Hadronic light-by-light phenomenology

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HISKP (Theorie) & BCTP

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International Physics School on

Muon Dipole Moments and Hadronic Effects

31/8/2021

Simon Eidelman (1948–2021)

picture credit: Zdeněk Doležal 2014

Outline

Hadronic light-by-light scattering: phenomenology

Part I

- overview of the problem
- methods: dispersion relations
- some illustrative examples

Part II

- in-depth analysis: π^0 pole contribution

Summary / Outlook

Part I:

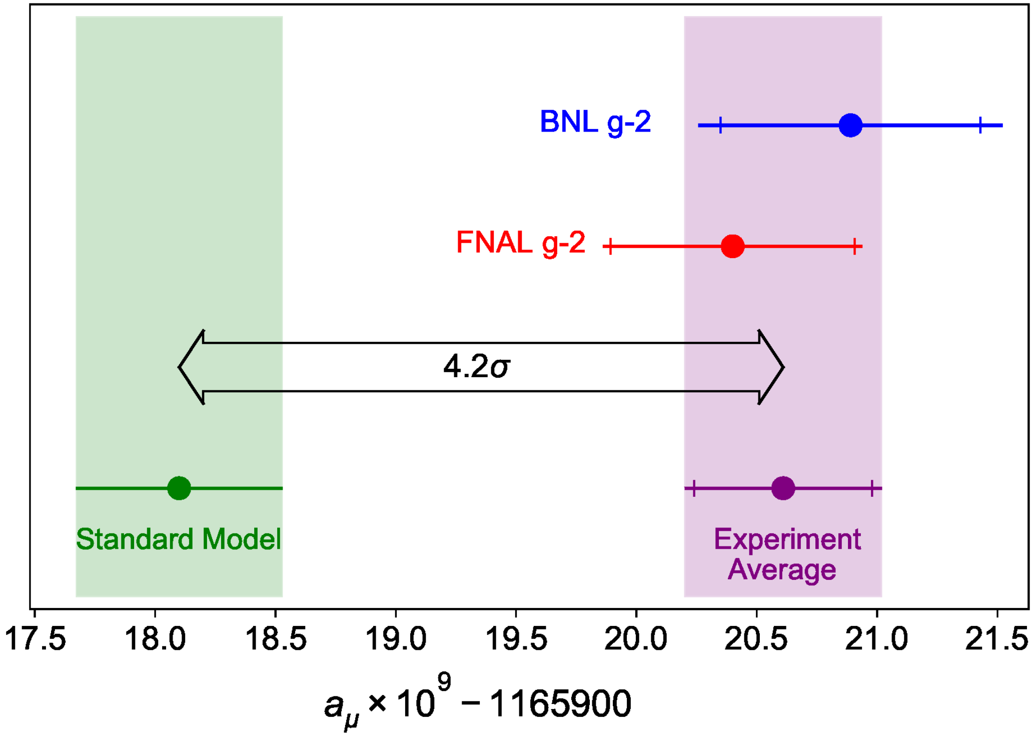
problem — methods — examples

The anomalous magnetic moment of the muon

- gyromagnetic ratio: magnetic moment $\leftrightarrow$ spin

$$\vec{\mu} = g\frac{e}{2m}\vec{S} \qquad \text{Dirac:} \quad g_\mu = 2$$

- rad. corr.: $g_\mu = 2(1 + a_\mu)$, $\quad a_\mu$ "anomalous magnetic moment"
- one of the most precisely measured quantities in particle physics



$\longrightarrow 4.2\sigma$ discrepancy: $\qquad a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (251 \pm 59) \times 10^{-11}$

BNL E821 2006, Fermilab 2021; Aoyama et al. 2020

Hadronic contributions to a_μ

	$a_\mu\ [10^{-11}]$	$\Delta a_\mu\ [10^{-11}]$
experiment	116 592 061.	41.
QED $\mathcal{O}(\alpha)$	116 140 973.321	0.023
QED $\mathcal{O}(\alpha^2)$	413 217.626	0.007
QED $\mathcal{O}(\alpha^3)$	30 141.902	0.000
QED $\mathcal{O}(\alpha^4)$	381.004	0.017
QED $\mathcal{O}(\alpha^5)$	5.078	0.006
QED total	116 584 718.931	0.030
electroweak	153.6	1.0
had. VP (LO)	6931.	40.
had. VP (NLO)	–98.3	0.7
had. LbL	92.	19.
total	116 591 810.	43.

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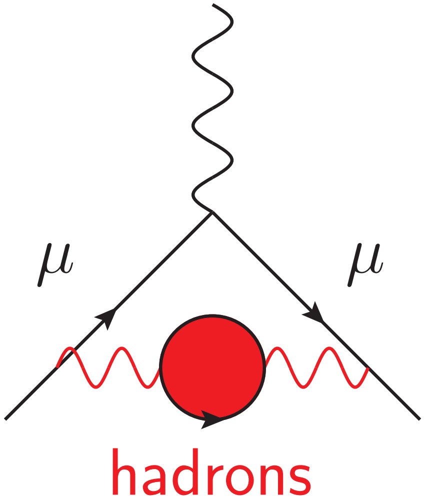

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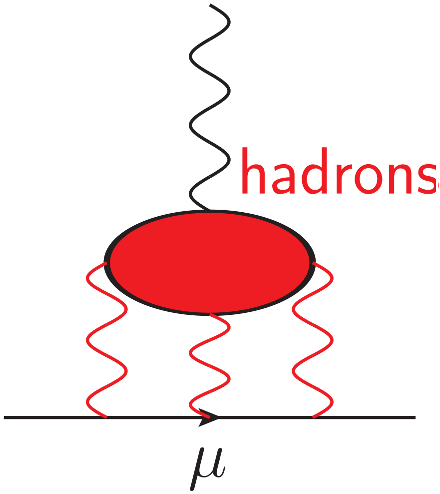
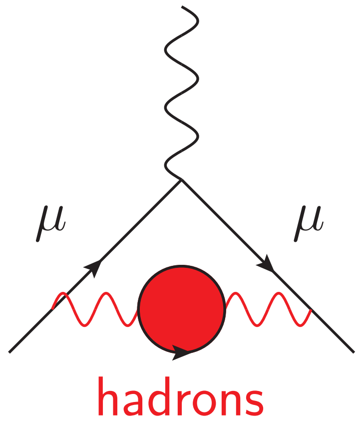
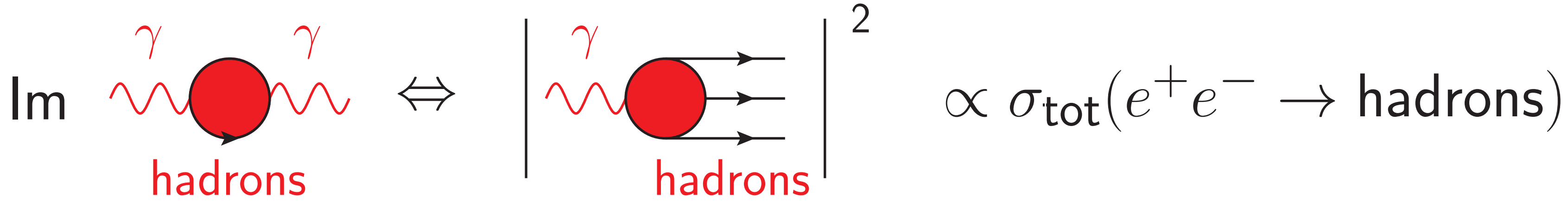


Hadronic vacuum polarisation

$\longrightarrow$ lecture by Daisuke Nomura

- how to control hadronic vacuum polarisation?
- characteristic scale set by muon mass
  $\longrightarrow$ this is not a perturbative QCD problem!
- dispersion relations to the rescue:
  use the optical theorem!





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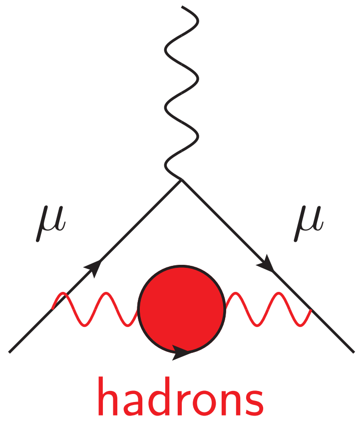
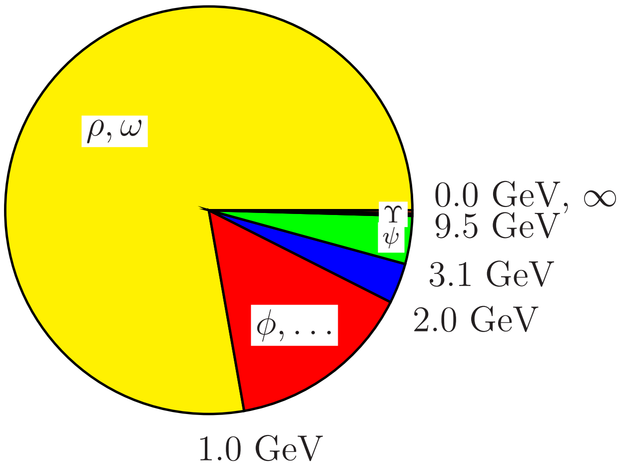


$$a_\mu^{\text{had VP}} \propto \int_{4M_\pi^2}^{\infty} ds\, K(s)\sigma_{\text{tot}}(e^+e^- \to \text{hadrons})$$

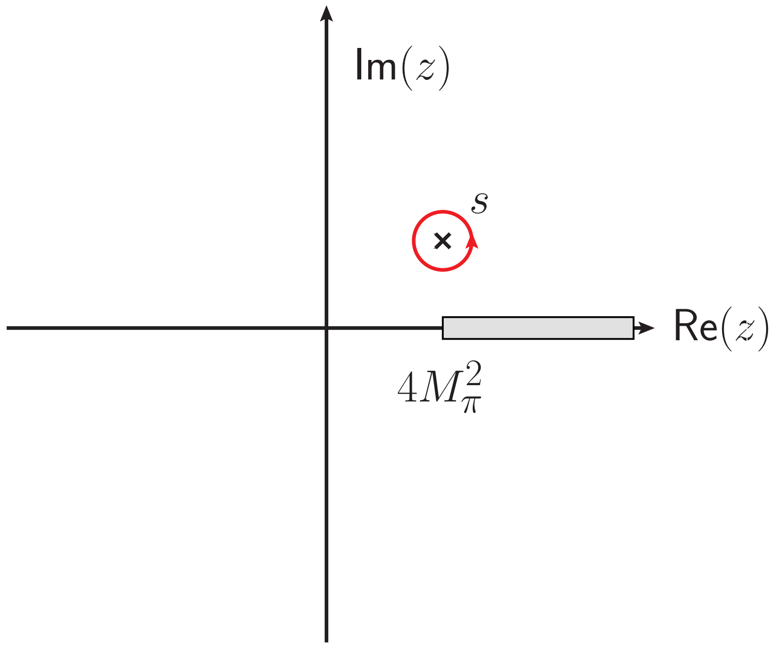
- $K(s)$: kinematical function, for large s: $K(s) \propto 1/s$,
  $\sigma_{\text{tot}}(e^+e^- \to \text{hadrons}) \propto 1/s$
- more than 75% of $a_\mu^{\text{had VP}}$ given by
  energies $s \leq 1\,\text{GeV}^2$ Jegerlehner, Nyffeler 2009
- well constrained by data

BABAR, BESIII, CMD, KLOE, SND, ...

