FROM FEYNMAN INTEGRALS TO SPECIAL FUNCTIONS (AND BACK…!)

Elliptics 2022
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DISCLAIMER

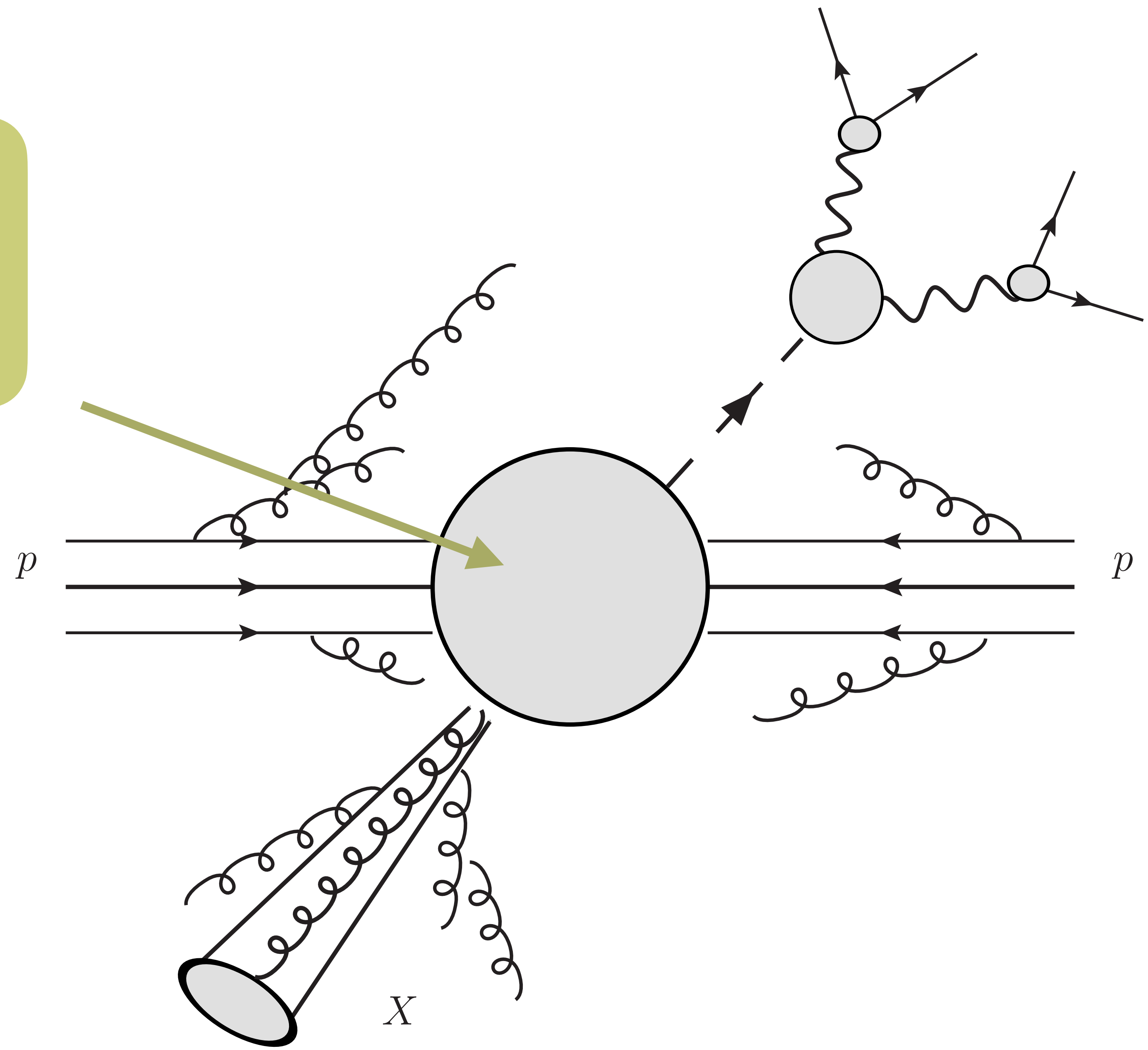
Try and set the stage of the many interesting developments of the last year(s)

Just my personal account, clearly *biassed* by having been educated as (and being still) a particle physicist…

FEYNMAN INTEGRALS IN QUANTUM FIELD THEORY

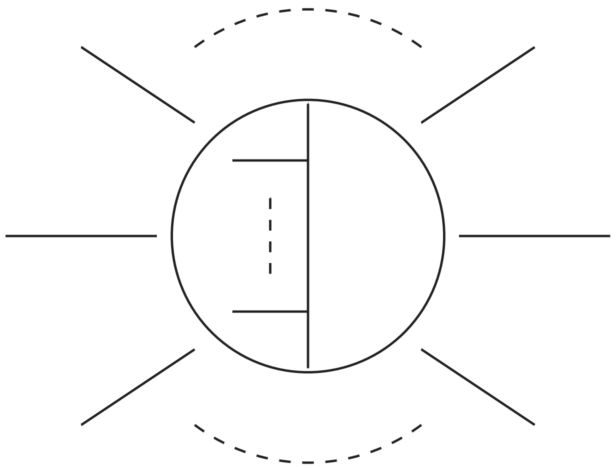
Feynman integrals are essential ingredients for physical predictions in QFT

Building blocks for
Scattering Amplitudes



FROM AMPLITUDES TO INTEGRALS

Scattering Amplitudes



$$\sim \qquad \mathscr{A} = \epsilon_1^{\mu_1} \cdots \epsilon_n^{\mu_n} \, \bar{v}(q) \, \Gamma_{\mu_1,\ldots,\mu_n} \, u(p)$$

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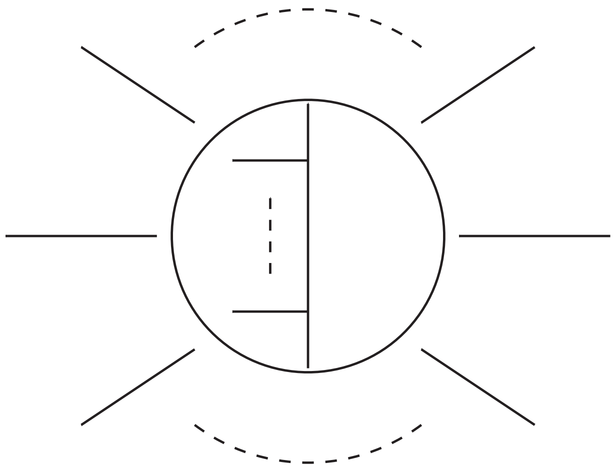

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Tensor integrals

$$\Gamma_{\mu_1 \ldots \mu_n} \sim \int \prod_{l=1}^{L} \frac{d^D k_l}{(2\pi)^D} \frac{k_{\mu_1} \ldots k_{\mu_n}}{D_1 \ldots D_n}$$

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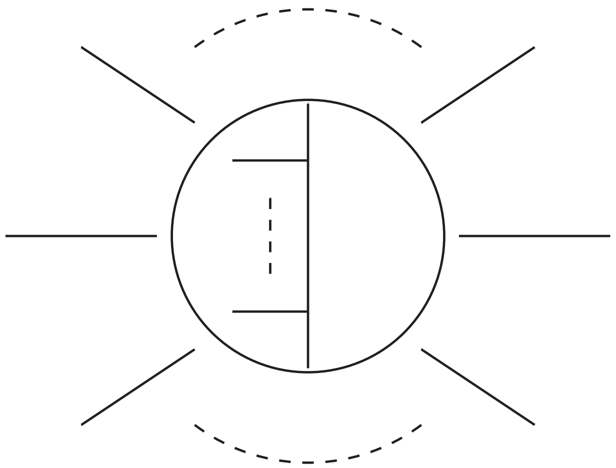

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tensor decomposition $\longrightarrow$

Scalar Feynman Integrals

$$\mathscr{I} = \int \prod_{l=1}^{L} \frac{d^D k_l}{(2\pi)^D} \frac{S_1^{b_1} \ldots S_m^{b_m}}{D_1^{a_1} \ldots D_n^{a_n}}$$

with $S_i \in \{k_i \cdot k_j, \ldots, k_i \cdot p_j\}$

SCALAR FEYNMAN INTEGRALS

From tensor reduction, huge number of scalar integrals ($gg \to gg$ @ 3 loops $\sim 10^7$ integrals!)

