

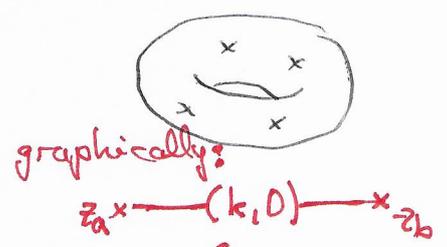
# Modular graph forms (MGFs) from equivariant iterated Eisenstein integrals (EIEIs)

based on: 2209.abcde with D. Dorigoni, M. Doroudiani, J. Druitt, M. Hidding, A. Kleinschmidt, N. Matthes, B. Verbeek

- Outline:
- 1) MGF background
  - 2) EIEI background
  - 3) Iterated integrals for MGFs
  - 4) Genus-zero inspiration
  - 5) Change of alphabet for genus 1
  - 6) EIEIs & MGFs

## 1) MGF background

Closed-string genus-1 amplitudes



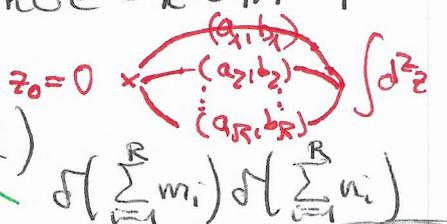
$\Rightarrow$  non-holo modular forms in  $\alpha'$ -expansion  
 due to  $\prod_{j=1}^n \int_{\text{torus}(\tau)} d^2 z_j$  (Kronecker-Eisenstein  $f(z_a - z_b | \tau)$  & cc)

• toy example (low #  $\alpha'$ ): non-holo Eisenstein series

$$E_k(\tau) = \left(\frac{\text{Im} \tau}{\pi}\right)^k \sum_{\substack{m, n \in \mathbb{Z} \\ (m, n) \neq (0, 0)}} \frac{1}{|\text{Im} \tau + n|^2k}$$

here  $k \in \mathbb{N} + 1$

• generalizes to MGFs (here: dihedral)



$$C \begin{bmatrix} a_1 & a_2 & \dots & a_R \\ b_1 & b_2 & \dots & b_R \end{bmatrix} (\tau) = \left( \prod_{j=1}^R \frac{(\text{Im} \tau)^{a_j}}{\pi^{b_j}} \right) \sum_{\substack{m_j, n_j \in \mathbb{Z} \\ (m_j, n_j) \neq (0, 0)}} \frac{1}{\prod_{j=1}^R (m_j \tau + n_j)^{a_j} (m_j \bar{\tau} + n_j)^{b_j}}$$

$a_i, b_i \in \mathbb{Z}$  (usually  $\geq 0$ )

[D'Hoker, Green, Gürdogan, Vanhove '15 & D'Hoker, Green '16]

- non-holo mod forms of wt  $(0, \sum_{j=1}^R (b_j - a_j))$  [hence  $C^+_-$  notation]

- myriads of  $\mathbb{Q}[(sv)MZV]$  relations, e.g.

$$C^+ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = E_3 + \sqrt{3} \quad [\text{Zagier}]$$

$\leadsto$  seek  $\mathbb{Q}[(sv)MZV]$  bases @ fixed  $(\sum_j a_j, \sum_j b_j)$

## 2) EIEI background

Holo Eisenstein series  $G_k(\tau) = \sum_{\substack{m, n \in \mathbb{Z} \\ (m, n) \neq (0, 0)}} \frac{1}{(m\tau + n)^k}$ ,  $k \geq 4$

$\leadsto$  mod forms of wt  $(k, 0)$ , but primitives are not

$$\int_{\tau}^{i\infty} d\tau_1 \tau_1^j G_k(\tau_1) \xrightarrow{\tau \rightarrow -\frac{1}{\tau}} (-j) \left( \int_{\tau}^{i\infty} - \int_0^{i\infty} \right) d\tau_1 \tau_1^{k-2-j} G_k(\tau_1)$$

additive const ("multiple modular value")  $\sim i\pi \zeta_{k-1}, \pi^k$  or 0

[tangential base-pt reg  $\int_{\tau}^{i\infty} d\tau_1 \tau_1^j = -\frac{1}{\tau^{j+1}} \tau^{j+1}$  on both sides]

Brown '14 & '17: modular / equiv. iterated Eisenstein int's

- build equivariant forms  $\underline{G}_k[X, Y; \tau_1] = (X - \tau Y)^{k-2} \int d\tau_1 G_k(\tau_1)$   
 $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sh}_2(\mathbb{Z})$   $\uparrow$  real indeterminates

$$\Rightarrow \underline{G}_k[aX + bY, cX + dY; \frac{a\tau_1 + b}{c\tau_1 + d}] = \underline{G}_k[X, Y; \tau_1]$$