$\longrightarrow$ *largely* an experimental task



Dispersion relations for pedestrians



analyticity ($\simeq$ causality)

& Cauchy's theorem:

$$T(s) = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{T(z)dz}{z-s}$$

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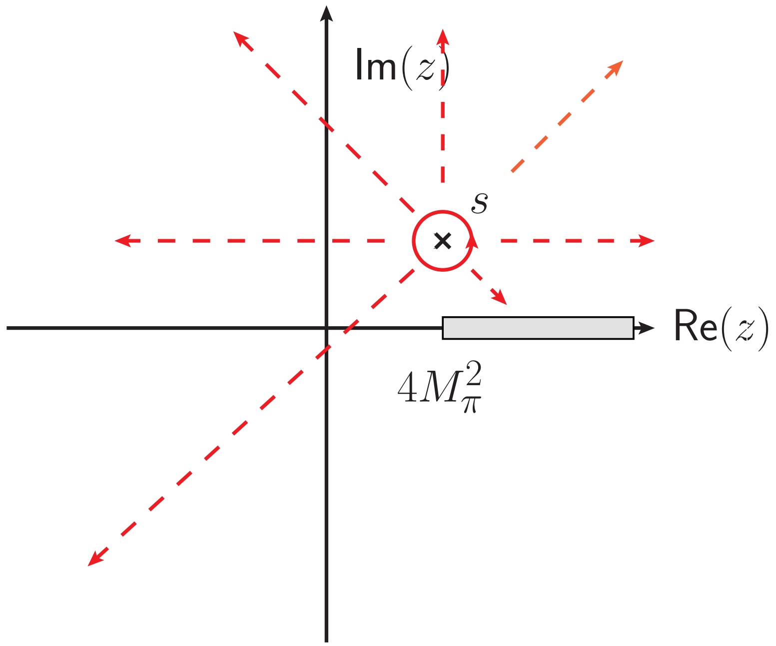

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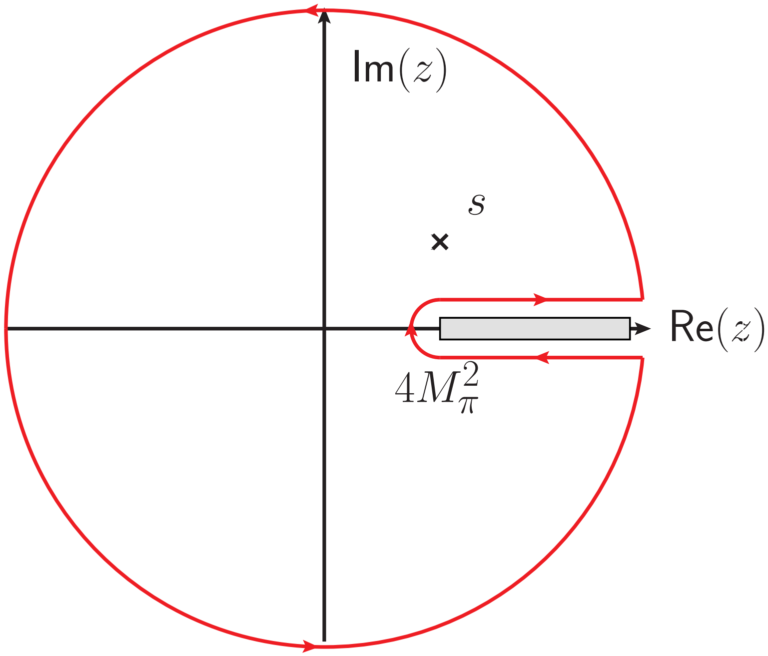

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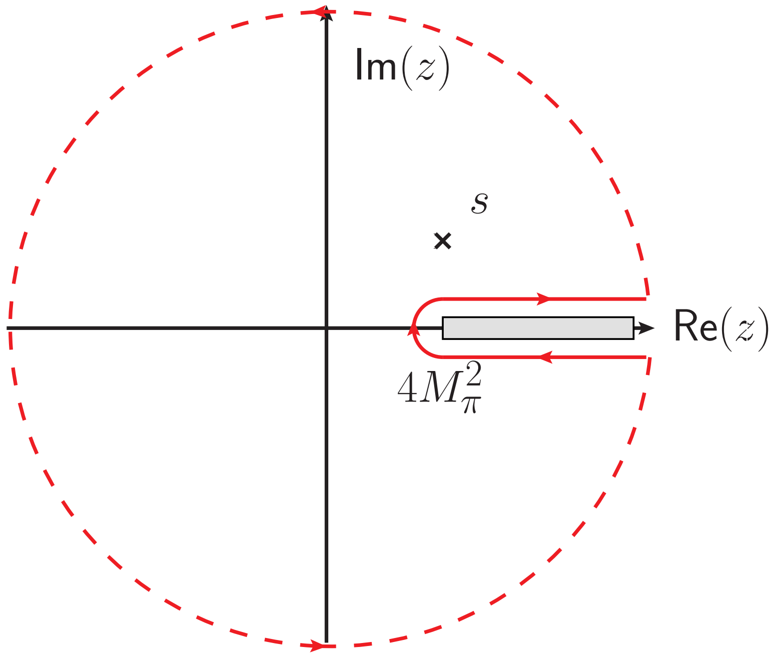

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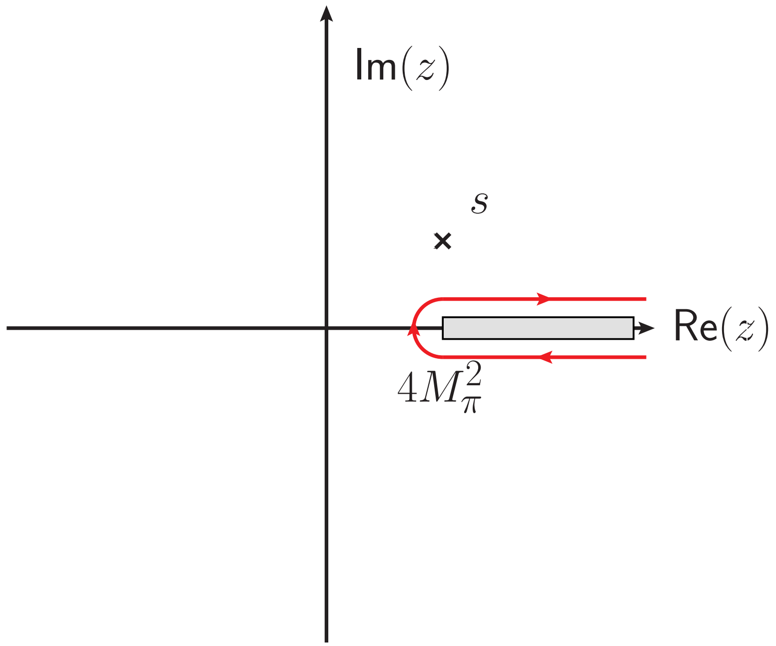

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$$\longrightarrow \frac{1}{2\pi i}\int_{4M_\pi^2}^{\infty}\frac{\text{disc}\, T(z)dz}{z-s}$$

$$= \frac{1}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\text{Im}\, T(z)dz}{z-s}$$

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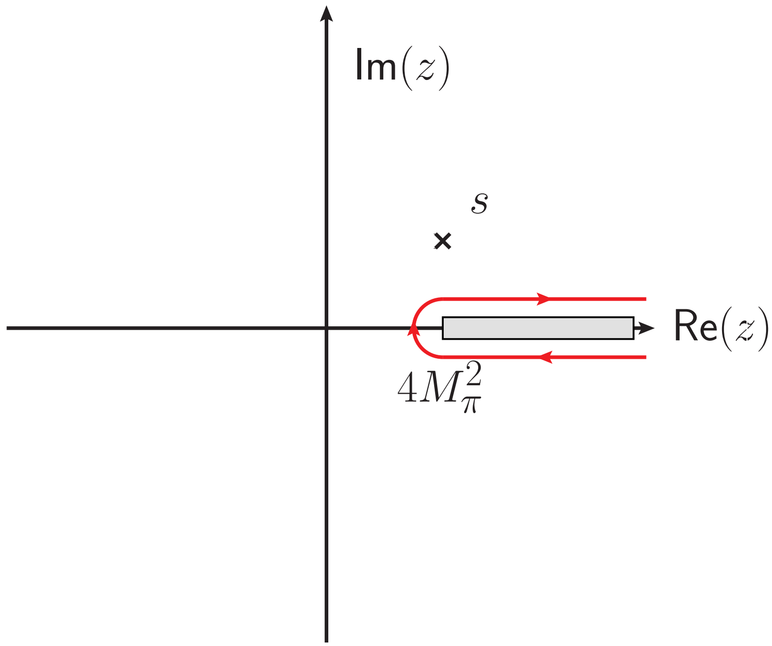

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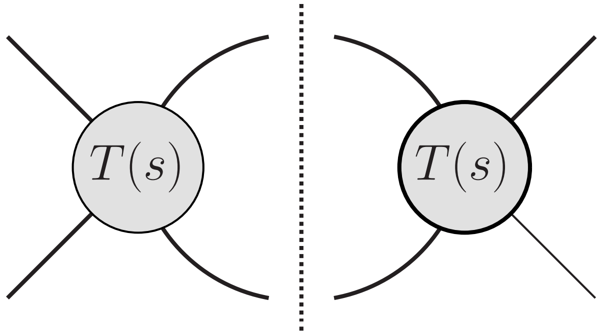

e.g. if $T(s)$ is a $\pi\pi$ partial wave $\longrightarrow$

$$\frac{\operatorname{disc} T(s)}{2i} = \operatorname{Im} T(s) = \frac{2q_\pi}{\sqrt{s}}\theta(s-4M_\pi^2)|T(s)|^2$$

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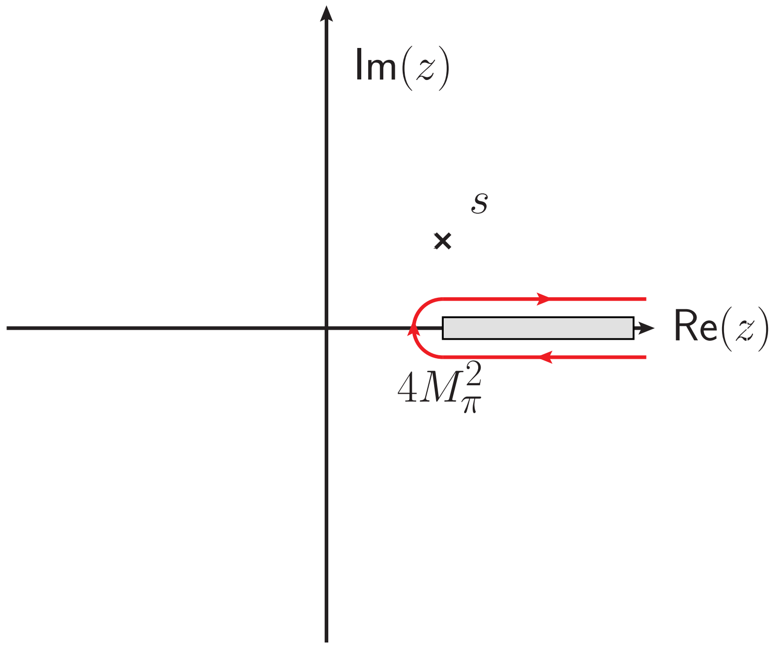

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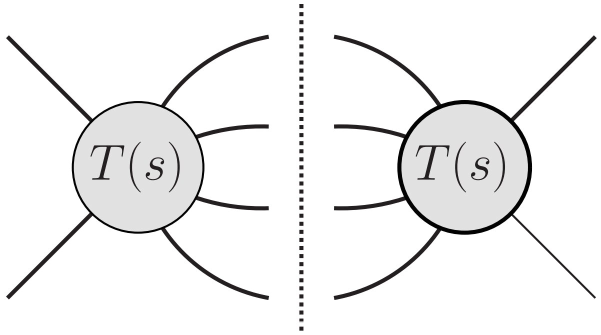
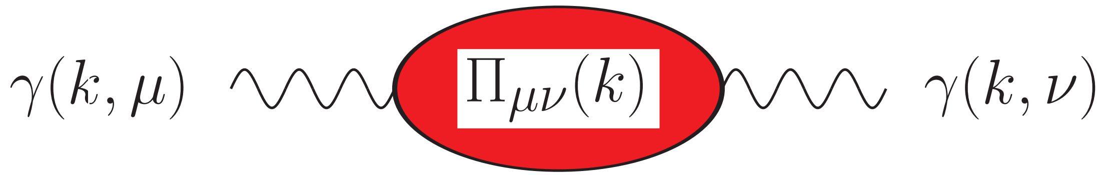


inelastic intermediate states ($K\bar{K}$, 4π)
suppressed at low energies
$\longrightarrow$ will often be neglected

Hadronic vacuum polarisation — why so simple?