Standard Approach: **divide et impera**

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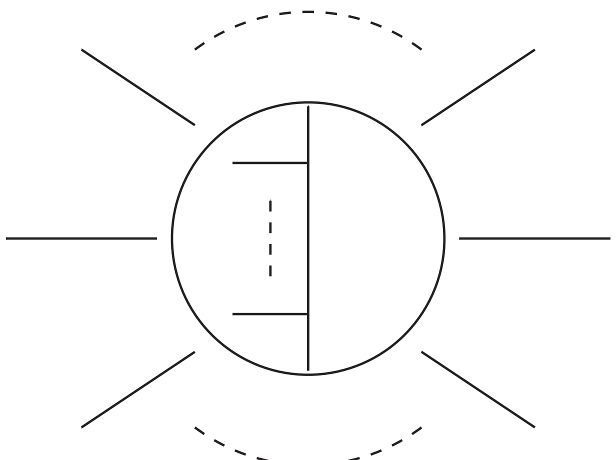

Integration by parts identities → **master integrals**

[Chetyrkin, Tkachov '81] **& many others: most recently finite fields, intersection theory etc**

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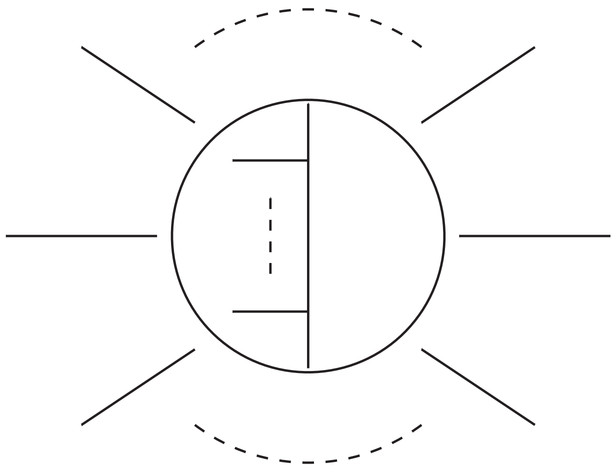

$$= \sum_{i=1}^{N} R_i(x_1, \ldots, x_r)\, \mathcal{I}_i(x_1, \ldots, x_n)$$

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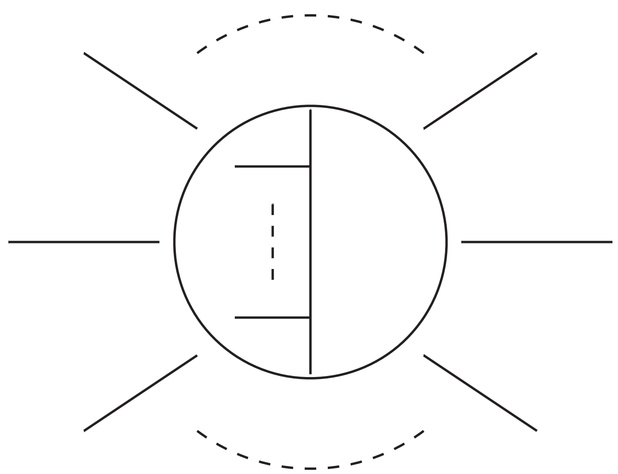

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Coefficients (process-dependent)

Very complicated rational functions, hundreds of **MBs** for complicated processes:

Algebraic Complexity

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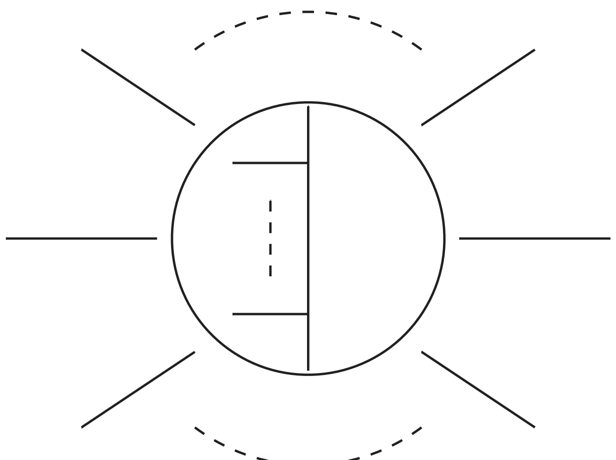

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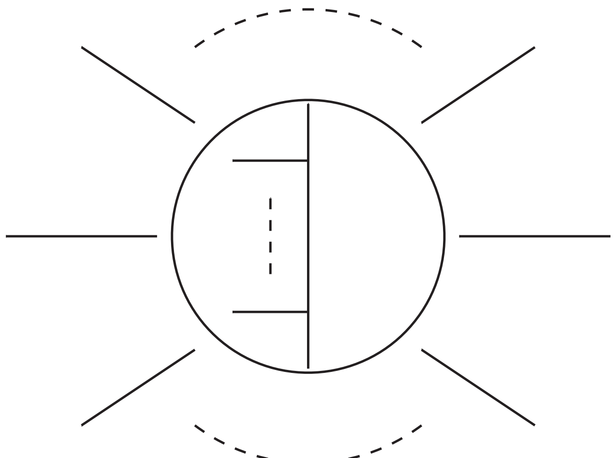
Process-independent building blocks: **Master Integrals**

Involved **special functions** with complicated mathematical properties:

Analytic complexity

MASTER INTEGRALS: ANALYTIC COMPLEXITY

Scattering amplitude has (poles and) ***branch cuts*** — encoded in master integrals!



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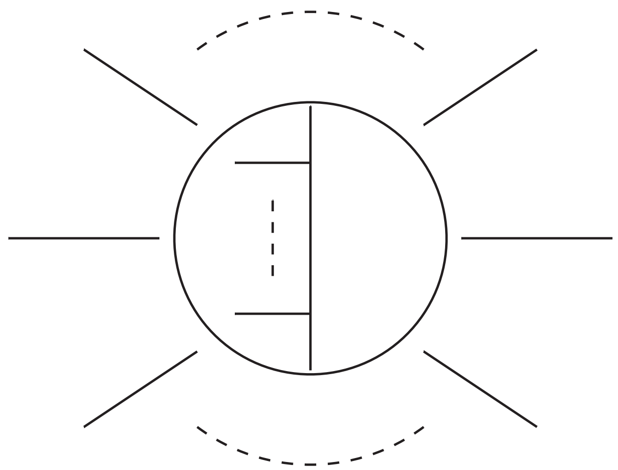

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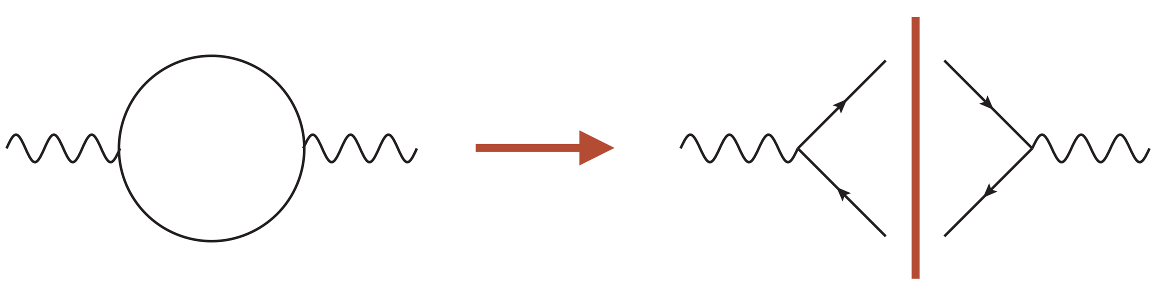


$$\sim \int_{4m^2}^{\infty} \frac{ds'}{s' - s - i\epsilon} \frac{1}{\sqrt{s'(s' - 4m^2)}}$$

$$\sim \frac{1}{\sqrt{s(s-4m^2)}} \ln\left(\frac{\sqrt{s-4m^2}+\sqrt{s}}{\sqrt{s-4m^2}-\sqrt{s}}\right)$$

Branch-cut structure dictated by **causality & unitarity** — **multivalued functions!**

ANALYTIC CALCULATIONS

Highly non-trivial job to make ***general statements on analyticity properties of the S-matrix***

- effort in the ‘60s to use analyticity properties to make non-perturbative statements:

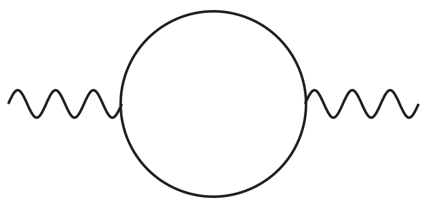
**Landau Equations, Unitarity cuts,
Dispersion relations etc**

- techniques saw resurgence in the past decades, applied to **perturbative calculations!**

**“New” technique: differential equations
inspired by these investigations!**

Goal: compute Feynman (master) integrals **analytically**

WHAT DOES IT MEAN TO BE ANALYTIC?