- coeff's of  $(X - \tau Y)^j (X - \bar{\tau} Y)^{k-2-j}$  pre-integrations  $\nu(\text{Im } \tau)^j$  times

$$w_+ \left[ \int_k i\tau, \tau_1 \right] = \frac{d\tau_1}{2\pi i} \left( \frac{\tau - \tau_1}{4y} \right)^{j-2} (\bar{\tau} - \tau_1)^j G_k(\tau_1)$$

$$w_- \left[ \int_k i\tau, \tau_1 \right] = \frac{-d\bar{\tau}_1}{2\pi i} \left( \frac{\tau - \bar{\tau}_1}{4y} \right)^{j-2} (\bar{\tau} - \bar{\tau}_1)^j \overline{G_k(\tau_1)}$$

here & below  
 $y = \pi \text{Im } \tau$

are mod forms @ wt  $(j, k-j-2)$  but  $\# X^a Y^b$  are not

- iterated integrals of  $\underline{G}_{ki}[X_i, Y_i; \tau_i]$  can be made modular by adding cplx. conjugates &  $\mathbb{Q}[MZV]$  lower depth [Brown] 2

### 3) Iterated integrals for MGFs

Homotopy-invariant iterated integrals  $\beta_{\pm}$  over  $\tau_j$  &  $\bar{\tau}_j$   
 (not  $\tau, \bar{\tau}$ )  $\leadsto$  see exhibit 1

$$=: \beta^{eqv} \begin{bmatrix} a-1 \\ a+b \end{bmatrix}$$

Need lower-depth corrections to attain mod. forms, e.g.

$$C^+ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} = - \frac{(2i)^{b-a} \Gamma(a+b)}{\Gamma(a) \Gamma(b)} \left( \beta_+ \begin{bmatrix} a-1 \\ a+b \end{bmatrix} + \beta_- \begin{bmatrix} a-1 \\ a+b \end{bmatrix} - \frac{2\zeta_{a+b-1}}{(a+b-1)(4y)^{b-1}} \right)$$

At higher depth, aim to construct **more tricky @ dpt  $l \geq 2$**

$$\beta^{eqv} \begin{bmatrix} j_1 & j_2 & \dots & j_l \\ k_1 & k_2 & \dots & k_l \end{bmatrix} = \sum_{i=0}^l \beta_- \begin{bmatrix} j_i & \dots & j_1 \\ k_i & \dots & k_1 \end{bmatrix} \beta_+ \begin{bmatrix} j_{i+1} & \dots & j_l \\ k_{i+1} & \dots & k_l \end{bmatrix} + \left( \text{MZV} \times \text{lower dpt} \right)$$

by itself  $\tau \rightarrow \tau+1$  invariant

as non-holo modular forms @ weights  $(0, \sum_{i=1}^l (k_i - 2j_i))$

Spoiler:  $\beta^{eqv}$  come from Brown's construction and yield MGFs,  $\leadsto$  see exhibit 2 for dpt 2 expls

### 4) Genus-zero inspiration

$$\text{MPLs } G(a_1, a_2, \dots, a_l | z) = \int_0^z \frac{dt}{t-a_1} G(a_2, \dots, a_l | t)$$

with  $a_i \in \{0, 1\}$  and  $G(\emptyset | z) = 1$  and  $u$ -reg

$\rightarrow$  build  $sv$ -versions from generating series

$$I_{\pm}(e_0, e_1 | z) = \sum_{l=0}^{\infty} \sum_{\substack{a_1, a_2, \dots, a_l \\ \in \{0, 1\}}} e_{a_1} e_{a_2} \dots e_{a_l} \times \begin{cases} G(a_1, \dots, a_l | z) = " + " \\ \frac{G(a_1, \dots, a_l | z)}{G(a_1, \dots, a_l | z)} = " - " \end{cases}$$

