Hypergeometric solutions of Feynman Integrals using Mellin-Barnes Integrals

Sumit Banik

(Joint Work with B. Ananthanarayan and Samuel Friot)

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Feynman Integrals

- In the perturbative framework, high-precision theoretical calculation relies on computing Feynman integrals.
- These integrals are hard to evaluate at higher-order.
- I focus on evaluation of Feynman integrals using the MB representation approach

Feynman Integrals

- In this talk, we discuss the connection between MB integrals and conic hulls.
- We also discuss possible connection between MB integrals and triangulation of point configuration.
- Show applications to the computation of conformal Feynman diagrams.

Multivariable-Hypergeometric Functions

- Feynman integrals can be expressed in terms of Hypergeometric functions.
- This is expected as hypergeometric functions have similar MB representations as Feynman diagrams.
- Multiple Hypergeometric functions [H.M.Srivastava, 1985] are generalizations of the Gauss hypergeometric function ${}_2F_1(x)$,

$$ {}_2F_1(x) = \sum_{n=0}^{\infty} \frac{\Gamma[a+n]\Gamma[b+n]}{\Gamma[c+n]} \frac{x^n}{n!}, \qquad |x| < 1 $$

Multivariable-Hypergeometric Series

- The series representations of multi-variable hypergeometric functions are numerically efficient.
- However, two difficulties in the theory of multiple hypergeometric series,

  1. Find their convergence region.
  2. Derive their analytic continuations.

  The conic-hull approach of MB integrals will address some of these difficulties.

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Conic-Hull Approach

- A multi-fold MB integral is of the form :

$$\int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \cdots \int_{-i\infty}^{+i\infty} \frac{dz_N}{2\pi i} \frac{\prod_{i=1}^{k} \Gamma^{a_i}(\mathbf{e}_i \cdot \mathbf{z} + g_i)}{\prod_{j=1}^{l} \Gamma^{b_j}(\mathbf{f}_j \cdot \mathbf{z} + h_j)} x_1^{z_1} \cdots x_N^{z_N}$$

  where a_i, b_j, k, l and N are positive integers.

- e_i, f_j are N-dimensional coefficient vectors and $z = (z_1, .., z_N)$.
- The above type of MB integral can be derived from the Schwinger or Feynman parametric representation of a Feynman diagram using AMBRE [J. Gluza, K. Kajda, T. Riemann 2007].

Conic-Hull Approach

There are two types of MB integrals:

- Degenerate Case: $\Delta = \sum e_i - \sum f_j = 0$. Several hypergeometric type series solutions coexist which are analytic continuations.
- Non-Degenerate Case: $\Delta = \sum e_i - \sum f_j \neq 0$. One or more convergent series converging for all values of parameters. Additionally, asymptotic series also arise.

The conic hull method works for both!

Conic-Hull Approach

MB integrals can be further classified based on the singularity structure:

- Non-resonant Case: Here, the number of singular hyper-planes intersecting at any pole is equal to the fold of the MB.
- Resonant Case: Here, the number of singular hyper-planes intersecting at some poles are greater than the fold of the MB. See [K J. Larsen, R. Rietkerk 2018] and [P. Griffiths, J. Harris 1994]

The conic hull method works for both!

Conic-Hull Approach

Conic Hull Method for N-fold MB: (Non-Resonant)

- **Step 1:** Find all possible N-combinations of numerator gamma functions and retain non-singular ones.
- **Step 2:** Associate a series (building block) with each combination.
- **Step 3:** Construct a conic hull for each combination/series.
- **Step 4:** Largest intersecting subsets of conic hulls give MB solutions.
- **Step 5:** The intersecting region gives the master conic hull.

Conic-Hull Approach

Illustrative Example:

- Consider the following 2-fold degenerate MB integral,

$$\int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{dz_2}{2\pi i} (-u_1)^{z_1} (-u_2)^{z_2} \Gamma(-z_1)\Gamma(-z_2) \times \frac{\Gamma(a+z_1+z_2)\Gamma(b_1+z_1)\Gamma(b_2+z_2)}{\Gamma(c+z_1+z_2)}$$

  associated with Appell F_1 double hypergeometric function.