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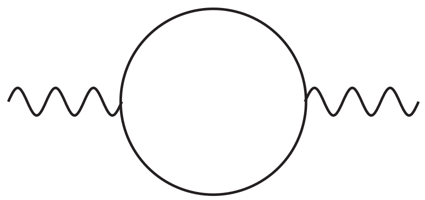

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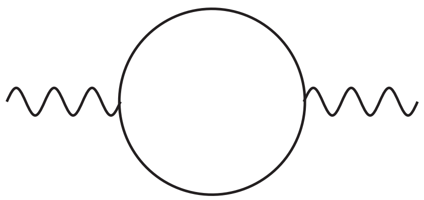

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$$\log 1/x + \log x = 0$$

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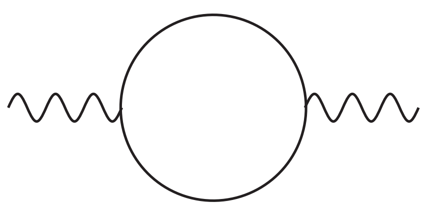

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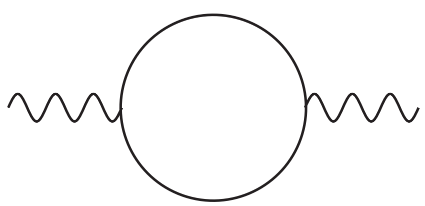

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Argument transformation and Series expansion for <u>numerical evaluation</u>

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O\left(x^5\right)$$

WHY IS THIS USEFUL?

One might wonder why this is useful… from the practical point of view of the *practitioner particle physicist*, **what we need at the end of a day is a number (a “cross section”)**

In fact, there are many very powerful (semi-) numerical methods that have been developed

1. Sector decomposition
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3. etc etc

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We *analytic people* like to claim, **analytic calculations give you that, and much more…**

- of course, a chance to obtain more control over (numerical!) cancellations, and therefore **precision and speed**
- more importantly (from my point of view), we get a glimpse at more general properties of amplitudes and get a chance to consider the question: "**what structures appear in pQFT?**"

A general statement, can help up to compute amplitudes

(possibly even "without computing them" —> Bootstrap… etc)

WHERE IT ALL BEGAN: THE “DISCOVERY OF SPECIAL FUNCTIONS IN PARTICLE PHYSICS”

The “most famous calculation” in pQFT: the **g-2 of the electron**

$$a_e^{QED} = C_1\left(\frac{\alpha}{\pi}\right) + C_2\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4 + C_5\left(\frac{\alpha}{\pi}\right)^5 + \ldots$$

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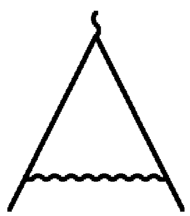
$C_1 =$  $= +0.50000000...$

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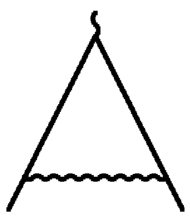
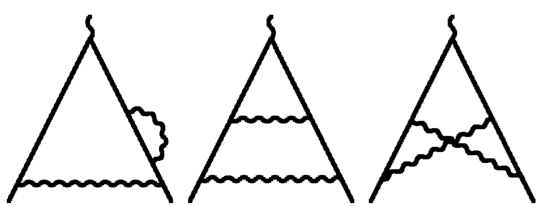
$C_1 =$  $= +0.50000000...$

$C_2 =$  $= -0.328478965...$

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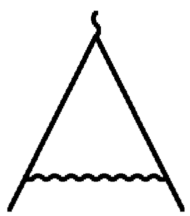
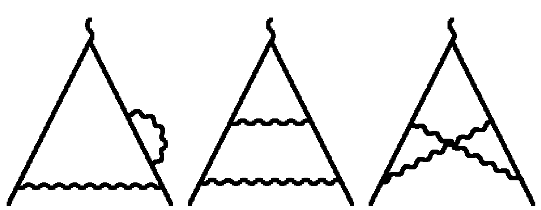
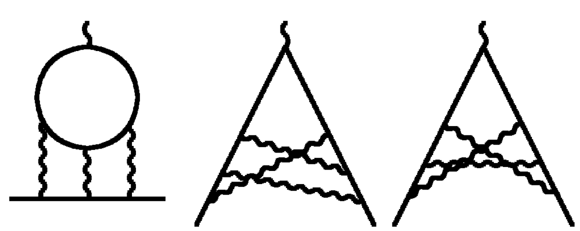
$C_1 =$  $= +0.50000000...$

$C_2 =$  $= -0.328478965...$

$C_3 =$  $= +1.181241456...$

$C_4 =$ $= -1.912245764...$

lots of Feynman diagrams

$C_5 =$ $= +6.737(159)$

Impressive numerical calculations by **Kinoshita et al**

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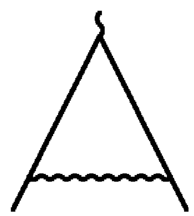
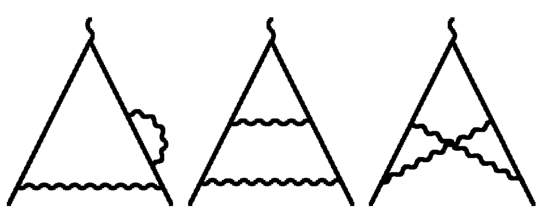
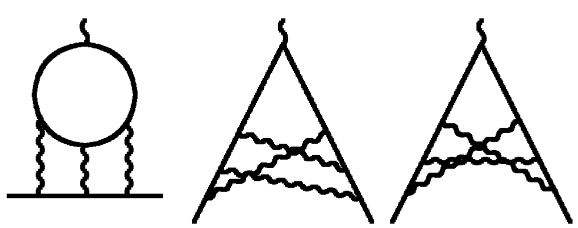
$C_1 =$  $= +0.50000000\ldots$

$C_2 =$  $= -0.328478965\ldots$

$C_3 =$  $= +1.181241456\ldots$

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As numbers, they don't say much

(except that the perturbative series seems to converge nicely once multiplied by 1/137 :-))

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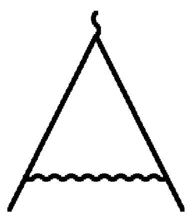
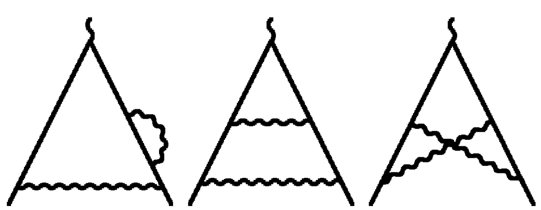
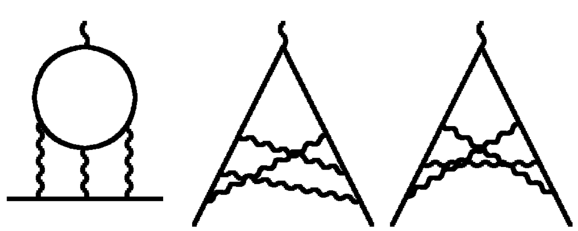
$C_1 =$  $= \frac{1}{2}$ **[Schwinger '48]**

$C_2 =$  $= \frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2 \ln 2 + \frac{3}{4}\zeta(3)$ **[Petermann, Sommerfield '57]**

$C_3 =$ 

$$= \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3}\left[\left(\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{\ln^4 2}{24}\right) - \frac{\pi^2 \ln^2 2}{24}\right]$$
$$- \frac{239}{2160}\pi^4 + \frac{139}{18}\zeta(3) - \frac{298}{9}\pi^2 \ln 2 + \frac{17101}{810}\pi^2 + \frac{28259}{5184}$$

[Laporta, Remiddi '97]

But if we look at *analytic results*, some sort of pattern seems to emerge:

rational numbers, Riemann zeta values, …, in general ***multiple polylogarithms*** *evaluated at special (rational) points*

SPECIAL FUNCTIONS IN PARTICLE PHYSICS: MULTIPLE POLYLOGARITHMS

The physicists approach (most of the times :-)): **from specific to general**

g-2 @ 3 loops one of many calculations in pQFT exhibiting clear iterated patterns

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Harmonic Sums

$$S_{m,j_1,\cdots,j_p}(n) \quad = \quad \sum_{i=1}^{n} \frac{1}{i^m} S_{j_1,\cdots,j_p}(i)$$

$$\int_0^1 dx \frac{\ln(x)\ \ln^2(1-x)\ \ln(1+x)}{x} \quad =$$

$$= -\frac{3}{8}\zeta_2\zeta_3 - \frac{2}{3}\zeta_2 \ln^3(2) + \frac{7}{4}\zeta_3 \ln^2(2) - \frac{7}{2}\zeta_5$$
$$+4\ln(2)\ \mathrm{Li}_4(1/2) + \frac{2}{15}\ln^5(2) + 4\ \mathrm{Li}_5\ (1/2)$$

[Vermaseren ’98; Blümlein et al ’98 …]

Harmonic Polylogarithms

$$\mathrm{H}(\vec{m}_w; x) = \int_0^x dx'\ f(a; x')\ \mathrm{H}(\vec{m}_{w-1}; x')$$

$$\mathrm{H}(\vec{0}_w; x) = \frac{1}{w!} \ln^w x$$

$$\begin{aligned} f(0;x) &= \frac{1}{x}\,, \\ f(1;x) &= \frac{1}{1-x}\,, \\ f(-1;x) &= \frac{1}{1+x}\,. \end{aligned}$$