$\nwarrow$   
 $\swarrow$   
 non-comm. braid operators

$$I^{sv}(e_0 e_1 z) = \sum_{l=0}^{\infty} \sum_{\substack{a_1, \dots, a_l \\ e_i \in \{0,1\}}} e_{a_1} e_{a_2} \dots e_{a_l} G^{sv}(a_1 a_2 \dots a_l z)$$

$$= \underbrace{I_+(e_0 e_1 z)}_{\substack{\text{by themselves generate} \\ \sum_{i=0}^l \overline{G(a_1, \dots, a_i, a_{i+1}, \dots, a_l, z)} \\ \text{cf. } \beta^{sv} \text{ @ leading depth}}} M^{sv} \underbrace{I_-(e_0 e_1 z)}_{\substack{\text{series on sv MZVs} \\ \text{with derivations } M_3, M_5, M_7, \dots}} (M^{sv})^{-1}$$

reversal  
...  $e_a e_b \dots = \dots e_b e_a \dots$

reformulation of Brown '04  
work in progress with  
proof by Deepak Kamlesh

$$M^{sv} = 1 + \sum_{k \in \mathbb{Z}_{N+1}} 2\zeta_k M_k$$

svfk →

$$+ \sum_{k_1, k_2 \in \mathbb{Z}_{N+1}} 2\zeta_{k_1} \zeta_{k_2} M_{k_1} M_{k_2} + \dots$$

sv( $f_{k_1} f_{k_2}$ ) →

(higher depth :  
sv [ $\geq 3$  letters  $f_{z+1}$ ]  
in  $f$ -alphabet)

$$M^{sv} \tilde{I}_- (M^{sv})^{-1} = \tilde{I}_- + \sum_{k \in \mathbb{Z}_{N+1}} 2\zeta_k [M_k, \tilde{I}_-]$$

$$+ \sum_{k_1, k_2 \in \mathbb{Z}_{N+1}} 2\zeta_{k_1} \zeta_{k_2} [M_{k_1}, [M_{k_2}, \tilde{I}_-]] + \dots$$

trade all  $M_k$  in  $I^{sv}$  for  $e_0 e_1$  by repeating

$$[e_0, M_k] = 0 \Rightarrow \left\{ \begin{aligned} & M^{sv} I_-(e_0 e_1 z) (M^{sv})^{-1} \\ & = I_-(e_0, e_1 = M^{sv} e_1 (M^{sv})^{-1}, z) \end{aligned} \right.$$

reverse before substituting  $e_1' = M^{sv} e_1 (M^{sv})^{-1}$

$$[e_1, M_k] = \left[ \underbrace{\Phi(e_0, e_1)}_{\substack{\text{Drinfeld assoc. } I_+(e_0 e_1 z=1)}} \Big|_{\zeta_k} e_1 \right]$$

only  $e$ 's

e.g.  $[e_1, M_3] = [[ [e_0 e_1], e_0 + e_1 ], e_1]$

$$\Rightarrow G^{sv}(0,0,1,1; z) = I^{sv}(e_0 e_1 z) |_{e_0 e_0 e_1}$$

$$\text{contains } 2\zeta_3 \overline{G(1; z)} [M_3, e_1] |_{e_0 e_0 e_1} = 2\zeta_3 \overline{G(1; z)}$$

## 5) Change of alphabet for genus 1

$$e_0, e_1 \xrightarrow{\text{generalize}} \underline{E_k^{(j)}} = \text{ad}_{E_0}^j(E_k) \text{ @ } k \geq 4 \text{ and } 0 \leq j \leq k-2$$

Tsunogai's definitions  $\{E_k, k \in 2\mathbb{N}_0\}$

subject to  $E_k^{(k-1)} = 0$  and  $[E_4, E_{10}] - 3[E_6, E_8] = 0$  and  $\infty$  more

genus-1 generating series

$$\mathcal{J}_{\pm}(\{E_k\}|\tau) = \sum_{l=0}^{\infty} \sum_{\substack{k_1 k_2 \dots k_l \\ E_k \in 2\mathbb{N}+2}}^{k_l-2} \dots \sum_{j_l=0}^{k_l-2} \left( \prod_{i=1}^l \frac{(-j_i)^{k_i-1}}{(k_i-j_i-2)!} \right)$$

$$\times \underbrace{E_{k_l}^{(j_l)} \dots E_{k_2}^{(j_2)} E_{k_1}^{(j_1)}}_{\text{words } P \text{ in } \{j_i\}} \beta_{\pm} \left[ \begin{matrix} j_1 & j_2 & \dots & j_l \\ k_1 & k_2 & \dots & k_l \end{matrix} \middle| \tau \right]$$

$$= \sum_P E[P] \beta_{\pm}[P|\tau] \quad \text{"words } P \text{ in } \{j_i\}"$$

generalize  $M^{\text{sv}} \tilde{I}_{-} (M^{\text{sv}})^{-1}$  to

$$M^{\text{sv}} \tilde{J}_{-}(\{E_k\}|\tau) (M^{\text{sv}})^{-1} = \tilde{J}_{-} + 2 \sum_{m \in 2\mathbb{N}+1} \tilde{J}_m[z_m, \tilde{J}_{-}] + \left( \begin{matrix} \text{above @} \\ \tilde{I}_{-} \rightarrow \tilde{J}_{-} \\ M_k \rightarrow z_k \end{matrix} \right)$$

with derivations  $z_m$  subject to  $[z_m, E_0] = 0$  and

$[z_m, E_k] = \text{nested } [0, \cdot]$  of  $E_{k_i}^{(j_i)}$  [Pollack '09]

$$\text{e.g. } [z_3, E_4] = \frac{1}{504} \left( [E_6^{(2)}, E_4] - 3[E_6^{(1)}, E_4^{(1)}] + 6[E_6, E_4^{(2)}] \right)$$