- photon two-point function:



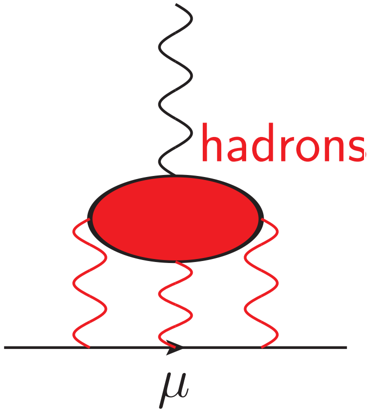
  - ▷ one single independent momentum k
  - ▷ symmetric rank-2 tensor: two structures $g_{\mu\nu}$, $k_\mu k_\nu$
  - ▷ scalar invariant can depend on one single invariant k^2
- gauge invariance: $k^\mu \Pi_{\mu\nu}(k) = 0 = k^\nu \Pi_{\mu\nu}(k)$

$$\boxed{\Pi_{\mu\nu}(k) = \big(k^2 g_{\mu\nu} - k_\mu k_\nu\big)\Pi(k^2)}$$

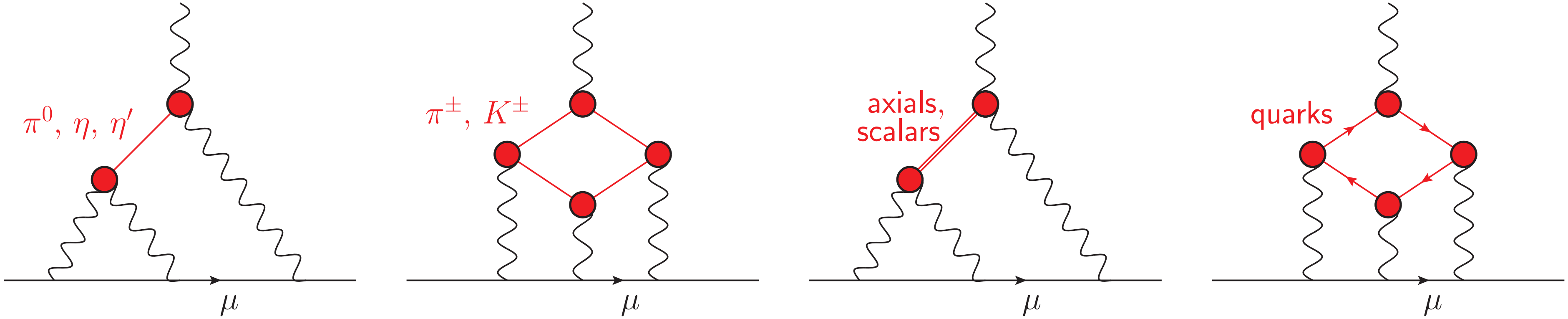
$\longrightarrow$ Lorentz + gauge invariance reduce HVP to one single function of a single variable!

Hadronic light-by-light scattering

- hadronic light-by-light:
  - ▷ subleading in α_{QED}
  - ▷ large relative uncertainty



- different contributions calculated or estimated (in 10^{-11}):



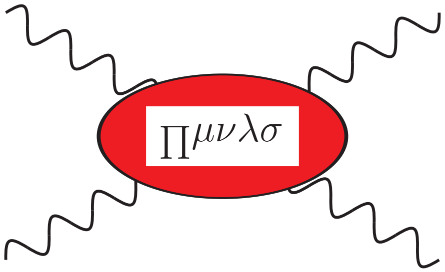
94±4	–24±1	5±7	18±10

$\longrightarrow$ increasing systematic control over HLbL using dispersion-theoretical approach

Aoyama et al. 2020

Hadronic light-by-light: dispersive approach

Colangelo, Hoferichter, Procura, Stoffer 2014, 2015



- HLbL tensor $\Pi^{\mu\nu\lambda\sigma}$: Lorentz invariance
  $\longrightarrow$ 138 (136) scalar functions Eichmann et al. 2014
- gauge invariance: Bardeen, Tung 1968; Tarrach 1975

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

$\longrightarrow$ 7 distinct structures, 47 related by crossing

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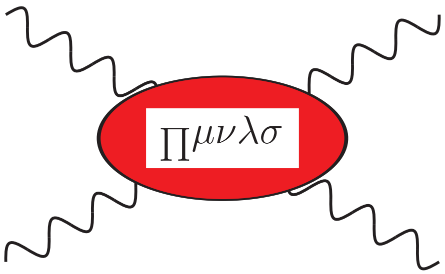

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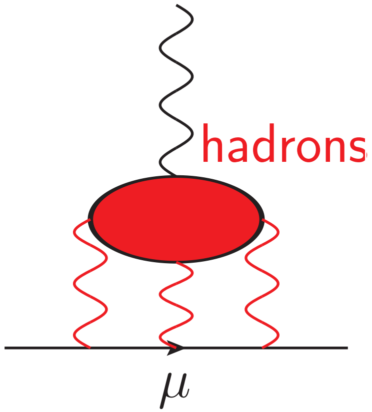


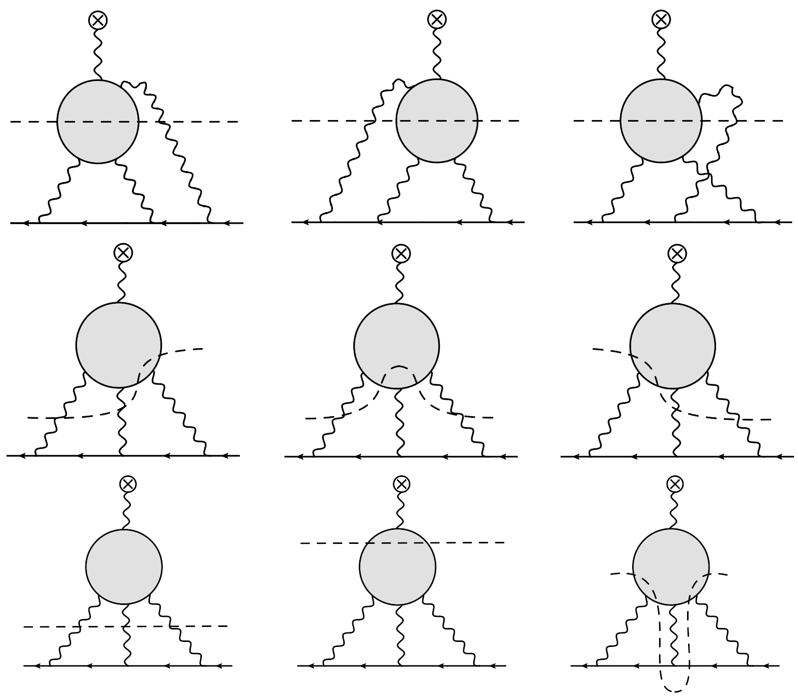
- master formula:

$$a_\mu^{\mathrm{HLbL}} = -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{\sum_{i=1}^{12} \hat{T}_i(q_1, q_2; p) \hat{\Pi}_i(q_1, q_2, -q_1 - q_2)}{q_1^2 q_2^2 (q_1 + q_2)^2 [(p + q_1)^2 - m_\mu^2][(p - q_2)^2 - m_\mu^2]}$$

- $\hat{T}_i$: known kernels
  $\hat{\Pi}_i$: dispersively $\leftrightarrow$ measurable form factors / scatt. amplitudes

Hadronic light-by-light: alternative dispersive approaches (1)

Pauk, Vanderhaeghen 2014



- idea: dispersion relation for muon's magnetic form factor F_2 instead of for the HLbL tensor

+ "all in one" approach, addresses directly the final observable

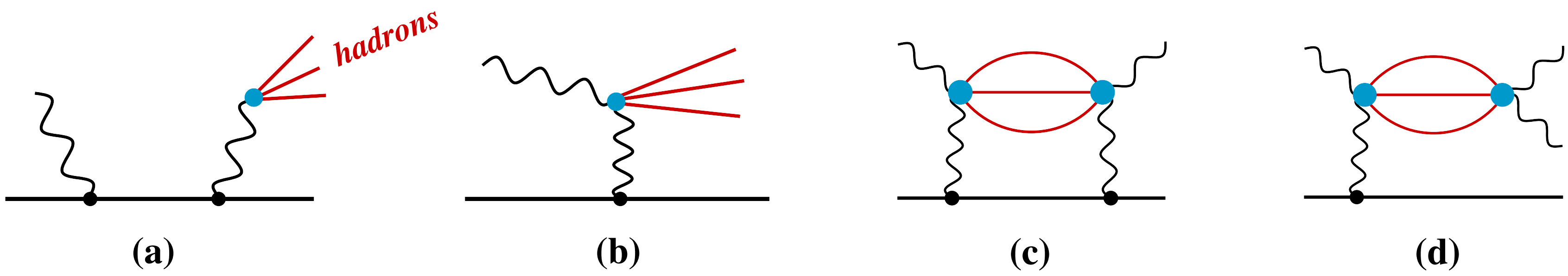
– intertwines hadronic and photon/lepton cuts

– practical difficulties in hadronic intermediate states not reduced

Hadronic light-by-light: alternative dispersive approaches (2)

Schwinger sum rule

Hagelstein, Pascalutsa 2017



- a_μ related to photoabsorption cross section on the muon:

$$a_\mu = \frac{m_\mu^2}{\pi^2\alpha} \int_{\nu_0}^{\infty} d\nu \left[\frac{\sigma_{LT}(\nu, Q^2)}{Q}\right]_{Q^2=0}$$

+ “all in one” approach, true generalisation of HVP dispersive (a)

– chance to measure these (polarised) cross sections

$$\gamma^* \mu^\pm \to \mu^\pm \text{ anything}$$

seems rather optimistic. . .

Pions. . . what about chiral perturbation theory?

Chiral perturbation theory (ChPT) Weinberg; Gasser, Leutwyler; . . .

- QCD near the chiral limit of two massless quarks: $m_{u,d} \ll \Lambda_{\text{QCD}}$
- chiral symmetry spontaneously broken:

$$\mathsf{SU}(2)_L \times \mathsf{SU}(2)_R \xrightarrow{\mathsf{SSB}} \mathsf{SU}(2)_V$$

$\longrightarrow$ pions are nearly massless Goldstone bosons
weakly interacting at low energies

- effective field theory: simultaneous expansion in

quark masses + small momenta

  - ▷ systematically improvable
  - ▷ well-established link to QCD: all symmetry constraints
  - ▷ interrelates many different observables

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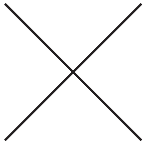
$\mathcal{O}(p^2)$



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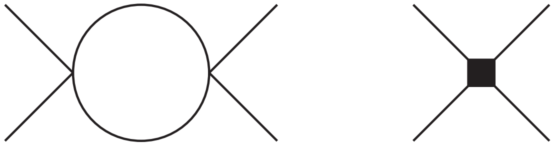

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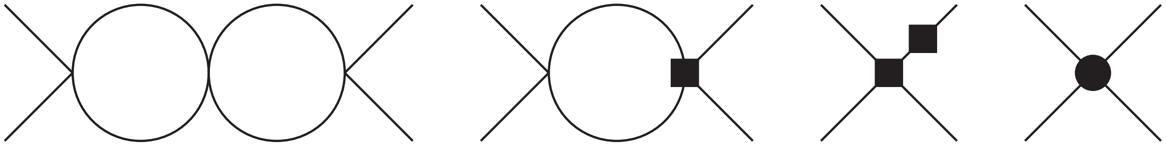
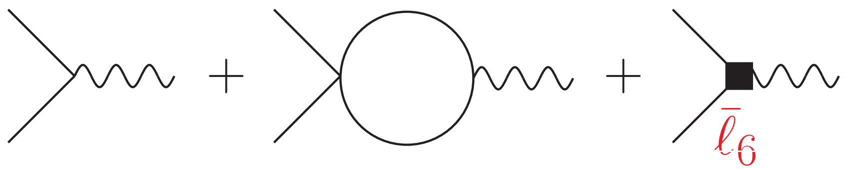


ChPT, example: pion vector form factor

Gasser, Leutwyler 1984

$$\langle \pi^+(p_+)\pi^-(p_-)|j_\mu^{\rm em}|0\rangle = -(p_+ - p_-)_\mu F_\pi^V(s) \; , \qquad s = (p_+ + p_-)^2$$

- ChPT at $\mathcal{O}(p^4)$:



- expansion at small s:

$$F_\pi^V(s) = 1 + \frac{1}{6}\langle r^2\rangle_\pi^V s + \mathcal{O}(s^2) \; , \qquad \langle r^2\rangle_\pi^V = \frac{1}{(4\pi F_\pi)^2}(\bar\ell_6 - 1)$$

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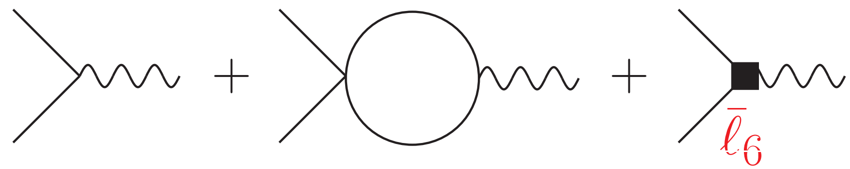

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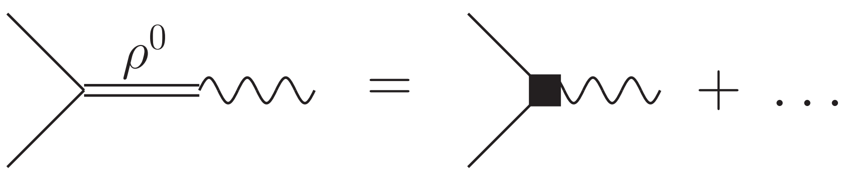
- contribution of the ρ-resonance:



$$\frac{s}{M_\rho^2 - s} = \frac{s}{M_\rho^2}\left(1 + \frac{s}{M_\rho^2} + \ldots\right) \longrightarrow \text{reproduces } \bar{\ell}_6 \text{ nicely!}$$

$\longrightarrow$ LECs incorporate effects of heavier states (resonances)

ChPT, problem: pion vector form factor in HLbL

ChPT and its limitations

- physics of pions (light pseudoscalars: π, K, η) only
  - ▷ (energy) range limited by resonances: $f_0(500)$, $\rho(770)$ ...
  - ▷ unitarity is only perturbatively fulfilled
- low-energy EFT; polynomial approximation to resonances
  $\longrightarrow$ bad high-energy behaviour:

$$[F_\pi^V(s)]^{\text{1-loop}}_{\text{ChPT}} \propto s\,, \qquad \text{expect} \qquad [F_\pi^V(s)]_{\text{QCD}} \propto 1/s$$

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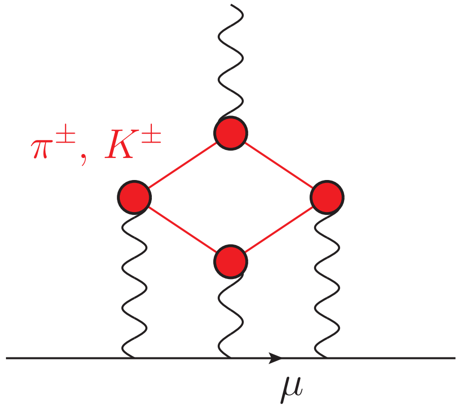
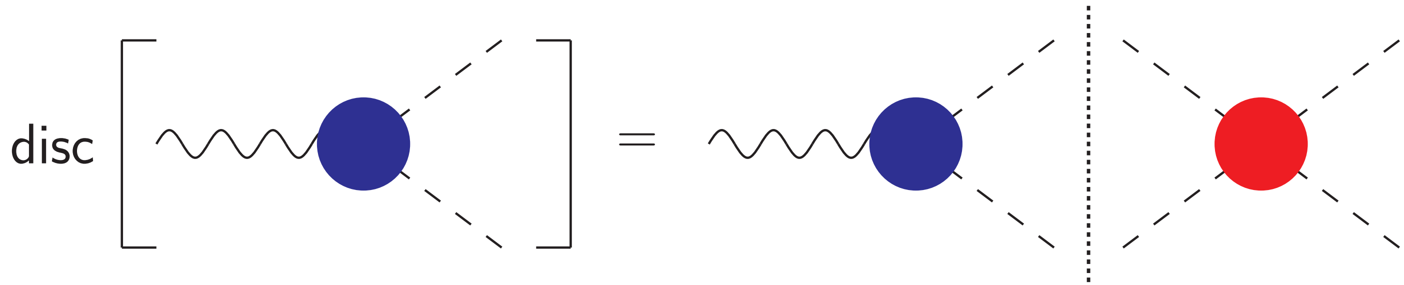
- consequences for charged-pion box in HLbL:
  - ▷ leading-order ChPT $\Leftrightarrow$ scalar QED
  - ▷ form factor effects make ChPT calculation divergent = non-predictive



$\longrightarrow$ restore unitarity/incorporate resonances dispersively

match to ChPT at lowest energies

Warm up: pion vector form factor



$$\frac{1}{2i}\,\text{disc}\,F_\pi^V(s) = \text{Im}\,F_\pi^V(s) \;=\; F_\pi^V(s)\times\theta(s-4M_\pi^2)\times\sin\delta_1^1(s)\,e^{-i\delta_1^1(s)}$$

$\longrightarrow$ final-state theorem: phase of $F_\pi^V(s)$ is just $\delta_1^1(s)$ Watson 1954

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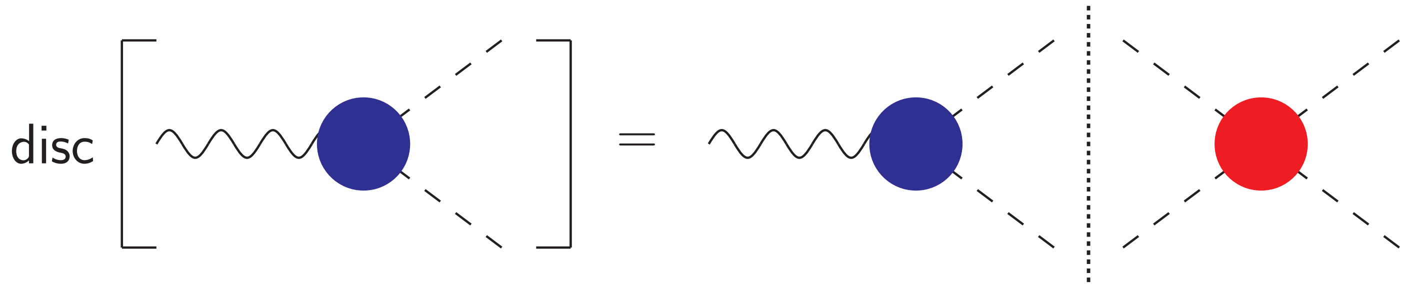

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- solution:

$$F_\pi^V(s) = P(s)\Omega(s)\;, \quad \Omega(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^\infty ds'\,\frac{\delta_1^1(s')}{s'(s'-s)}\right\}$$

$P(s)$ polynomial, $\Omega(s)$ Omnès function Omnès 1958

▷ $\pi\pi$ phase shifts from Roy equations

Ananthanarayan et al. 2001, García-Martín et al. 2011

▷ $P(0)=1$ from symmetries (gauge invariance)

- below 1 GeV: $F_\pi^V(s) \approx (1+0.1\,\text{GeV}^{-2}s)\Omega(s)$

slope due to inelastic resonances ρ', ρ''... Hanhart 2012

Form factors constrained by analyticity and unitarity

For illustration, let's briefly derive this solution!

- use $F_\pi^V(s) = P(s)\Omega(s)$: $\Omega(s)$ free of zeros, $\Omega(0) = 1$
- begin with the following simple manipulations:

$$\begin{aligned}
\text{disc}\,\Omega(s) &= 2i\,\Omega(s+i\epsilon) \times \sin\delta(s)\,e^{-i\delta(s)} \\
\Omega(s+i\epsilon) - \Omega(s-i\epsilon) &= \Omega(s+i\epsilon) \times (1 - e^{-2i\delta(s)}) \\
\Omega(s+i\epsilon) &= \Omega(s-i\epsilon) \times e^{2i\delta(s)} \\
\text{disc}\,\log\Omega(s) &= 2i\,\delta(s)
\end{aligned}$$

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\text{disc}\,\log\Omega(s) &= 2i\,\delta(s)
\end{aligned}$$

- this allows to write a dispersion relation for disc $\log\Omega(s)$:

$$\begin{aligned}
\log\Omega(s) &= \frac{s}{2\pi i}\int_{4M_\pi^2}^\infty ds' \frac{\text{disc}\,\log\Omega(s')}{s'(s'-s)} = \frac{s}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{\delta(s')}{s'(s'-s)} \\
\Omega(s) &= \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{\delta(s')}{s'(s'-s)}\right\}
\end{aligned}$$

Some homework for you: Omnès function, properties

1. Show

$$\arg \Omega(s) = \delta(s) .$$

2. Assume $\delta(s > s_0) = c \times \pi =$ const. (above some s_0).
   Demonstrate

$$\Omega(s \to \infty) \propto s^{-c} .$$

3. Assume the phase shift of an infinitely narrow resonance,

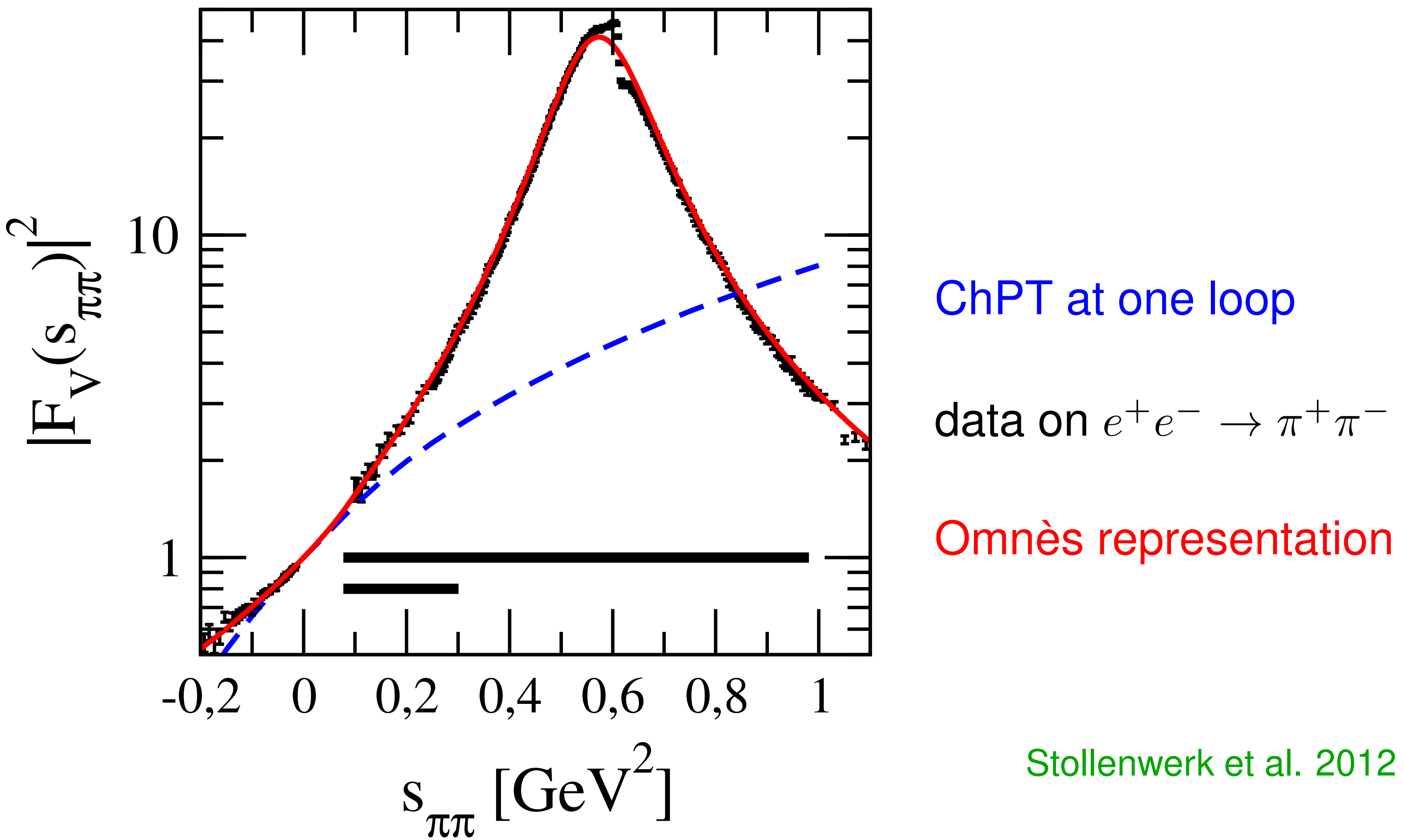
$$\delta(s) = \pi \times \theta(s - M_R^2) .$$

   What is the resulting Omnès function?