- $\Delta = (-1,0)+(0,-1)+(1,1)+(1,0)+(0,1)-(1,1) = (0,0)$.
- Taking a, b_1, b_2, c to be generic.

Conic-Hull Approach

- Integrand:

$$(-u_1)^{z_1}(-u_2)^{z_2}\frac{\Gamma(-z_1)\Gamma(-z_2)\Gamma\left(a+z_1+z_2\right)\Gamma\left(b_1+z_1\right)\Gamma\left(b_2+z_2\right)}{\Gamma\left(c+z_1+z_2\right)}$$

- Coefficient vectors of numerator gamma functions $\Gamma(\mathbf{e}_i \cdot \mathbf{z} + g_i)$:

i	Γ function	$\mathbf{e}_i$
1	$\Gamma(-z_1)$	$(-1,0)$
2	$\Gamma(-z_2)$	$(0,-1)$
3	$\Gamma(a+z_1+z_2)$	$(1,1)$
4	$\Gamma(b_1+z_1)$	$(1,0)$
5	$\Gamma(b_2+z_2)$	$(0,1)$

Conic-Hull Approach

Step 1 : Find all possible 2-combinations of numerator gamma functions and retain non-singular ones.

- There are $\binom{5}{2} = 10$ possible 2-combinations.
- $\{i_1, i_2\}$ denotes a 2-combination, where i_1 and i_2 are the labels of the gamma functions.
- Example: $\{1, 3\}$ denotes $\{\Gamma(-z_1), \Gamma(a + z_1 + z_2)\}$.

Conic-Hull Approach

- Retain 2-combinations with non-singular matrix $A_{i_1,i_2} = \begin{pmatrix} e_{i_1} \\ e_{i_2} \end{pmatrix}$.
- The 8 retained 2-combinations for the Appell F_1's MB are,

$$\left[\{1,2\}, \{1,3\}, \{1,5\}, \{2,3\}, \{2,4\}, \{3,4\}, \{3,5\}, \{4,5\}\right]$$

- One of the omitted 2-combination is $\{1,4\} : \{\Gamma(-z_1), \Gamma(b_1+z_1)\}$ as $A = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}$ is singular.

Conic-Hull Approach

Step 2: Associate a series (building block) with each combinations.

- Series/Building block is obtained by adding the 2-dimensional residues of only those poles associated with gamma functions in $\{i_1, i_2\}$.
- We label them by B_{i_1,i_2}, which are Horn-type hypergeometric functions here.
- We have 8 building blocks in this example..

Conic-Hull Approach

- The poles of $\{1,3\} : \{\Gamma(-z_1), \Gamma(a+z_1+z_2)\}$ are at $(z_1, z_2) = (n_1, -a-n_1-n_2)$ for $n_i \geq 0$.

- Associated building block:

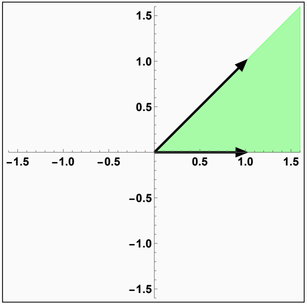
$$B_{1,3} = (-u_2)^{-a} \sum_{n_1,n_2=0}^{\infty} \left(-\frac{u_1}{u_2}\right)^{n_1} \left(\frac{1}{u_2}\right)^{n_2} \times \frac{\Gamma(a+n_1+n_2)\,\Gamma(b_1+n_1)\,\Gamma(-a+b_2-n_1-n_2)}{\Gamma(n_1+1)\,\Gamma(n_2+1)\,\Gamma(-a+c-n_2)} \propto F_1\left(a, b_1, a-c+1; a-b_2+1; \frac{u_1}{u_2}, \frac{1}{u_2}\right)$$

- Linear combinations of building blocks yield series solutions.

Conic-Hull Approach

Step 3: Construct a conic hull for each combination/series.

- Assign a **conic hull** to B_{i_1,i_2} or $\{i_1, i_2\}$, denoted by C_{i_1,i_2}.
- Edges of C_{i_1,i_2} should be parallel to the vectors e_{i_1} and e_{i_2} with vertex at the origin.
- $C_{1,3}$ is associated with $\{1,3\} : \{\Gamma(-z_1), \Gamma(a + z_1 + z_2)\}$ and edges parallel to $e_1 = (-1, 0)$ and $e_3 = (1, 1)$.