## 6) EIEIs & MGFs

• new ingredient @ genus 1

→ series  $B^{\text{sv}}(\{E_k\}|\tau)$  rational in  $\tau, \bar{\tau}$  such that...

$$\mathcal{J}^{\text{eqv}}(\{E_k\}|\tau) = \mathcal{J}_{+}(\{E_k\}|\tau) B^{\text{sv}}(\{E_k\}|\tau) M^{\text{sv}} \tilde{J}_{-}(\{E_k\}|\tau) (M^{\text{sv}})^{-1}$$

$$= \sum_P E[P] \beta^{\text{eqv}}[P|\tau] \quad \text{implicit [Brown '17]}$$

... generates mod-forms  $\beta^{\text{eqv}}$  after  $E_k^{(j)}$ -rel's

• unpack  $B^{sv}(\{\epsilon_k\}; \tau) = \sum_P E[P] b^{sv}[P; \tau]$  with e.g.

$$b^{sv} \begin{bmatrix} j \\ k; \tau \end{bmatrix} = -\frac{2\zeta_{k-1}}{(k-1)(4y)^{k-2}}, \quad b^{sv} \begin{bmatrix} 2 & 1 \\ 4 & 4; \tau \end{bmatrix} = \frac{\zeta_3(\pi\tau)^2}{540} + \frac{5\zeta_5}{108} + \frac{\zeta_3^2}{18y}$$

• more generally, svMZVs  $c^{sv} \begin{bmatrix} j_1 \dots j_\ell \\ k_1 \dots k_\ell \end{bmatrix}$  @ wt  $\sum_{i=1}^{\ell} (1+j_i)$

$$\Rightarrow b^{sv} \begin{bmatrix} \dots j_i \dots \\ \dots k_i \dots; \tau \end{bmatrix} = \sum_{p_i=0}^{k_i-j_i-2} \sum_{l_i=0}^{j_i+p_i} \binom{k_i-2-j_i}{p_i} \binom{j_i+p_i}{l_i}$$

from  $c^{sv} = \# X^a Y^b$   
to  $b^{sv} =$

$$\#(X-\tau Y)^a (X-\bar{\tau} Y)^b$$

constructed

$$\frac{(-2\pi i \tau)^{l_i}}{(4y)^{p_i}} c^{sv} \begin{bmatrix} \dots j_i - l_i + p_i \dots \\ \dots k_i \dots \end{bmatrix}$$

e.g.  $c^{sv} \begin{bmatrix} 2 & 2 & 4 \\ 4 & 4 & 6 \end{bmatrix} = -\frac{\zeta_{3,5,3}^{sv}}{450} - \frac{2}{45} \zeta_3^2 \zeta_5 - \frac{221}{21600} \zeta_{11}^{sv}$

• MGIs from real-analytic  $G_{k_i}(\tau_i)$ -integrals  $\beta^{sv} \begin{bmatrix} j_1 \dots j_\ell \\ k_1 \dots k_\ell \end{bmatrix}$

with  $\mathbb{Q}[(sv)MZV, y^{-1}]$  coeffs [Gerken, Kleinschmidt, QS '20]

contain  $\mathbb{R}_{\pm}$ , Admsv

$\rightarrow$  all modular combinations from & parts of  $B^{sv}$

$$\beta^{equiv} \begin{bmatrix} j_1 \dots j_\ell \\ k_1 \dots k_\ell; \tau \end{bmatrix} = \sum_{i=0}^{\ell} d^{sv} \begin{bmatrix} j_1 \dots j_i \\ k_1 \dots k_i; \tau \end{bmatrix} \beta^{sv} \begin{bmatrix} j_{i+1} \dots j_\ell \\ k_{i+1} \dots k_\ell; \tau \end{bmatrix}$$

where  $d^{sv}[P; \tau] = b^{sv}[P; \tau] \Big|_{\text{keep } 1/y} \Big|_{\tau \rightarrow 0} \sim \mathbb{Q}[y^{-1}] \times c^{sv}$