4. Look up a parameterisation of the $\pi\pi$ P-wave phase shift $\delta_1^1(s)$.
   e.g., García-Martín et al., `arXiv:1102.2183`

   [Above the maximum energy for which $\delta_1^1(s)$ is given, continue it smoothly towards π.] Calculate $\Omega_1^1(s)$ numerically!

Pion vector form factor (again)

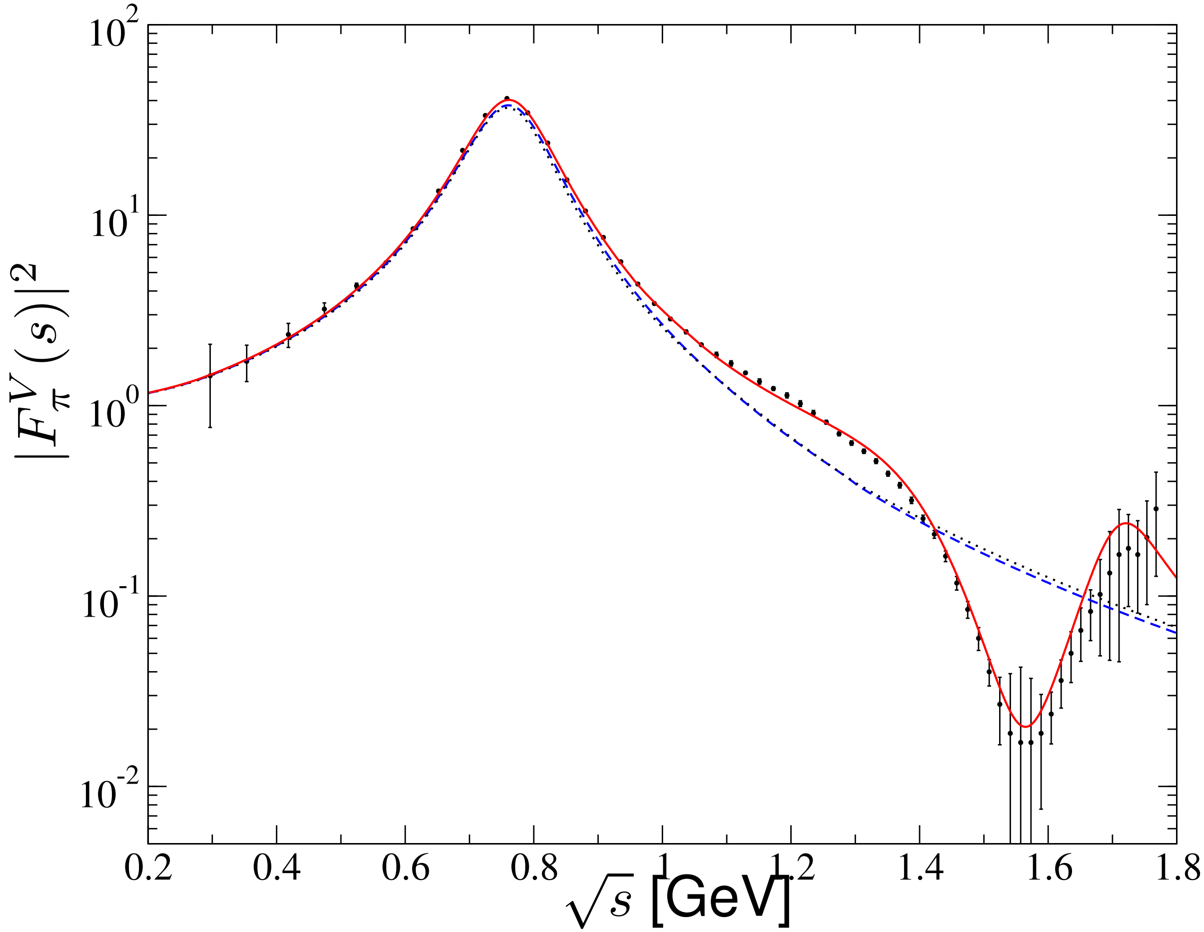


Stollenwerk et al. 2012

$\longrightarrow$ Omnès representation vastly extends range of applicability

Pion vector form factor vs. Omnès representation

Data on pion form factor in $\tau^- \to \pi^- \pi^0 \nu_\tau$ Belle 2008



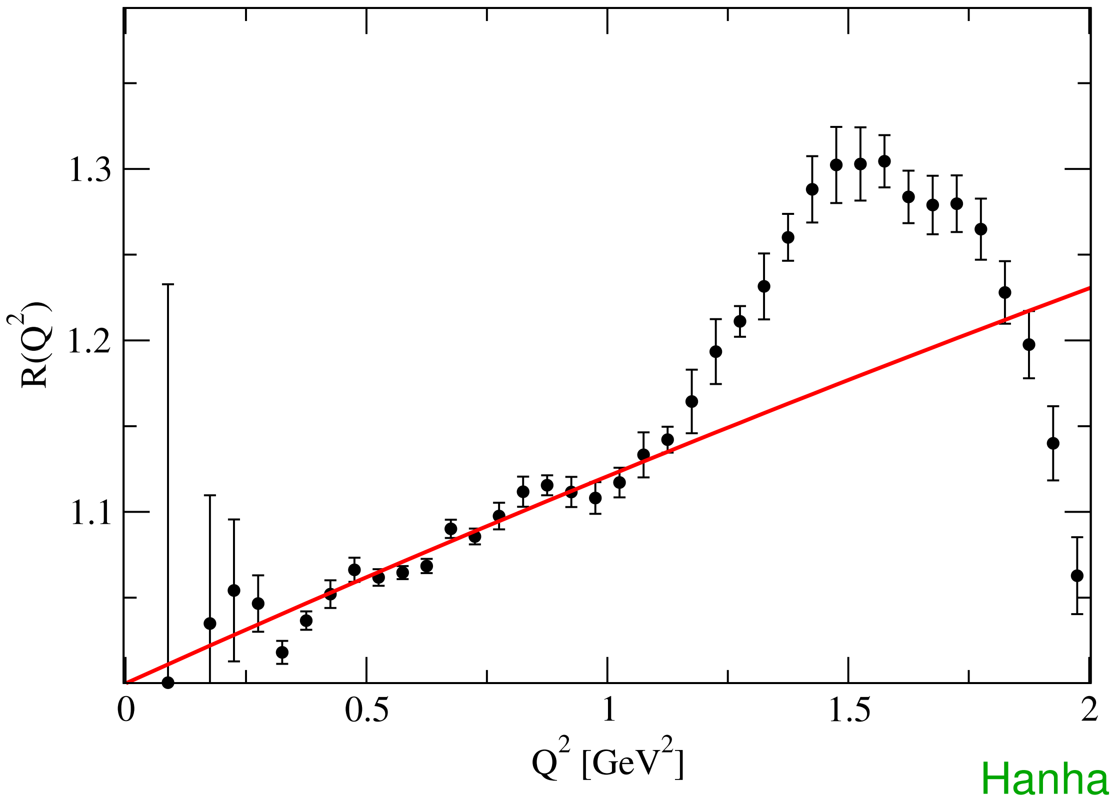
$\pi\pi$ P-wave phase shift / effective form factor phase incl. ρ', ρ''

Schneider et al. 2012

Pion vector form factor vs. Omnès representation

Data on pion form factor in $\tau^- \to \pi^-\pi^0\nu_\tau$ Belle 2008

- divide $\tau^- \to \pi^-\pi^0\nu_\tau$ form factor by Omnès function:



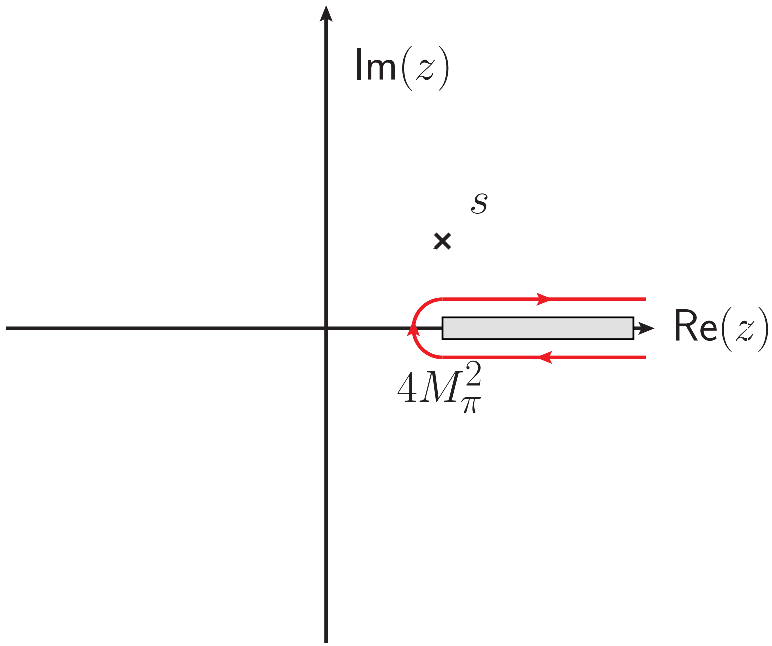
Hanhart et al. 2013

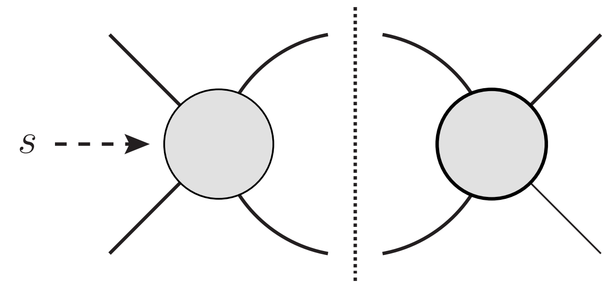
$\longrightarrow$ linear below 1 GeV: $F_\pi^V(s) \approx (1 + 0.1\,\text{GeV}^{-2}s)\Omega(s)$

$\longrightarrow$ above: inelastic resonances ρ', $\rho''\ldots$

What are left-hand cuts?

Example: pion–pion scattering





- right-hand cut due to unitarity: $s \geq 4M_\pi^2$

What are left-hand cuts?