[Remiddi, Vermaseren ’99]

Technology developed to perform integrals in g-2, deep inelastic scattering, Drell Yan integrals etc

SPECIAL FUNCTIONS IN PARTICLE PHYSICS: MULTIPLE POLYLOGARITHMS

HPLs and Harmonic Sums found immediate applications in numerous problems in particle physics

The natural instinct for generalisation lead physicists to (re-)discover **multiple polylogarithms**

$$
\begin{aligned}
G(c_1, c_2, ..., c_n, x) &= \int_0^x \frac{dt_1}{t_1 - c_1} G(c_2, ..., c_n, t_1) \\
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To study their algebraic and analytic properties, and use them to devise algorithms for their **numerical evaluation** [Vollinga, Weinzierl '04]

Connection to "pure math": properties of multiple polylogarithms can be derived starting from their **Hopf Algebra**

- The symbol map [Goncharov, Spradlin, Volovich '10; Duhr, Gangl, Rhodes '11]
- Coaction and coproduct [Duhr '12]

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The natural instinct for generalisation lead physicists to (re-)discover **multiple polylogarithms**

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$$= \int_0^x \frac{dt_1}{t_1 - c_1} \int_0^{t_1} \frac{dt_2}{t_2 - c_2} ... \int_0^{t_{n-1}} \frac{dt_n}{t_n - c_n}$$

To study their algebraic and analytic properties, and use them to devise algorithms for their **numerical evaluation** [Vollinga, Weinzierl '04]

Connection to "pure math": properties of multiple polylogarithms can be derived starting from their **Hopf Algebra**

- The symbol map [Goncharov, Spradlin, Volovich '10; Duhr, Gangl, Rhodes '11]
- Coaction and coproduct [Duhr '12]

This was only the beginning … just discovering the **entrance of the rabbit hole!**

HYPERLOGARITHMS: RATIONAL FUNCTIONS ON THE RIEMANN SPHERE

In fact, multiply polylogarithms had been known for more than a century already to mathematicians

1840 E.E. Kummer, *Über die Transcendenten, welche aus wiederholten Integrationen rationaler Formeln entstehen*

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↓

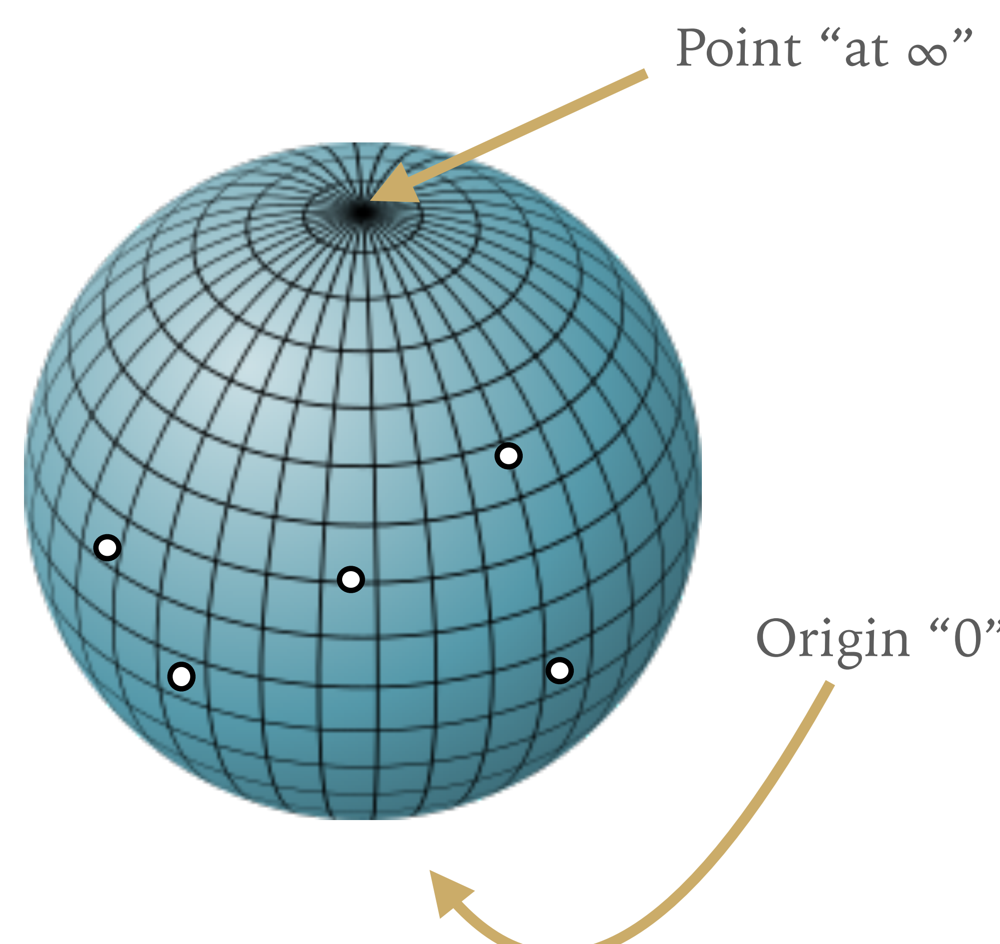
Modern point of view: Iterated integrations of rational functions on the **Riemann sphere (genus 0)**

$$R(z) = \frac{P(z)}{Q(z)}$$

$$\begin{cases} P(z) & = a_n z^n + \ldots + 1 \\ Q(z) & = b_m z^m + \ldots + 1 \end{cases}$$

A rational function has no **branch cuts**

But it has **poles**



HYPERLOGARITHMS: RATIONAL FUNCTIONS ON THE RIEMANN SPHERE

Given any rational function R(x), by **factorising poles** and **partial fractioning** we get

$$\int dx\, R(x) = \int dx\, \frac{p(x)}{q(x)} \sim \left\{ \int dx\, x^n \quad , \quad \int \frac{dx}{(x-c)^k} \right\}$$

$$\int \frac{dx}{(x-c)^k} = -\frac{1}{k-1}\frac{1}{(x-c)^{k-1}}\,, \quad k > 1$$

$$\int \frac{dx}{x-c} = \log\left(x-c\right) \quad \longleftrightarrow \quad \oint_{\gamma_c} \frac{dx}{x-c} = 2\pi i$$

Residue non zero → **multivalued function**

Modern point of view: the algebra generated by MPLs on the field of rational functions with a set of singularities on the Riemann sphere is closed under differentiation and integration

[Goncharov; Brown; …]

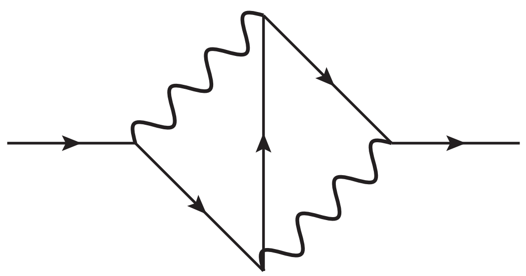
ONE STEP FURTHER: THE SUNRISE INTEGRAL AND THE "ELLIPTIC WORLD"

Realising profound connection with old and new mathematics very appealing "conceptually"

How useful could this be to go beyond?

ONE STEP FURTHER: THE SUNRISE INTEGRAL AND THE "ELLIPTIC WORLD"

Electron self-energy in QED @ 2 loops [first computation attempted in **1961** by A. Sabry]



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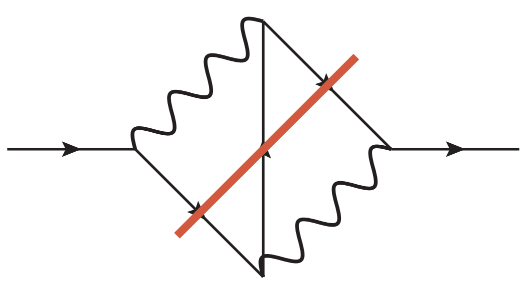
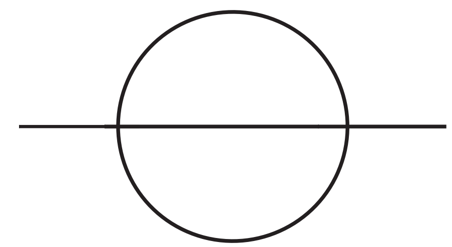
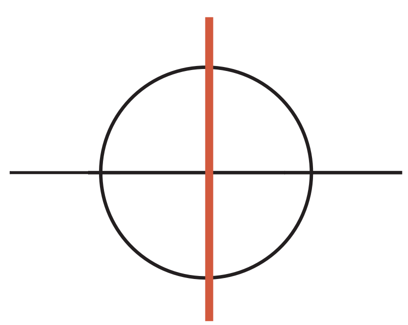




