dz\right]$$

What about the length?

At $l = 3$, a different symbol exists due to an eMPL expression from [Broedel, Duhr, Dulat, Marzucca, Penante, Tancredi '19], with length **three**.

Remember, though, our functions were "washed". If we "un-wash" them, using $\underline{J} = W\underline{L}$,

$$\begin{aligned} J_{3,1} &= \varpi_{3,0} L_{3,1} + \varpi_{3,1} L_{3,2} + \varpi_{3,2} L_{3,3} \\ &= \varpi_{3,2}\tau_0 L_{3,1} + \varpi_{3,2}\tau_1 L_{3,2} + \varpi_{3,2} L_{3,3} \end{aligned}$$

get length three.

What about the length???

But . . . the length never gets above three!

We might want higher length at higher loops, to expose higher transcendental weight...but that doesn't happen in this formalism.

One way of thinking about it

Remember how we made a choice to add τ_i?

Could add more functions, get longer symbols.

Another way of thinking about it

Our symbol entries aren't "simple", they have structure.

Need something like symbol-prime? [Wilhelm, Zhang '22]

- We have symbols and coactions for the sunrise integral to any loop
- (Also know structure for generic-mass, see bonus slides)
- Would be nice to see explicitly how to connect to known examples
  - Polylogarithmic limits
  - Known elliptic symbols
- Lots of choices...can we fine-tune them well?

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Thank You

Bonus: Generic-Mass Σ Matrices

Frobenius basis has $\lambda = 2^{l+1} - \binom{l+2}{\lfloor \frac{l+2}{2} \rfloor}$ elements

Σ different for odd and even loops:

$$\Sigma_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & -1 & 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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Bonus: Generic-Mass Differential Equations

Using Griffiths, inverse Wronskian is once again linear in each ϖ and derivatives, but now can be more complicated linear combinations:

$$dL_k(\underline{z}) = \sum_{i=0}^{l-1} c_{k,i}(X_i(\underline{z})\tau_i(\underline{z})(\underline{z}) + D_i(\underline{z}))$$

$$X_k = \alpha_{k,\underline{0}}(z)\varpi_\lambda(\underline{z}) + \sum_{\underline{s}\in T} \alpha_{k,\underline{s}}(\underline{z})\partial_{\underline{z}}^{\underline{s}}\varpi_\lambda(\underline{z})$$

$$D_k = \sum_{\underline{s}\in T} \sum_{\substack{\underline{m},\underline{s} \\ n\neq\underline{0} \\ \underline{m}+\underline{n}=\underline{s}}} \alpha_{k,\underline{s}}(\underline{z})\partial_{\underline{z}}^{\underline{m}}\varpi_\lambda\partial_{\underline{z}}^{\underline{n}}\tau_k(\underline{z})$$

- $c_{k,i} \in \{-1, 0, 1\}$
- T = vectors $\underline{s}$ such that $\partial_{\underline{z}}^{\underline{s}} J_{l,0}(\underline{z})$ is a master integral
- $\alpha(\underline{z}) = \sum_i^{l+1} \omega_i(\underline{z})dz_i$ for ω_i rational

Augment basis as before:

$$T_l(\underline{z}) = (L_{l,1}(\underline{z}), ..., L_{l,\lambda}(\underline{z}), \tau_0(\underline{z}), ..., \tau_{\lambda-1}(\underline{z}), 1)$$

Bonus: Generic-Mass Unipotent Differential Equations

Get $d\underline{T}_I(\underline{z}) = M_I(\underline{z})\underline{T}_I(\underline{z})$, with for example:

$$M_3(\underline{z}) = \begin{pmatrix} 0 & \overbrace{\ldots}^{\times 5} & 0 & 0 & 0 & 0 & 0 & X_5 + D_5 \\ 0 & \ldots & 0 & 0 & -X_2 & -X_3 & -X_4 & -D_2 - D_3 - D_4 \\ 0 & \ldots & 0 & -X_1 & 0 & -X_3 & -X_4 & -D_1 - D_3 - D_4 \\ 0 & \ldots & 0 & -X_1 & -X_2 & 0 & -X_4 & -D_1 - D_2 - D_4 \\ 0 & \ldots & 0 & -X_1 & -X_2 & -X_3 & 0 & -D_1 - D_2 - D_3 \\ 0 & \ldots & X_0 & 0 & 0 & 0 & 0 & D_0 \\ 0 & \ldots & 0 & 0 & 0 & 0 & 0 & d\tau_0 \\ 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \vdots \\ 0 & \ldots & 0 & 0 & 0 & 0 & 0 & d\tau_{\lambda-1} \\ 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Bonus: Generic-Mass Coaction, $l = 3$

$$
\begin{aligned}
\Delta(L_{3,1}(\underline{z}) =& 1 \otimes D_5 + 1 \otimes X_5 + L_{3,1}(\underline{z}) \otimes 1 \\
\Delta(L_{3,2}(\underline{z}) =& L_{3,2}(\underline{z}) \otimes 1 - 1 \otimes [d\tau_2|X_2] - 1 \otimes [d\tau_3|X_3] - 1 \otimes [d\tau_4|X_4] \\
& - 1 \otimes (D_2 + D_3 + D_4) - \tau_2 \otimes X_2 - \tau_3 \otimes X_3 - \tau_4 \otimes X_4 \\
\Delta(L_{3,3}(\underline{z}) =& L_{3,3}(\underline{z}) \otimes 1 - 1 \otimes [d\tau_1|X_1] - 1 \otimes [d\tau_3|X_3] - 1 \otimes [d\tau_4|X_4] \\
& - 1 \otimes (D_1 + D_3 + D_4) - \tau_1 \otimes X_1 - \tau_3 \otimes X_3 - \tau_4 \otimes X_4 \\
\Delta(L_{3,4}(\underline{z}) =& L_{3,4}(\underline{z}) \otimes 1 - 1 \otimes [d\tau_1|X_1] - 1 \otimes [d\tau_2|X_2] - 1 \otimes [d\tau_4|X_3] \\
& - 1 \otimes (D_1 + D_2 + D_4) - \tau_1 \otimes X_1 - \tau_2 \otimes X_2 - \tau_4 \otimes X_4 \\
\Delta(L_{3,5}(\underline{z}) =& L_{3,5}(\underline{z}) \otimes 1 - 1 \otimes [d\tau_1|X_1] - 1 \otimes [d\tau_2|X_2] - 1 \otimes [d\tau_3|X_3] \\
& - 1 \otimes (D_1 + D_2 + D_3) - \tau_1 \otimes X_1 - \tau_2 \otimes X_2 - \tau_3 \otimes X_3 \\
\Delta(L_{3,6}(\underline{z}) =& L_{3,6}(\underline{z}) \otimes 1 + 1 \otimes [d\tau_0|X_0] + 1 \otimes D_0 + \tau_0 \otimes X_0 \\
\Delta(\tau_k) =& \tau_k \otimes 1 + 1 \otimes [d\tau_k]
\end{aligned}
$$