Conic-Hull Approach

8 conic hulls for each building blocks associated with Appell F_1.

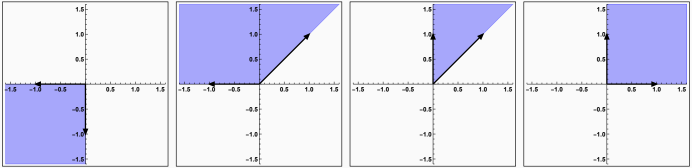


Figure 1: Conic hulls of: $B_{1,2}, B_{1,3}, B_{3,5}, B_{4,5}$

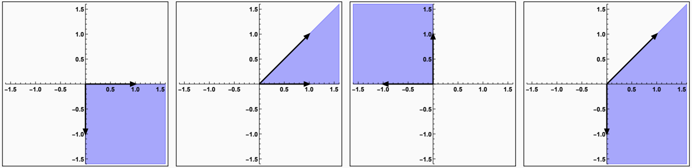


Figure 2: Conic hulls of: $B_{2,4}, B_{3,4}, B_{1,5}, B_{2,3}$

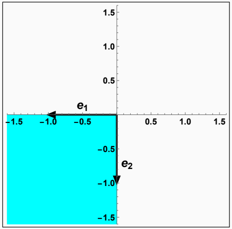
Conic-Hull Approach

Step 4: Largest intersecting subsets of conic hulls give MB solutions.

- Find the largest subsets of conic hulls having a common intersection.
- Add the associated building blocks to get series solutions of the MB.

Conic-Hull Approach

The first conic hull $C_{1,2}$ does not intersect with any other conic hulls.



Therefore, its associated building block

$$B_{1,2} = \sum_{n_1,n_2=0}^{\infty} \frac{\Gamma(a+n_1+n_2)\Gamma(b_1+n_1)\Gamma(b_2+n_2)}{\Gamma(c+n_1+n_2)} \frac{u_1^{n_1} v_2^{n_2}}{n_1! n_2!}$$

is a series solution of the Appell F_1 MB.

Solution

Next, consider the conic hulls associated with three building blocks $B_{1,3}$, $B_{3,5}$ and $B_{4,5}$ of the Appell F_1 series:

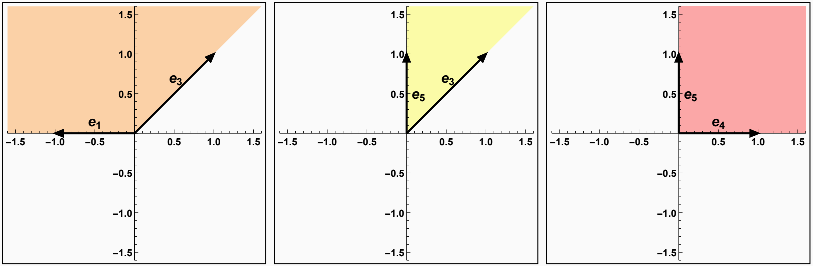
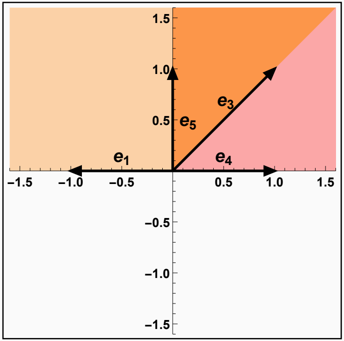


Figure 3: Conic hulls $C_{1,3}$ (left), $C_{3,5}$ (center) and $C_{4,5}$ (right).

Conic-Hull Approach

The conic hulls $C_{1,3}$, $C_{3,5}$ and $C_{4,5}$ have a common intersection.

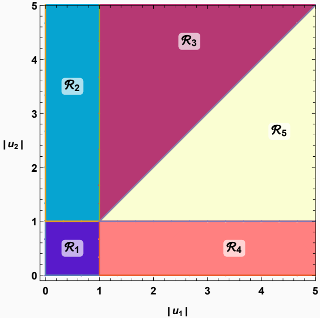


Thus, one of the MB solution is $B_{1,3} + B_{3,5} + B_{4,5}$.