The sunrise integral



$$= \frac{1}{\sqrt{(3m-\sqrt{s})(\sqrt{s}+m)^3}}\,\mathrm{K}\left(\frac{16m^3\sqrt{s}}{(3m-\sqrt{s})(\sqrt{s}+m)^3}\right)$$

It's imaginary part "contains" an **elliptic integral**

$$\mathrm{K}(x) = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-x\,z^2)}}$$

ONE STEP FURTHER: THE SUNRISE INTEGRAL AND THE "ELLIPTIC WORLD"

Some history and early attempts to understand the sunrise

1961
Sabry

1990
Broadhurst
(dispersion relations)

1995
Bauberger, Berends, Bohm, Buza
Bauberger, Bohm
(one-fold integral representation, hypergeometric and Lauricella functions)

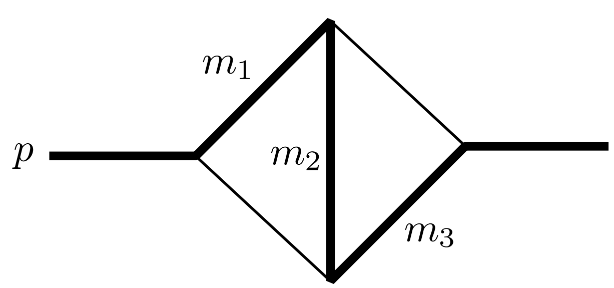
2005
Laporta, Remiddi
(differential equations)
Pozzorini, Remiddi
(numerical diff equations)

2009
Caffo, Czyz, Gunia, Remiddi
(numerical diff equations different masses)

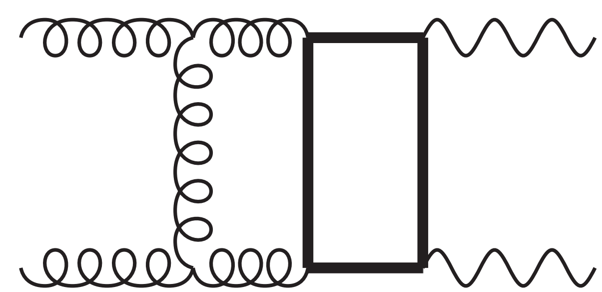
From a practical point of view, the problem of calculating the sunrise graph was "solved"

ONE STEP FURTHER: THE SUNRISE INTEGRAL AND THE “ELLIPTIC WORLD”

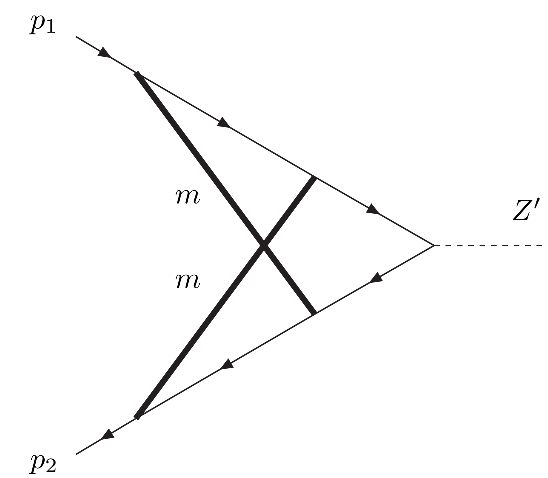
General understanding still missing, and number of examples recognised to involve **elliptic integrals** was growing



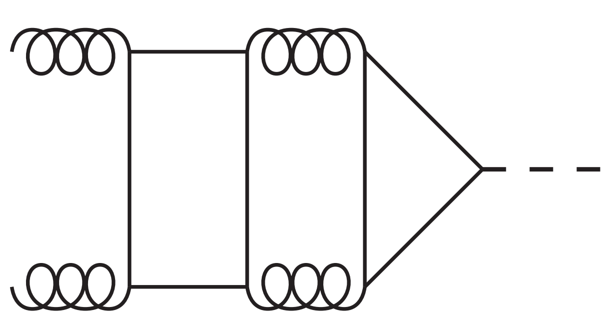
Electron propagator



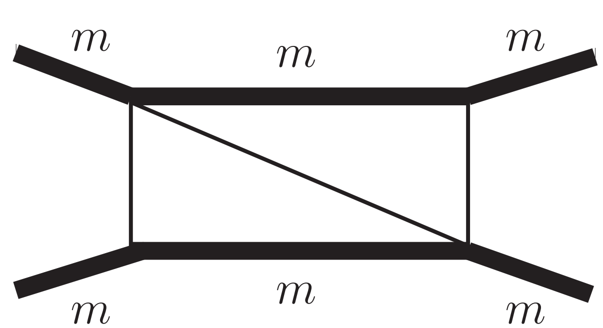
top quark corrections



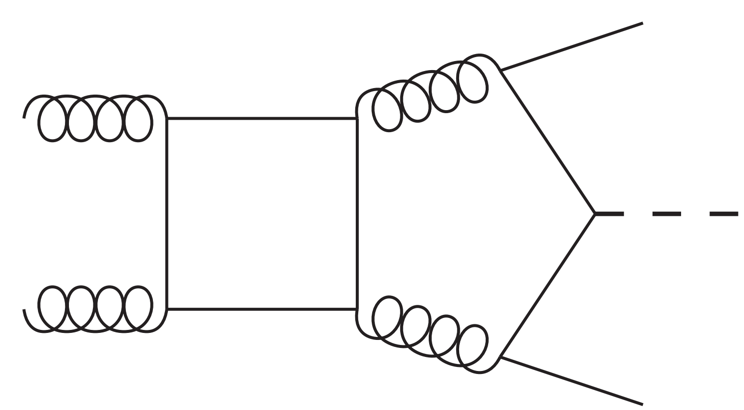
ew form factor



Higgs production



Bhabha scattering in QED



$t\bar{t}(+X)$ production in QCD

NEW DEVELOPEMENTS: THE GEOMETRY OF THE SUNRISE GRAPH

New hints towards a more general point of view from maths and string theory

2007-2011

Levin, Racinet

Brown, Levin

(mathematical foundations of elliptic multiple polylogs)

2012-2013

Adams, Weinzierl
Remiddi, Tancredi

(2nd order diff equation for sunrise)

Block, Vanhove

(Sunrise as elliptic d-log)

Henn

(canonical dlog integrals)

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(cuts and homogeneous differential equation beyond mpls)

2017-2018

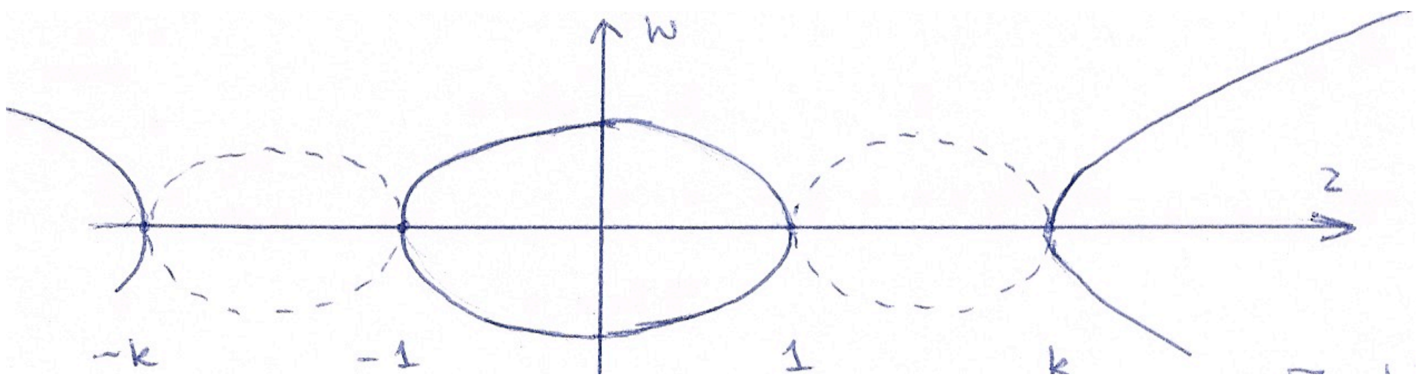
Broedel, Penante, Duhr, Dulat, Tancredi

(connection of different languages - definition of eMPLs with logarithmic singularities)