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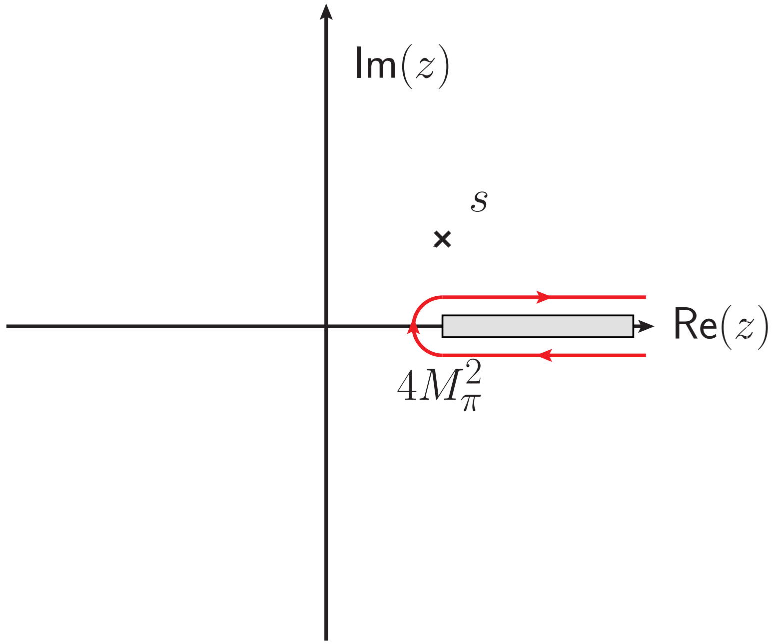

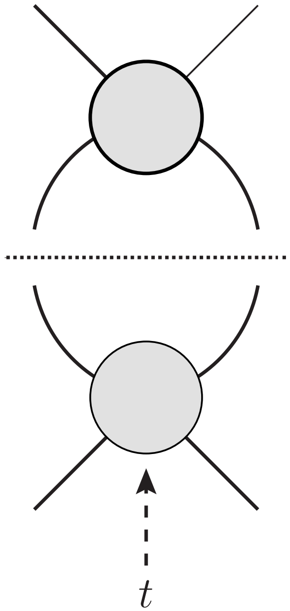

- right-hand cut due to unitarity: $s \geq 4M_\pi^2$
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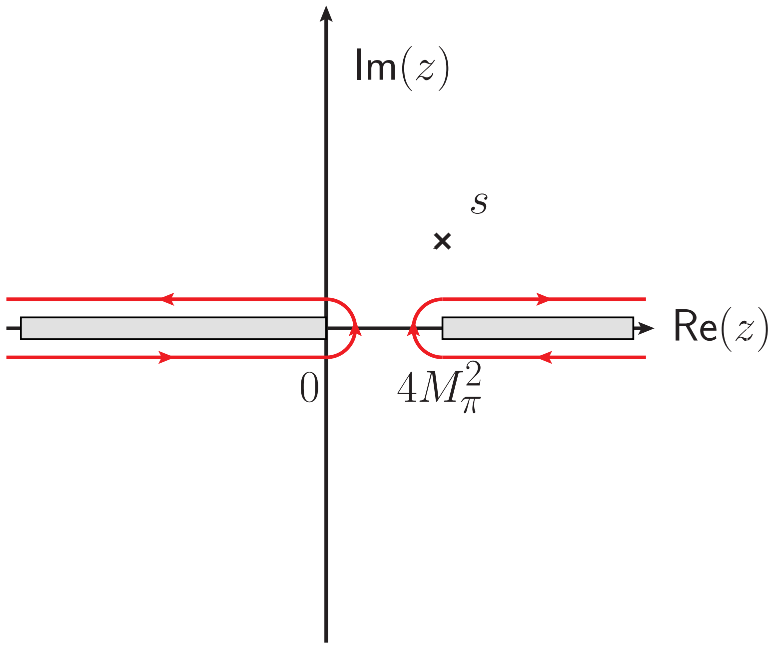

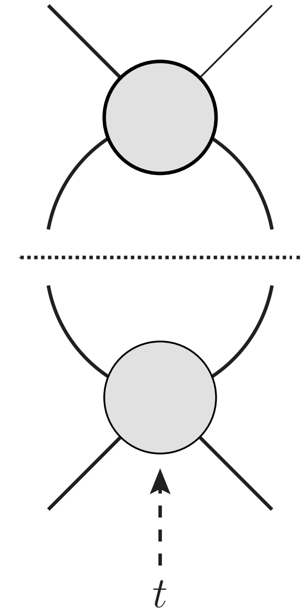

- right-hand cut due to unitarity: $s \geq 4M_\pi^2$
- crossing symmetry: cuts also for $t,\ u \geq 4M_\pi^2$
- partial-wave projection: $\quad T(s,t) = 32\pi \sum_i T_i(s) P_i(\cos\theta)$

$$t(s,\cos\theta) = \frac{1-\cos\theta}{2}(4M_\pi^2 - s)$$

$\longrightarrow$ cut for $t \geq 4M_\pi^2$ becomes cut for $s \leq 0$ in partial wave

$\pi\pi$ scattering constrained by analyticity and unitarity

Roy equations = coupled system of partial-wave dispersion relations
+ crossing symmetry + unitarity

- twice-subtracted fixed-t dispersion relation:

$$T(s,t) = c(t) + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} ds' \left\{ \frac{s^2}{s'^2(s'-s)} + \frac{u^2}{s'^2(s'-u)} \right\} \mathrm{Im}T(s',t)$$

- subtraction function $c(t)$ determined from crossing symmetry

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- subtraction function $c(t)$ determined from crossing symmetry
- project onto partial waves $t_J^I(s)$ (angular momentum J, isospin I) $\longrightarrow$ coupled system of partial-wave integral equations

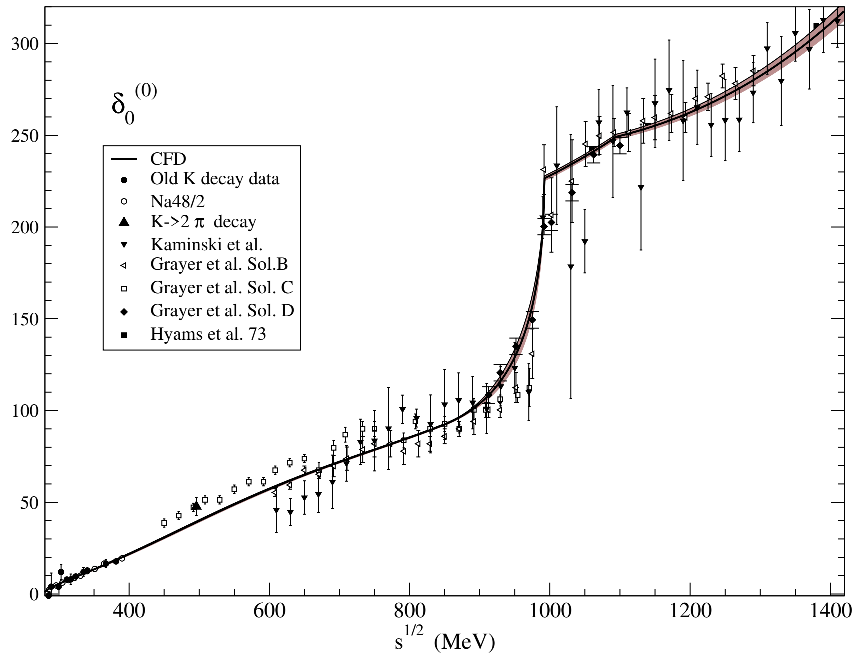
$$t_J^I(s) = k_J^I(s) + \sum_{I'=0}^{2}\sum_{J'=0}^{\infty}\int_{4M_\pi^2}^{\infty} ds' K_{JJ'}^{II'}(s,s')\mathrm{Im}t_{J'}^{I'}(s')$$

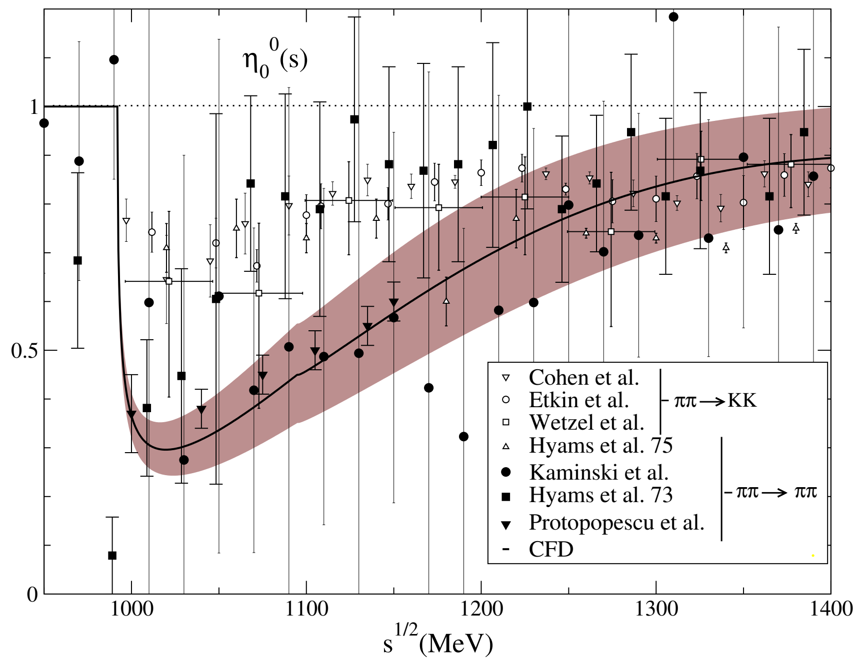
Roy 1971

- subtraction polynomial $k_J^I(s)$: $\pi\pi$ scattering lengths
  can be matched to chiral perturbation theory   Colangelo et al. 2001
- kernel functions $K_{JJ'}^{II'}(s,s')$ known analytically

$\pi\pi$ scattering constrained by analyticity and unitarity

- elastic unitarity $\longrightarrow$ coupled integral equations for phase shifts
- modern precision analyses:
  - ▷ $\pi\pi$ scattering — Ananthanarayan et al. 2001, García-Martín et al. 2011
  - ▷ πK scattering — Büttiker et al. 2004, Peláez, Rodas 2020
- example: $\pi\pi$ $I = 0$ S-wave phase shift & inelasticity



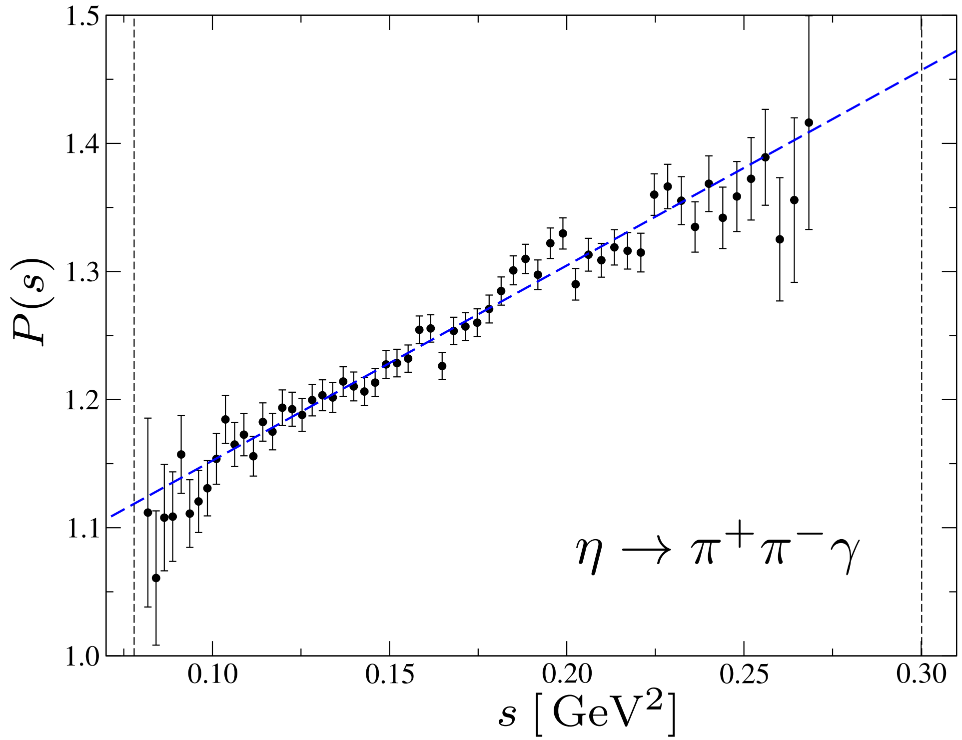


García-Martín et al. 2011

- strong constraints on data from analyticity and unitarity!