Conic-Hull Approach

Analysing all the 8 conic hulls gives us 5 series solutions.

$$
= \begin{cases}
B_{1,2} & |u_1| < 1 \cap |u_2| < 1 \\
B_{1,3} + B_{1,5} & |u_1| < 1 \cap |u_2| > 1 \\
B_{1,3} + B_{3,5} + B_{4,5} & |u_1| > 1 \cap \left|\frac{u_1}{u_2}\right| < 1 \\
B_{2,3} + B_{2,4} & |u_1| > 1 \cap |u_2| < 1 \\
B_{2,3} + B_{3,4} + B_{4,5} & \left|\frac{u_2}{u_1}\right| < 1 \cap |u_2| > 1
\end{cases}
$$



Conic-Hull Approach

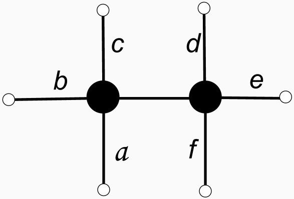
Step 5: **The intersecting region gives the master conic hull**.

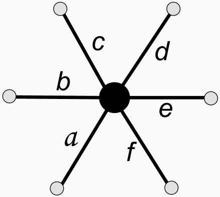
- The common intersection region of conic hulls is called master conic hull.
- We map back the master conic hull to a series called master series.
- Each series solution will have a master series.
- The convergence region of the series solution will be same as the master series. (Conjecture!)

Conformal Feynman Integrals

Conformal Feynman Integrals

We computed the previously unsolved dual-conformal fishnet Double-Box and Hexagon diagrams,





both of which have a nine-fold MB representation.

- The total number of building blocks for Double-Box and Hexagon are 4834 and 2530, respectively.
- We solved both as a linear combination of 44 and 26 building blocks, respectively.
- We also solved it for the non-trivial unit propagator case.

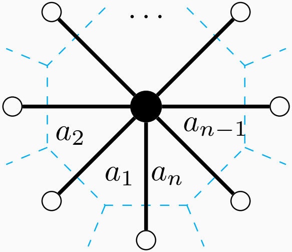
Conformal Feynman Integrals

Numerical Comparison for Hexagon		
Upper Sum Limit	Series Representation (Time)	Feynman Parametrization (Time)
2	636.76884 (14 sec)	636.76882 (9 Hours)

Table 1: Computed for $u_1 = 1, u_2 = 10^{12}, u_3 = 1/10^{12}, u_4 = 1, u_5 = 1, u_6 = 100, u_7 = 1/100, u_8 = 10000, u_9 = 1/10^8$ for propagator powers $a = 42/100, b = 11/100, c = 15/100, d = 32/100, e = 59/100, f = 55/100$

Conformal Feynman Integrals

A conjecture was made based on the Yangian bootstrap analysis [F. Loebbert, J. Miczajka, D. Müller , H. Münkler 2021], states that dual-conformal n-point one-loop Feynman integrals can be written as a single hypergeometric series.



The corresponding $\frac{n(n-1)}{2}$ MB-representation can be written as:

$$I_{n\bullet}^{m_1\ldots m_n} = \frac{\pi^{D/2+1/2}}{2^{D-1}\prod_{i=1}^{n}\Gamma(a_i)m_i^{a_i}} \prod_{\alpha\in B_n}\left(\int_{-i\infty}^{+i\infty}\frac{\mathrm{d}\,z_\alpha}{2\pi i}\Gamma(-z_\alpha)(-u_\alpha)^{z_\alpha}\right) \times \frac{\prod_{i=1}^{n}\Gamma\left(a_i+\sum_{\alpha\in B_{n|i}} z_\alpha\right)}{\Gamma\left(\frac{D+1}{2}+\sum_{\alpha\in B_n} z_\alpha\right)}$$

where $B_n = \{12, 13, 23, \cdots, (n-1, n)\}$ is the set of pairs of distinct integers (written in increasing order) in $\{1, \cdots, n\}$ and $B_{n|j}$ is the subset of B_n with pairs containing j.

- The conic hull associated with $\Gamma(-z_\alpha)$ is in the $(- - \cdots - -)$ quadrant.
- Therefore, this conic hull will not intersect with any other conic hulls.
- Hence, the single series associated with this conic hull itself gives us a solution to the MB.