CHANGING THE GEOMETRY: FROM GENUS 0 TO GENUS 1

Elliptic curves ~ complex Tori → repeat construction of MPLs on genus 1 surface

$$y(z) = \sqrt{(1-z^2)(k^2-z^2)}$$



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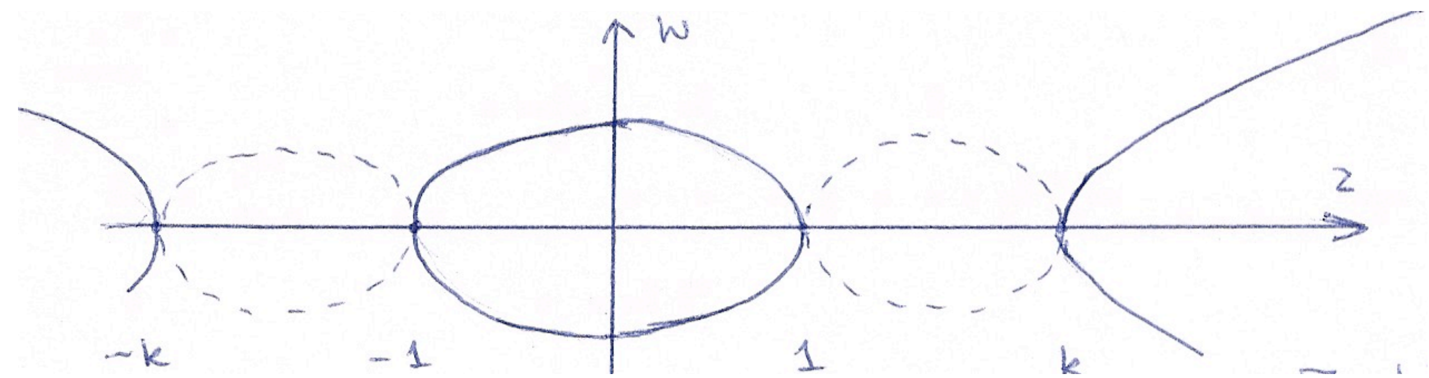
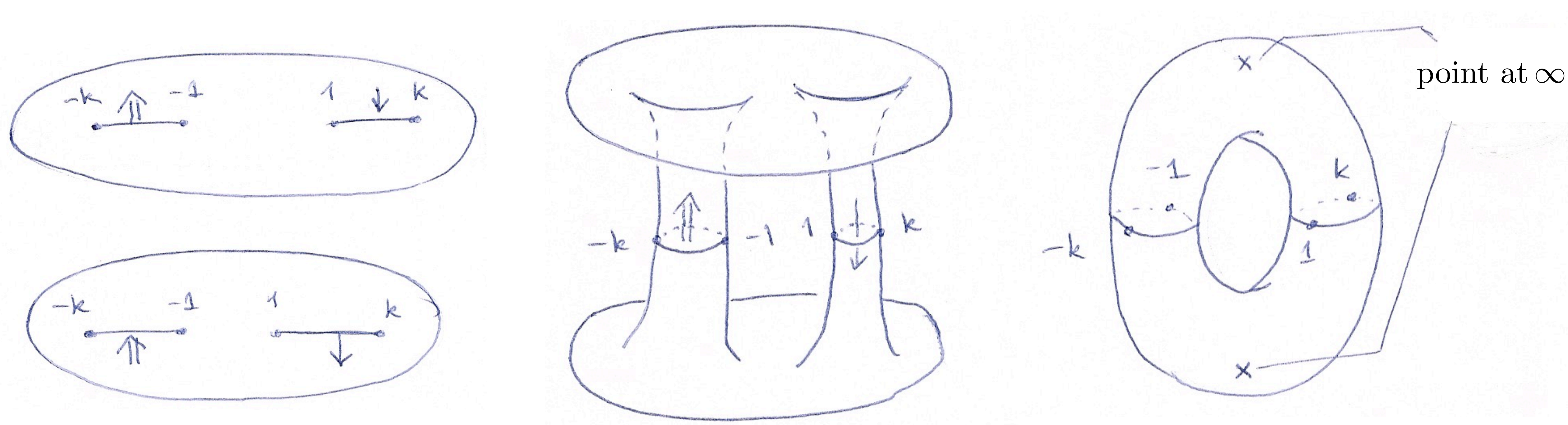




[Drawings by C. Teleman, Riemann Surfaces]

CHANGING THE GEOMETRY: FROM GENUS 0 TO GENUS 1

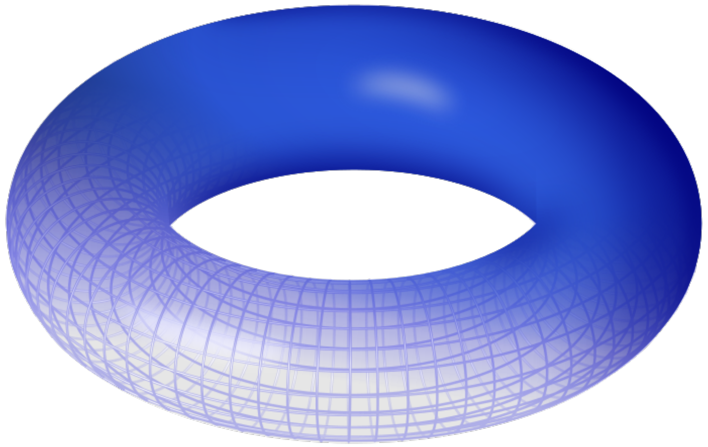
Some definitions. Take a completely general elliptic curve:

$$y^2 = (x-a_1)(x-a_2)(x-a_3)(x-a_4)$$

We define the two periods as

$$\omega_1 = 2c_4 \int_{a_2}^{a_3} \frac{dx}{y} = 2\,\mathrm{K}(\lambda)\,, \quad \omega_2 = 2c_4 \int_{a_1}^{a_2} \frac{dx}{y} = 2\,i\,\mathrm{K}(1-\lambda)\,,$$

$$\lambda = \frac{(a_1-a_4)(a_2-a_3)}{(a_1-a_3)(a_2-a_4)}\,, \quad c_4 = \frac{1}{2}\sqrt{(a_1-a_3)(a_2-a_4)}$$



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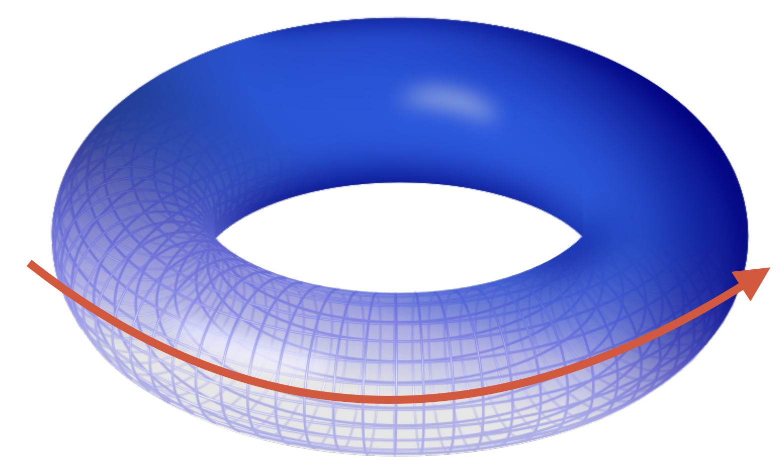

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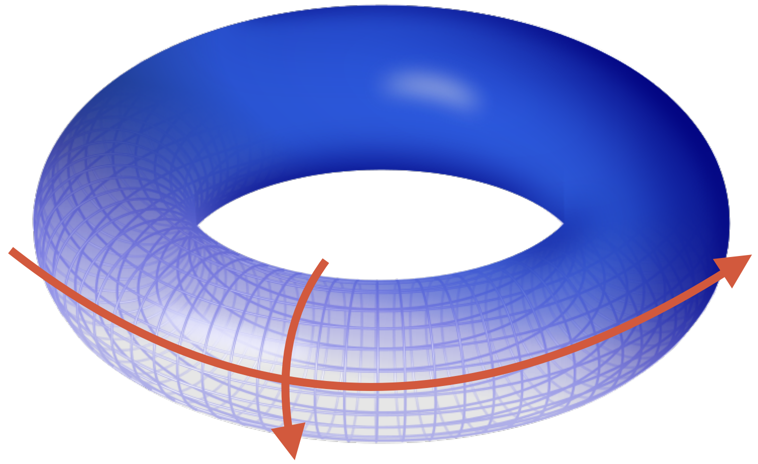
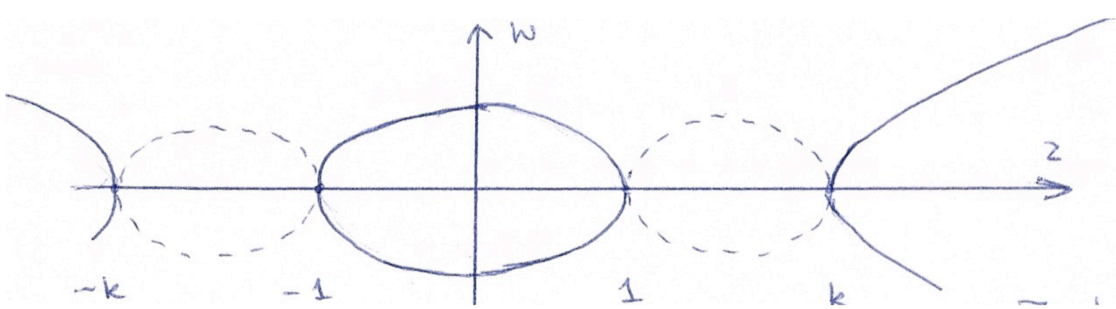