Final-state universality: $\eta,\ \eta' \to \pi^+\pi^-\gamma$

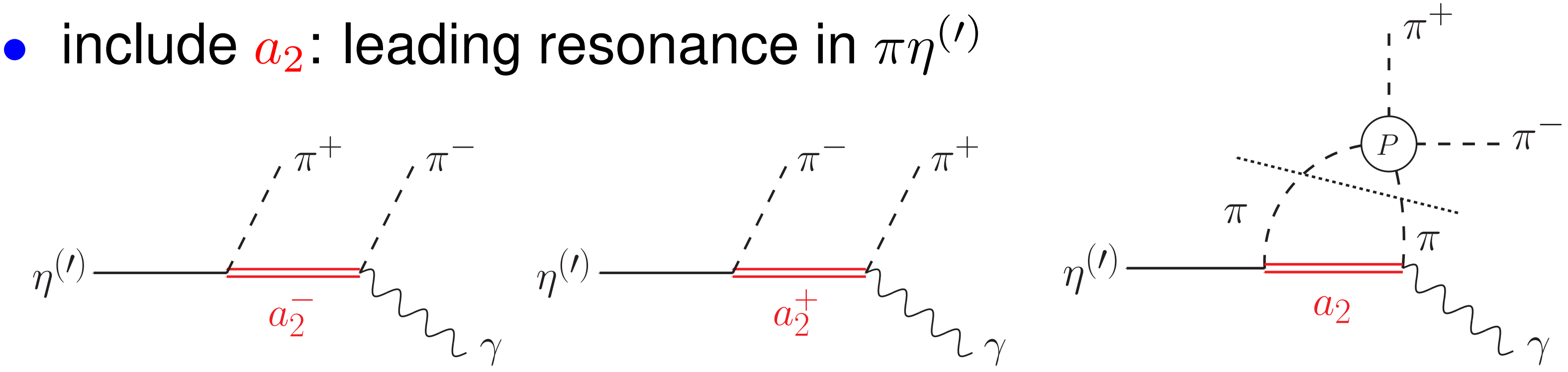
- $\eta^{(\prime)} \to \pi^+\pi^-\gamma$ driven by the chiral anomaly, $\pi^+\pi^-$ in P-wave
  $\longrightarrow$ final-state interactions the same as for vector form factor
- ansatz: $\mathcal{F}^{\eta^{(\prime)}}_{\pi\pi\gamma} = A \times P(s) \times \Omega(s),\ \ P(s) = 1 + \alpha^{(\prime)} s,\ \ s = M^2_{\pi\pi}$
- divide data by pion form factor $\longrightarrow P(s)$ Stollenwerk et al. 2012

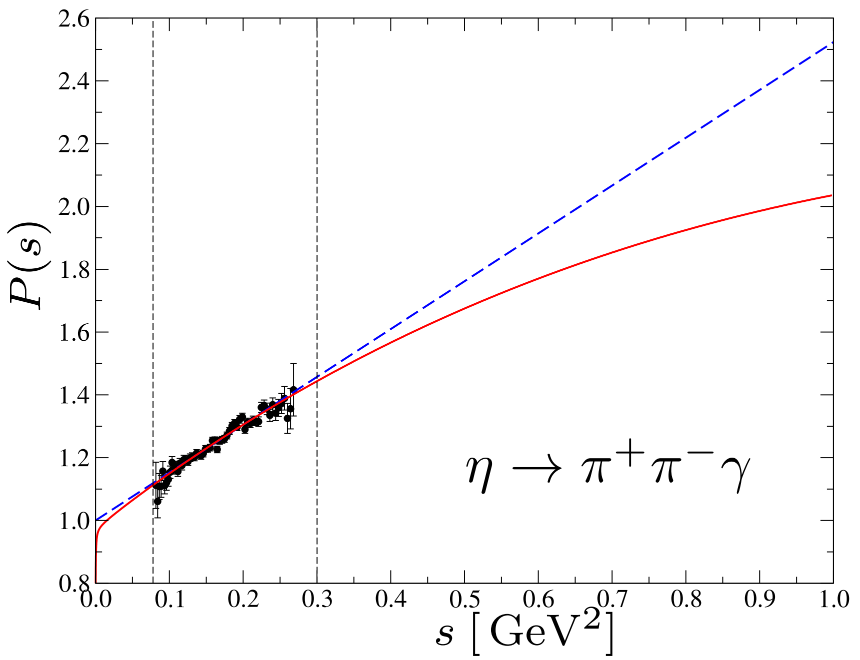


$\longrightarrow$ exp.: $\alpha_{\text{KLOE}} = (1.52 \pm 0.06)\ \text{GeV}^{-2}$ cf. KLOE 2013

$\eta,\ \eta' \to \pi^+\pi^-\gamma$ with left-hand cuts

- include a_2: leading resonance in $\pi\eta^{(\prime)}$





KLOE 2013; BK, Plenter 2015

- induces curvature in $P(s)$

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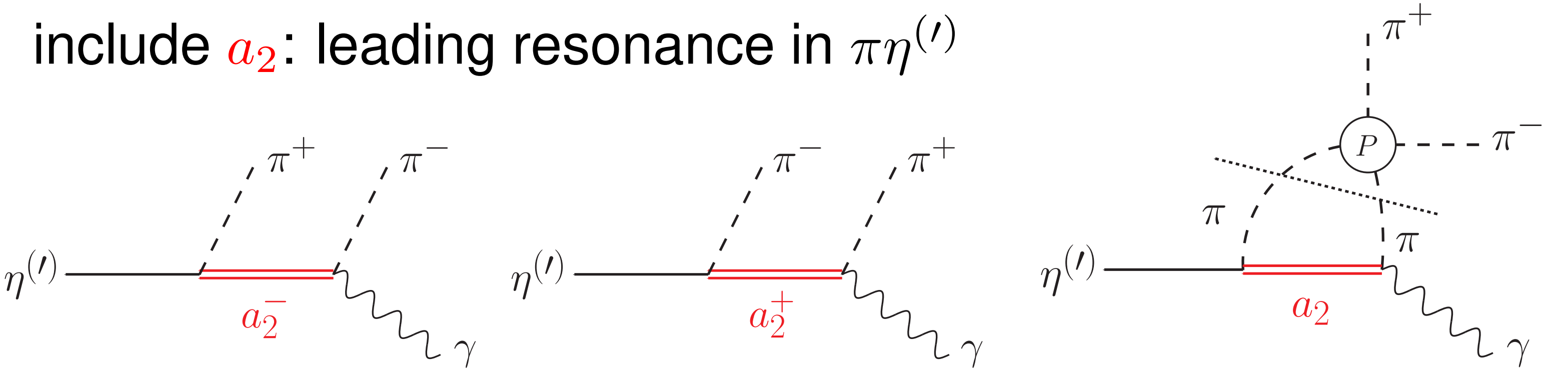

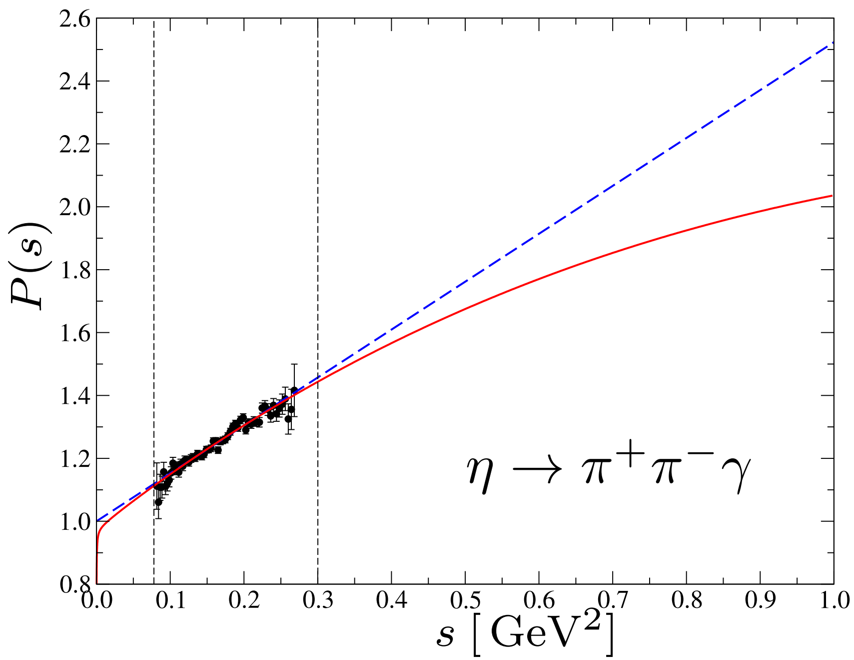

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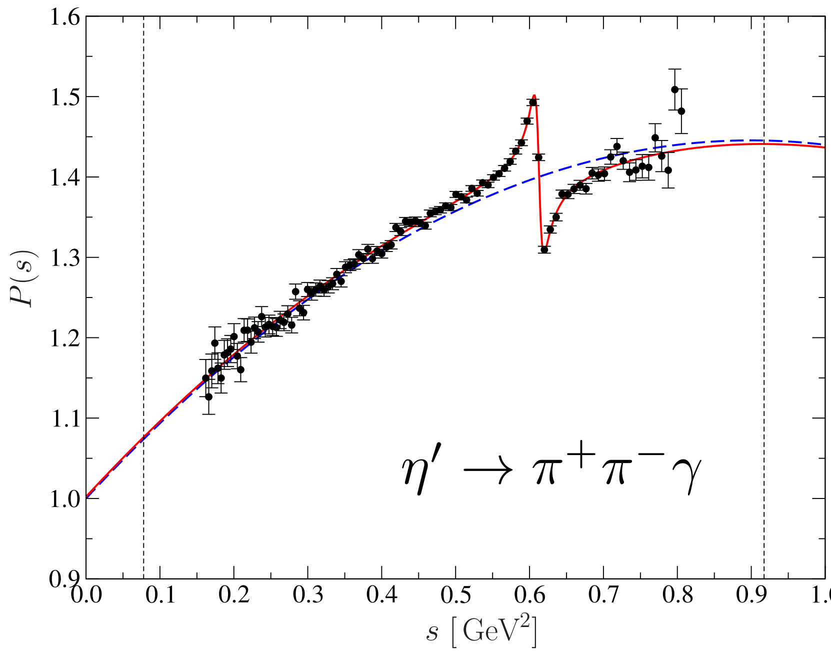


BESIII 2017; Hanhart et al. 2017

- induces curvature in $P(s)$
- curvature, plus ρ–ω mixing

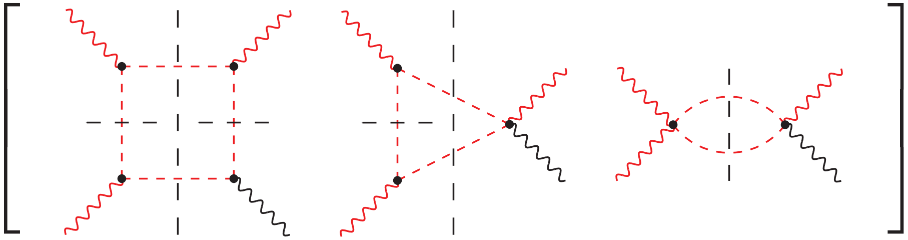
Pion loop contributions / $\pi\pi$ intermediate states

Colangelo, Hoferichter, Procura, Stoffer 2017 [figs. courtesy of M. Hoferichter]

Decompose light-by-light scattering tensor $\Pi_{\mu\nu\lambda\sigma}$ into

- form factor scalar QED part $\longrightarrow$ preserves gauge invariance

$$\Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} = F_\pi^V(q_1^2)F_\pi^V(q_2^2)F_\pi^V(q_3^2)\times$$



$$\longrightarrow \boxed{a_\mu^{\pi\text{-box}} = -15.9(2)\times 10^{-11}}$$

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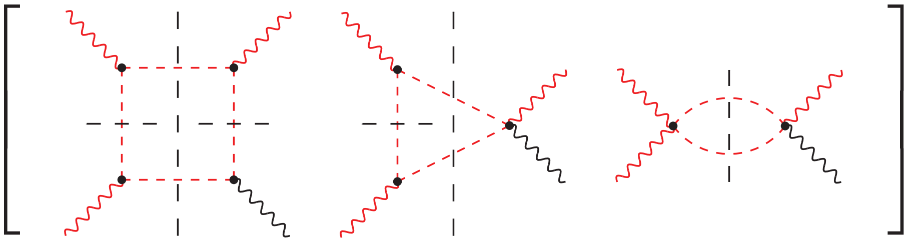
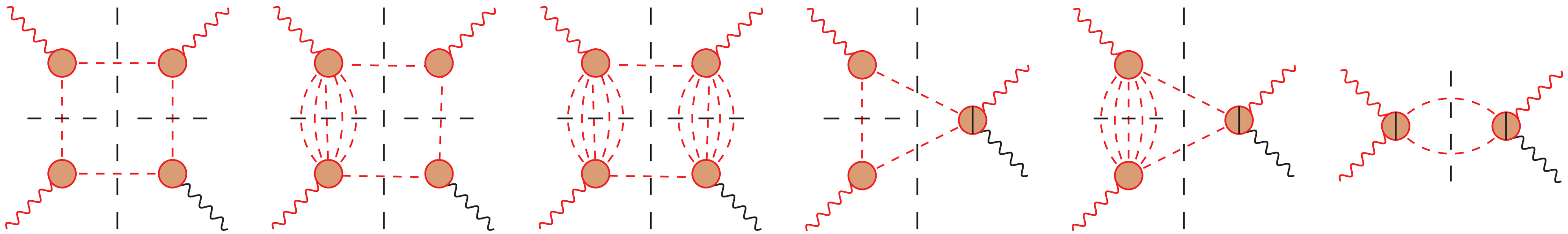


$$\longrightarrow \boxed{a_\mu^{\pi\text{-box}} = -15.9(2)\times 10^{-11}}$$

- + remainder $\bar{\Pi}_{\mu\nu\lambda\sigma}$ expanded in $\gamma^*\gamma^*\to\pi\pi$ helicity partial waves



organised according to left-hand-cut structure

S-wave rescattering $\longrightarrow \boxed{a_\mu^{\pi\pi,S\text{-wave}} = -8(1)\times 10^{-11}}$