• in general,  $\beta^{sv} \neq$  cpts of Brown's sv iterated Eisenstein integrals generated by  $J^{sv} = J^{equiv} (B^{sv})^{-1}$

$$\sum_P E[P] \beta^{sv}[P; \tau] = J^{sv}(\{\epsilon_k\}; \tau) \underbrace{B^{sv}(\{\epsilon_k\}; \tau) D^{sv}(\{\epsilon_k\}; \tau)^{-1}}_{1 + \mathcal{O}(\epsilon_{k_1}^{(j_1)} \epsilon_{k_2}^{(j_2)})}$$

morally a  $(\tau \rightarrow \tau+1)$ -invariant version

of Brown's sv iterated Eisenstein int's

• with  $N^{sv} = B^{sv}(\{\epsilon_k\}; \tau) M^{sv}$ , can write

$$J^{sv}(\{\epsilon_k\}; \tau) = J_+(\{\epsilon_k\}; \tau) N^{sv} J_-(\{\epsilon_k\}; \tau) (N^{sv})^{-1}$$

$\sim$  closer to genus -1 counterpart, but cpts are not modular forms

## 7) Outlook / open questions

- can make coeff's of  $J^{equiv}$  modular \*before\* Ek-rel's  
by adding  $\int_{\tau}^{i\infty} d\tau_1 \tau_1 \Delta_k(\tau_1)$  with holo' cusp forms  $\Delta_k$   
[coeff's are ratios of  $\Delta(\Delta_k, \cdot)$ -values & beyond]  
[Brown '17; 2109.05018, 2209.abcde]
- revisit relation closed strings  $\leftrightarrow$  sv(open strings)  
in light of  $\beta^{equiv}$  [1803.00527, 2010.10558]
- change of alphabet via  $X^{sv} * (X^{sv})^{-1}$  closely connects  
with coactions & should be very general  
( $g=0$  polylogs in  $N$  var's, sv elliptic polylogs)  
[in progress] start from elliptic MGFs [2208.11116]
- higher genus, apply to modular graph tensors

Exhibit 1: iterated integrals  $\beta_{\pm}$  for MGFs

$$\beta_+ \left[ \begin{matrix} j_1 & j_2 & \dots & j_l \\ k_1 & k_2 & \dots & k_l \end{matrix} ; i\tau \right] = \int_{\tau}^{i\infty} d\tau_1 \omega_+ \left[ \begin{matrix} j_l \\ k_l \end{matrix} ; i\tau, \tau_1 \right] \int_{\tau_1}^{i\infty} d\tau_2 \omega_+ \left[ \begin{matrix} j_{l-1} \\ k_{l-1} \end{matrix} ; i\tau_1, \tau_2 \right] \dots$$

$$\beta_- \left[ \begin{matrix} j_1 & j_2 & \dots & j_l \\ k_1 & k_2 & \dots & k_l \end{matrix} ; i\tau \right] = \int_{\tau}^{-i\infty} d\tau_1 \omega_- \left[ \begin{matrix} j_l \\ k_l \end{matrix} ; i\tau, \tau_1 \right] \int_{\tau_1}^{-i\infty} d\tau_2 \omega_- \left[ \begin{matrix} j_{l-1} \\ k_{l-1} \end{matrix} ; i\tau_1, \tau_2 \right] \dots$$

spectator

Exhibit 2: MGFs  $C^+$  from modular forms  $\beta^{\text{equiv}}$

in Brown's construction, e.g. [depth 2]

$$C^+ \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} = -18 \beta^{\text{equiv}} \begin{bmatrix} 2 & 0 \\ 4 & 4 \end{bmatrix} - 126 \beta^{\text{equiv}} \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

$$\text{Im } C^+ \begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 0 & 3 \end{bmatrix} = 60 \left( \beta^{\text{equiv}} \begin{bmatrix} 0 & 3 \\ 4 & 6 \end{bmatrix} - \beta^{\text{equiv}} \begin{bmatrix} 1 & 2 \\ 6 & 4 \end{bmatrix} \right) - 270 \left( \beta^{\text{equiv}} \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix} - \beta^{\text{equiv}} \begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix} \right)$$

$$+ 390 \left( \beta^{\text{equiv}} \begin{bmatrix} 2 & 1 \\ 4 & 6 \end{bmatrix} - \beta^{\text{equiv}} \begin{bmatrix} 3 & 0 \\ 6 & 4 \end{bmatrix} \right) - 3 \sqrt{3} \beta^{\text{equiv}} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$