CHANGING THE GEOMETRY: FROM GENUS 0 TO GENUS 1

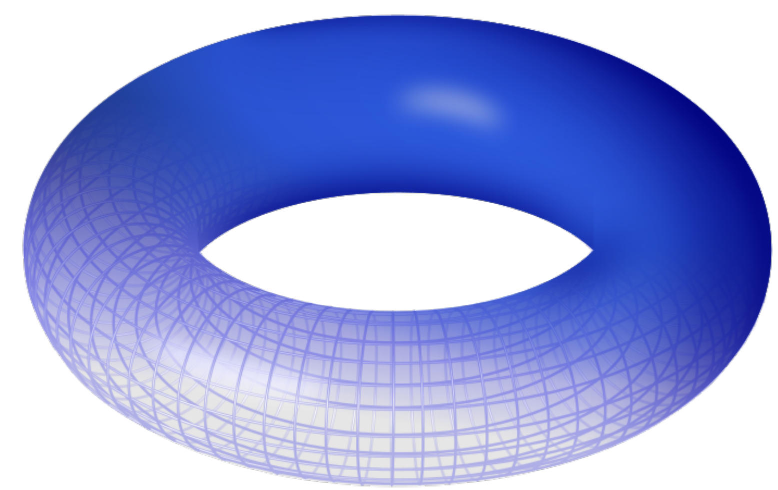
Dual description of the same problem

Elliptic curve as algebraic equation

$$y^2 = (x - a_1)(x - a_2)(x - a_3)(x - a_4)$$



Genus one complex surface - Torus



Move between the two using ***Abel's Map***

Or its inverse (kappa-function)

$$z_x = \frac{c_4}{\omega_1} \int_{a_1}^{x} \frac{dt}{y(t)}$$

ELLIPTIC MULTIPLE POLYLOGARITHMS

$$G(c_1, ..., c_k; x) = \int_0^x dt\, r(c_1, t) G(c_2, ..., c_k; t)\,, \qquad r(c,t) = \frac{1}{t-c} \quad c \in \mathbb{C}$$

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$$\begin{aligned}
G_*(\vec{a}) &= \left(\frac{2\eta_1}{\omega_1} - \frac{\lambda}{3} + \frac{2}{3}\right) \mathrm{F}(\sqrt{\alpha}|\lambda) \\
&\quad - \mathrm{E}(\sqrt{\alpha}|\lambda) + \sqrt{\frac{\alpha(\alpha\lambda - 1)}{\alpha - 1}}
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Transcendental integration kernels

Mimic MPLs, only **simple poles**, **logarithmic singularities of scattering amplitudes**

WE KIND OF UNDERSTAND EMPLS…

Many interesting problems solved in terms of eMPLs (kite, form factors, many more)

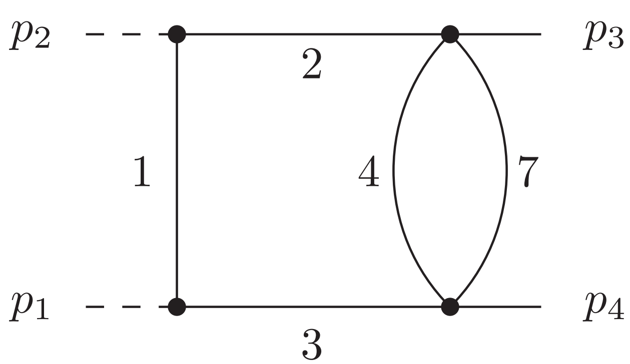
[See Melih's talk]

Are we done, then?

BEYOND EMPLS: TOWARDS CALABY YAU GEOMETRIES

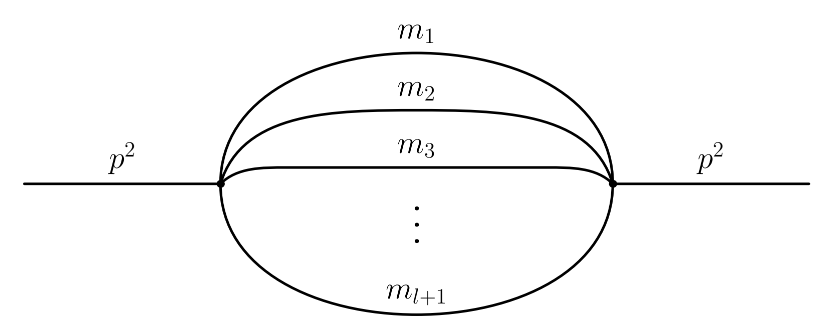
eMPLs are not enough, (not even at 2 loops, *at least naively*…)

Multiple elliptic curves



[Adams, Ekta, Weinzierl 2018]
[Müller, Weinzierl 2022]

K3 and Calabi Yau Geometries



[Block, Kerr, Vanhove 2017] [Primo, Tancredi 2017]
[Bourjaily, He, McLeod, von Hippel, Wilhelm 2018]
[Bourjaily, McLeod, Vergu, Volk, von Hippel, Wilhelm 2019]
[Brödel, Duhr, Dulat, Marzucca, Penante, Tancredi 2019]
[Bönisch, Duhr, Fischback, Klemm, Nega 2021]

We will here a lot about these exciting advancements in this workshop…

See seminars by Matt, Franziska, Christoph…

THE MISSING LINK: (CANONICAL?) DIFFERENTIAL EQUATIONS

Henn's canonical differential equations — paradigm change for MPLs Feynman Integrals

$$\mathrm{d}f = \epsilon\, A\, f \quad \text{with} \quad A = \sum_i A_i \, \mathrm{d}\log g_i$$

Integrals with *unit leading singularities* → *logarithmic singularities only!*

Natural concept of **(uniform) transcendental weight** inherited from algebra of MPLs

What about Feynman integrals with elliptic (or more general kernels?)

- is there a concept of transcendental weight?
- Uniform weight & purity? [Brödel, Duhr, Dulat, Penante, Tancredi 2018]
- if so, how is uniform weight connected to structure of differential equations (ϵ dep…)?

THE MISSING LINK: (CANONICAL?) DIFFERENTIAL EQUATIONS

Recently progress in finding ***epsilon-factorised differential equations*** for some problems

- Sunrise graph & modular forms [Adams, Weinzierl 2016]
- 3 loop Banana graph [Pögel, Wang, Weinzierl 2022]

[See talk by Hjalte for more general considerations]

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Special case: ***univariate problems based on 1 single elliptic curve*** (sunrise, 3 loop banana, …) can sometimes be expressed as ***iterated integrals over modular forms***

$$\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_k \\ z_1 & \dots & z_k \end{smallmatrix}; z_{k+1}, \tau\right) = \int_0^{z_{k+1}} dw\, g^{(n_1)}(w - z_1; \tau)\, \tilde{\Gamma}\left(\begin{smallmatrix} n_2 & \dots & n_k \\ z_2 & \dots & z_k \end{smallmatrix}; w, \tau\right) \qquad \text{with} \qquad z_i = \frac{r_i}{N} + \tau \frac{s_i}{N}$$

can be written as **iterated ints of Eisenstein series of $\Gamma(N)$**

$$\begin{aligned} I\left(\begin{smallmatrix} n_1 & N_1 \\ r_1 & s_1 \end{smallmatrix} \middle| \dots \middle| \begin{smallmatrix} n_k & N_k \\ r_k & s_k \end{smallmatrix}; \tau\right) &\equiv I(h^{(n_1)}_{N_1,r_1,s_1}, \dots, h^{(n_k)}_{N_k,r_k,s_k}; \tau) \\ &= \int_{i\infty}^{\tau} d\tau'\, h^{(n_1)}_{N_1,r_1,s_1}(\tau')\, I\left(\begin{smallmatrix} n_2 & N_2 \\ r_2 & s_2 \end{smallmatrix} \middle| \dots \middle| \begin{smallmatrix} n_k & N_k \\ r_k & s_k \end{smallmatrix}; \tau'\right) \end{aligned}$$

$$h^{(k)}_{N,r,s}(\tau) = -\sum_{\substack{(\alpha,\beta)\in\mathbb{Z}^2 \\ (\alpha,\beta)\neq(0,0)}} \frac{e^{2\pi i(s\alpha - r\beta)/N}}{(\alpha + \beta\tau)^{2n}}$$

THE MISSING LINK: (CANONICAL?) DIFFERENTIAL EQUATIONS

In those cases, all properties of these integrals are understood at a similar level as standard MPLs: series (q-)expansions, numerical evaluation, numerical evaluation

[Duhr, Tancredi 2019]

It is also clearer that **there is only 1 natural variable, τ,** and it is natural to study **differential equations in τ**

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Already "which differential equations" is not an obvious question for a more general eMPL which depends on many variables

$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \Psi_{n_1}(c_1, t, \vec{a})\, \mathcal{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

$\tau(\vec{a})$ determines the shape of the elliptic curve

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$\tau(\vec{a})$ determines the shape of the elliptic curve

There is an extra, different dependence on how we move along the curve and on punctures

THE MISSING LINK: (CANONICAL?) DIFFERENTIAL EQUATIONS

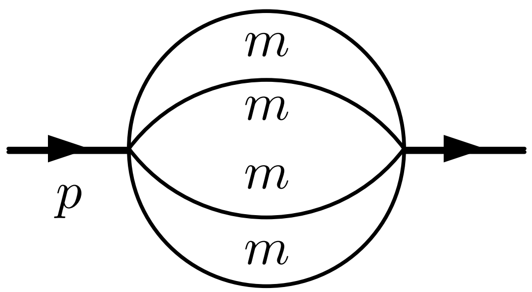
What can we say in general?