Limitations

- The method works only when the number of scales is equal to the fold of the MB integral.
- We encounter white zones where none of the series solutions converge.
- The package *MBConicHulls.wl* is often too slow.

Triangulation Method

Triangulation Method

- MB integrals can be alternatively solved using triangulation.
- We associate a point-configuration to each MB integral.
- Series solution can be derived using triangulation and multivariate residue .

Triangulation Method

- Appell F_1:

$$(-u_1)^{z_1}(-u_2)^{z_2}\frac{\overset{1}{\Gamma}(-z_1)\overset{2}{\Gamma}(-z_2)\overset{3}{\Gamma}(a+z_1+z_2)\overset{4}{\Gamma}(b_1+z_1)\overset{5}{\Gamma}(b_2+z_2)}{\Gamma(c+z_1+z_2)}$$

- We associate the point-configuration: $P=\{P_1,P_2,P_3,P_4,P_5\}$

$$P_1=\begin{pmatrix}1\\1\\0\end{pmatrix},\quad P_2=\begin{pmatrix}1\\0\\1\end{pmatrix},\quad P_3=\begin{pmatrix}1\\0\\0\end{pmatrix},\quad P_4=\begin{pmatrix}0\\1\\0\end{pmatrix},\quad P_5=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

which is

$$\mathcal{A}=\begin{pmatrix}1&1&1&0&0\\1&0&0&1&0\\0&1&0&0&1\end{pmatrix}$$

Triangulation Method

We use TOPCOM [J. Rambau, 2002] we obtain the following triangulations:

1. $\mathcal{T}_1 = \{(P_3, P_4, P_5)\}$,
2. $\mathcal{T}_2 = \{(P_1, P_2, P_3), (P_1, P_2, P_4), (P_2, P_4, P_5)\}$,
3. $\mathcal{T}_3 = \{(P_2, P_4, P_5), (P_2, P_3, P_4)\}$,
4. $\mathcal{T}_4 = \{(P_1, P_4, P_5), (P_1, P_3, P_5)\}$,
5. $\mathcal{T}_5 = \{(P_1, P_2, P_3), (P_1, P_4, P_5), (P_1, P_2, P_5)\}$

Triangulation Method

- From the triangulation's, we can identify the poles whose multivariate residue yields series solutions:

  1. $\mathcal{S}_1 = \left[\{1,2\}\right] = \left[\overset{1}{\Gamma}(-z_1)\overset{2}{\Gamma}(-z_2)\right]$, from $\mathcal{T}_1 = \{(P_3, P_4, P_5)\}$
  2. $\mathcal{S}_2 = \left[\{4,5\}, \{3,5\}, \{1,3\}\right]$, from $\mathcal{T}_2 = \{(P_1, P_2, P_3), (P_1, P_2, P_4), (P_2, P_4, P_5)\}$.
  3. $\mathcal{S}_3 = \left[\{1,3\}, \{1,5\}\right]$,
  4. $\mathcal{S}_4 = \left[\{2,3\}, \{2,4\}\right]$,
  5. $\mathcal{S}_5 = \left[\{4,5\}, \{2,3\}, \{3,4\}\right]$

  where, $\{a, b\}$ denotes the 2-combination $\{\overset{a}{\Gamma}, \overset{b}{\Gamma}\}$.

- We obtain the same set of solutions from the conic hull method.

Triangulation Method

- For all examples tried so far, triangulation method yields same solutions as the conic hull method.
- However, the triangulation method is computationally more efficient than the conic hull method.
- Triangulation method helps us derive simpler solutions for the hexagon and double box, which is a sum of 25 series for both cases.

Summary and Future Work

Summary

- N-fold MB integrals can be solved by introducing conic hulls or using triangulation.
- The conic hull method is automatized in the package MBConicHulls.wl.
- Analytic continuations of hypergeometric series can be derived systematically.
- Solutions of the conformal hexagon and double box were derived using both conic hull and triangulation method.

Future Work

- Automatize the triangulation method.
- To understand how intersection of conic hull is related to regular triangulation of point configuration.
- Interpretation of master conic hull in the picture of the triangulation method.
- To develop efficient ways for ϵ-expansion of series solutions.