THE MISSING LINK: (CANONICAL?) DIFFERENTIAL EQUATIONS

What can we say in general?

Hint: Maximal cuts, leading singularities and independent integration contours

[Primo, Tancredi 2016, 2017]
[Bosma, Sogaard, Zhang 2017]



$$= I_{a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9}\Big|_{a_5,\cdots,a_9<0}$$

$$= \int \frac{\mathfrak{D}^d k_1\, \mathfrak{D}^d k_2 \mathfrak{D}^d k_3 \;\; (k_3^2)^{-a_5}(k_1\cdot p)^{-a_6}(k_2\cdot p)^{-a_7}(k_3\cdot p)^{-a_8}(k_1\cdot k_2)^{-a_9}}{[k_1^2-m^2]^{a_1}[k_2^2-m^2]^{a_2}[(k_1-k_3)^2-m^2]^{a_3}[(k_2-k_3-p)^2-m^2]^{a_4}} \,,$$

$$\begin{aligned}
\mathcal{I}_1(\epsilon;s) =& (1+2\epsilon)(1+3\epsilon)(m^2)^{-2} I_{1,1,1,1,0,0,0,0,0}\,,\\
\mathcal{I}_2(\epsilon;s) =& (1+2\epsilon)(m^2)^{-1} I_{2,1,1,1,0,0,0,0,0}\,,\\
\mathcal{I}_3(\epsilon;s) =& I_{2,2,1,1,0,0,0,0,0}\,,
\end{aligned}$$

THE MISSING LINK: (CANONICAL?) DIFFERENTIAL EQUATIONS

What can we say in general?

Hint: Maximal cuts, leading singularities and independent integration contours

[Primo, Tancredi 2016, 2017]
[Bosma, Sogaard, Zhang 2017]

$$\frac{d}{dx}\begin{pmatrix}\mathcal{I}_1(\epsilon;x)\\ \mathcal{I}_2(\epsilon;x)\\ \mathcal{I}_3(\epsilon;x)\end{pmatrix} = B(x)\begin{pmatrix}\mathcal{I}_1(\epsilon;x)\\ \mathcal{I}_2(\epsilon;x)\\ \mathcal{I}_3(\epsilon;x)\end{pmatrix} + \epsilon\, D(x)\begin{pmatrix}\mathcal{I}_1(\epsilon;x)\\ \mathcal{I}_2(\epsilon;x)\\ \mathcal{I}_3(\epsilon;x)\end{pmatrix} + \begin{pmatrix}0\\ 0\\ -\frac{1}{2(4x-1)}\end{pmatrix}$$

Define a "solution matrix" from the maximal cuts evaluated along ***independent integration contours***:

$$G(x) = \begin{pmatrix}\mathrm{Cut}_{\mathcal{C}_1}(\mathcal{I}_1(x)) & \mathrm{Cut}_{\mathcal{C}_2}(\mathcal{I}_1(x)) & \mathrm{Cut}_{\mathcal{C}_3}(\mathcal{I}_1(x))\\ \mathrm{Cut}_{\mathcal{C}_1}(\mathcal{I}_2(x)) & \mathrm{Cut}_{\mathcal{C}_2}(\mathcal{I}_2(x)) & \mathrm{Cut}_{\mathcal{C}_3}(\mathcal{I}_2(x))\\ \mathrm{Cut}_{\mathcal{C}_1}(\mathcal{I}_3(x)) & \mathrm{Cut}_{\mathcal{C}_2}(\mathcal{I}_3(x)) & \mathrm{Cut}_{\mathcal{C}_3}(\mathcal{I}_3(x))\end{pmatrix}$$

(Pairing integrand - contour, see intersection theory etc)

See also talk by Hjalte !

THE MISSING LINK: (CANONICAL?) DIFFERENTIAL EQUATIONS

What can we say in general?

Hint: Maximal cuts, leading singularities and independent integration contours

[Primo, Tancredi 2016, 2017]
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Then the new basis of master integrals $\begin{pmatrix} \mathcal{I}_1(x) \\ \mathcal{I}_2(x) \\ \mathcal{I}_3(x) \end{pmatrix} = G(x) \begin{pmatrix} M_1(x) \\ M_2(x) \\ M_3(x) \end{pmatrix}$

By construction has diagonal matrix of ***"unit leading singularities" (on the max cut!)***

$$\begin{aligned}
&\mathrm{Cut}_{\mathcal{C}_1}(M_1(x)) = 1\,, \quad \mathrm{Cut}_{\mathcal{C}_2}(M_1(x)) = 0\,, \quad \mathrm{Cut}_{\mathcal{C}_3}(M_1(x)) = 0\,, \\
&\mathrm{Cut}_{\mathcal{C}_1}(M_2(x)) = 0\,, \quad \mathrm{Cut}_{\mathcal{C}_2}(M_2(x)) = 1\,, \quad \mathrm{Cut}_{\mathcal{C}_3}(M_2(x)) = 0\,, \\
&\mathrm{Cut}_{\mathcal{C}_1}(M_3(x)) = 0\,, \quad \mathrm{Cut}_{\mathcal{C}_2}(M_3(x)) = 0\,, \quad \mathrm{Cut}_{\mathcal{C}_3}(M_3(x)) = 1\,.
\end{aligned}$$

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THE MISSING LINK: (CANONICAL?) DIFFERENTIAL EQUATIONS

Basis built using this recipe
$$\begin{pmatrix} \mathcal{I}_1(x) \\ \mathcal{I}_2(x) \\ \mathcal{I}_3(x) \end{pmatrix} = G(x) \begin{pmatrix} M_1(x) \\ M_2(x) \\ M_3(x) \end{pmatrix}$$

Fulfils by construction ϵ-factorised differential equations, if the basis we started from only had linear dependence on ϵ

$$\mathrm{d}f = B\,f + \epsilon\,A\,f \quad \text{rotating out } B \text{ as above, becomes} \quad df' = \epsilon\,G^{-1}AG\,f'$$

THE MISSING LINK: (CANONICAL?) DIFFERENTIAL EQUATIONS

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BUT Notice:

- This is true **whatever is in the matrix A**… in the MPLs case this rotation works in the same way, but if done starting from "any A", then $G^{-1}AG$ will in general **not be in dlog form!** → **In general more information is needed on singularities of the integrals!**
- Moreover, to the best of my knowledge, it is not even obvious that **a basis linear in ϵ exists** in general beyond 2 or 3 loops! At least, many cases where **I don't know how to find one** :-)

OPEN QUESTIONS: FROM SPECIAL FUNCTIONS BACK TO FEYNMAN INTEGRALS!

From this perspective, a set of questions that I find particularly interesting:

- Is there a natural **generalisation of $\mathrm{d}\log$ forms** for CYs?
- What **“form” are the differential equations for those integrals expected to have**?
- Can we always find differential equations with **linear dependence in ϵ** when general CY geometries are involved? And if so, **how restrictive is this requirement**?
- If that is possible, which **criterion** Feynman integrals/integrands have to satisfy?

OPEN QUESTIONS: FROM SPECIAL FUNCTIONS BACK TO FEYNMAN INTEGRALS!

How do we move forward?

From experience with MPLs, many things have to come into place at the same time

- Collect as much data as possible (compute more integrals! :-))
- Understanding of **mathematical properties of new functions**, iterated integrals over CY periods and rational functions **See Christoph's talk**
- Generalisation of **symbol calculus**

  [Brödel, Duhr, Dulat, Penante, Tancredi 2018]
  [Wilhem, Zhang 2022]
  [Forum, von Hippel 2022] **See Chi's talk**

- Connection to leading singularities…

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I think it's fair to say that we are witnessing advances in all these directions

It is very exciting!

THE END (OF THIS INVITATION)

Thank you very much for your attention

and looking forward to learn about many of these things during this week!