Pion loop contributions / $\pi\pi$ intermediate states

Colangelo, Hoferichter, Procura, Stoffer 2017 [figs. courtesy of M. Hoferichter]

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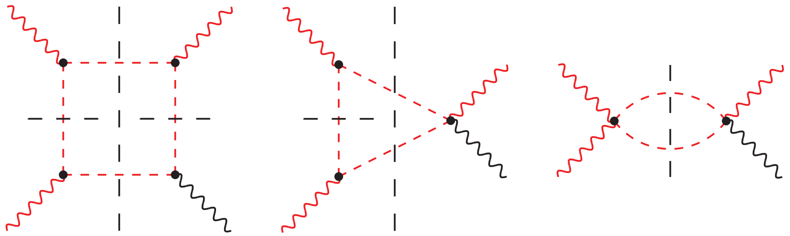
$$\Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} = F^V_\pi(q_1^2)F^V_\pi(q_2^2)F^V_\pi(q_3^2)\times \Big[$$  $$\Big]$$

$\longrightarrow$ $\boxed{a_\mu^{\pi\text{-box}} = -15.9(2)\times 10^{-11}}$

- + remainder $\bar{\Pi}_{\mu\nu\lambda\sigma}$ expanded in $\gamma^*\gamma^* \to \pi\pi$ helicity partial waves
- contains automatically
  - ▷ polarisability effects — Engel, Patel, Ramsey-Musolf 2012
  - ▷ $\pi\pi$ resonances: $f_0(500)$ [$f_2(1270)$]
  - ▷ can be extended to $K\bar{K}$ ($\longrightarrow f_0(980)$)

    Danilkin, Deineka, Vanderhaeghen 2019; Danilkin, Hoferichter, Stoffer 2021

Part II:

π^0 pole contribution

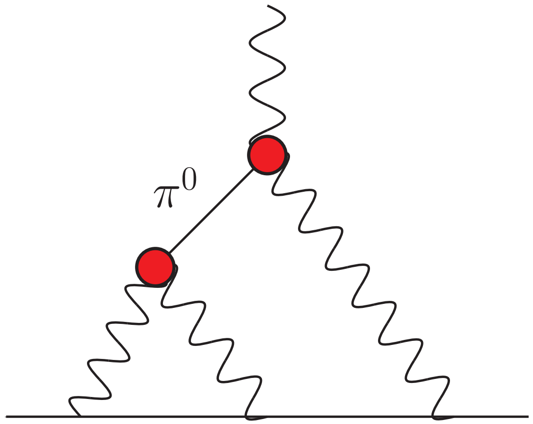
Hadronic light-by-light: the π^0 pole

- largest individual HLbL contribution:

  π^0 pole term

  singly / doubly virtual transition form factors (TFFs)

  $F_{\pi^0\gamma^*\gamma^*}(q^2, 0)$ and $F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$



- normalisation fixed by Wess–Zumino–Witten (WZW) anomaly (= full leading-order ChPT prediction):

$$F_{\pi^0\gamma^*\gamma^*}(0,0) = \frac{1}{4\pi^2 F_\pi}$$

  $\longrightarrow$ measured at 0.75% (F_π: pion decay constant) PrimEx 2020

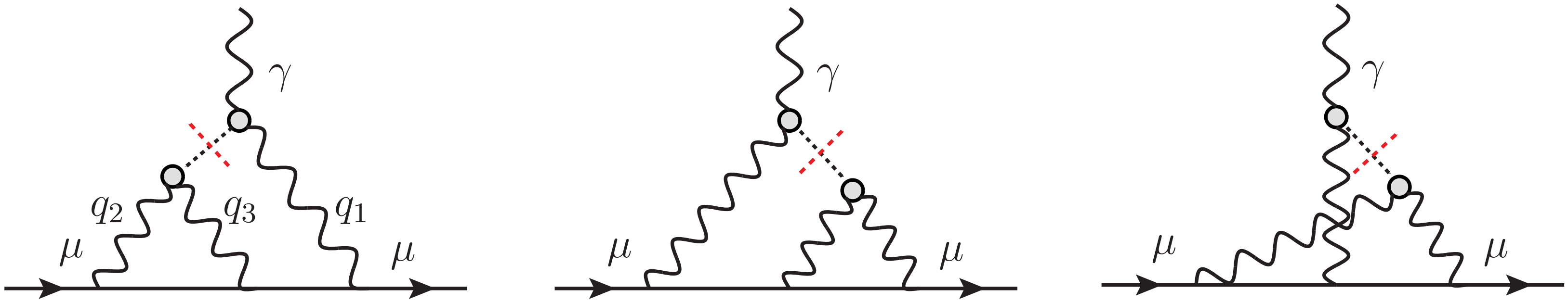
- two-loop integral with constant form factors does not converge

  $\longrightarrow$ no full prediction from e.g. chiral perturbation theory

  $\longrightarrow$ sensible high-energy behaviour required!

Pion-pole contribution to a_μ

- 3-dimensional integral representation: Jegerlehner, Nyffeler 2009

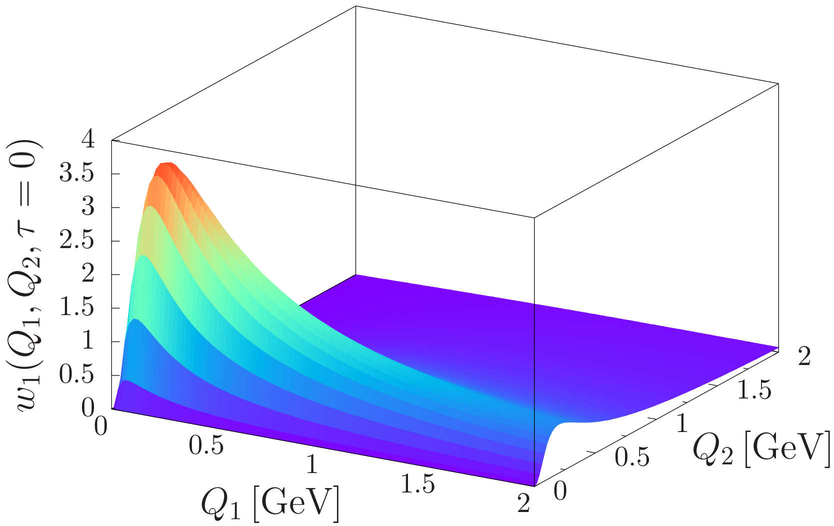


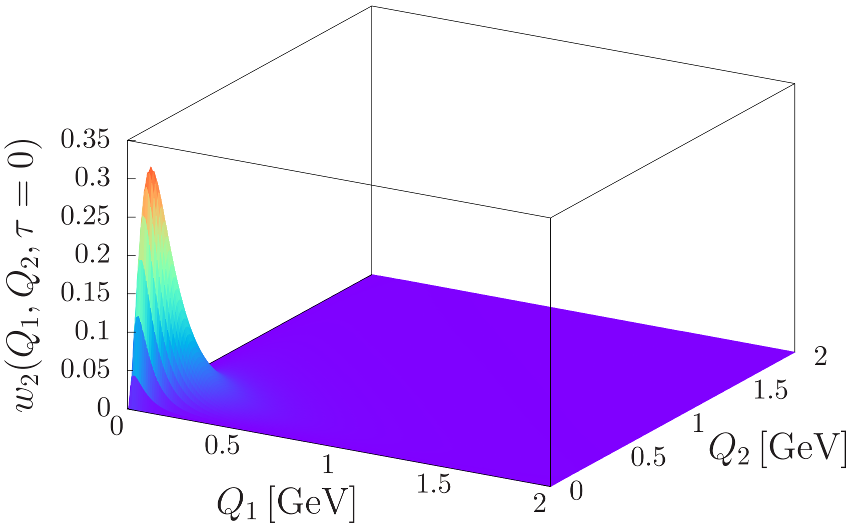
$$a_\mu^{\pi^0\text{-pole}} = \left(\frac{\alpha}{\pi}\right)^3 \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau$$
$$\times \Big[w_1(Q_1,Q_2,\tau)\, F_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_3^2)\, F_{\pi^0\gamma^*\gamma^*}(-Q_2^2,0)$$
$$+\, w_2(Q_1,Q_2,\tau)\, F_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)\, F_{\pi^0\gamma^*\gamma^*}(-Q_3^2,0)\Big]$$

- $w_{1/2}(Q_1,Q_2,\tau)$: kinematical weight functions, $\tau = \cos\theta$
- $F_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)$: space-like on-shell π^0 TFF

Pion-pole contribution to a_μ

- weight functions $w_{1/2}(Q_1, Q_2, \tau = 0)$: Nyffeler 2016





- concentrated for $Q_i \leq 0.5$ GeV

  $\longrightarrow$ pion-pole contribution dominantly from low-energy region

  $\longrightarrow$ pion transition form factor can be determined model-independently and with high precision using dispersion relations

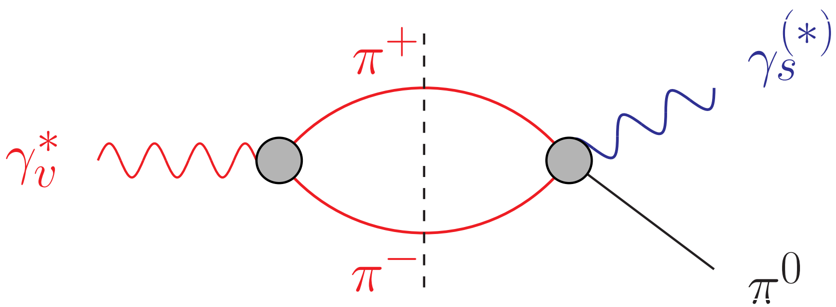
Dispersive analysis of $\pi^0 \to \gamma^*\gamma^*$

- isospin decomposition:

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = F_{vs}(q_1^2, q_2^2) + F_{vs}(q_2^2, q_1^2)$$

- analyse the leading hadronic intermediate states:

Hoferichter et al. 2014



▷ isovector photon: 2 pions

$\propto$ pion vector form factor — well known from $e^+e^- \to \pi^+\pi^-$

$\times$ $\gamma^* \to 3\pi$ P-wave amplitude — Khuri–Treiman formalism

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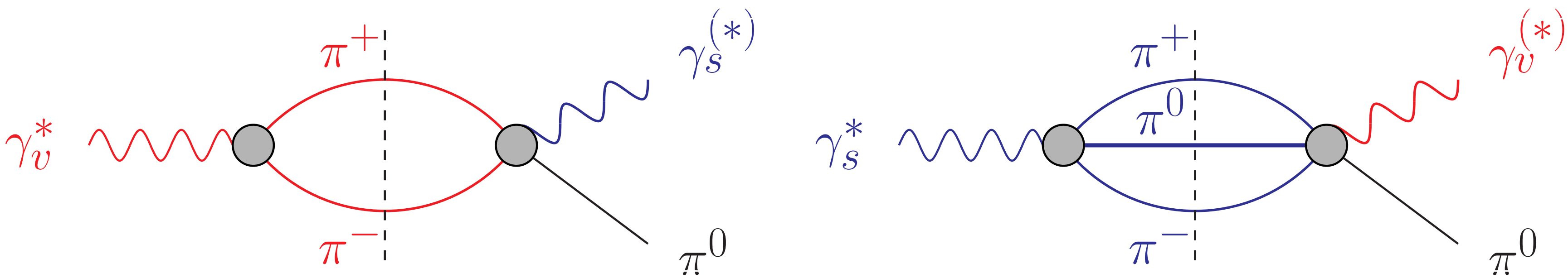

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▷ isoscalar photon: 3 pions

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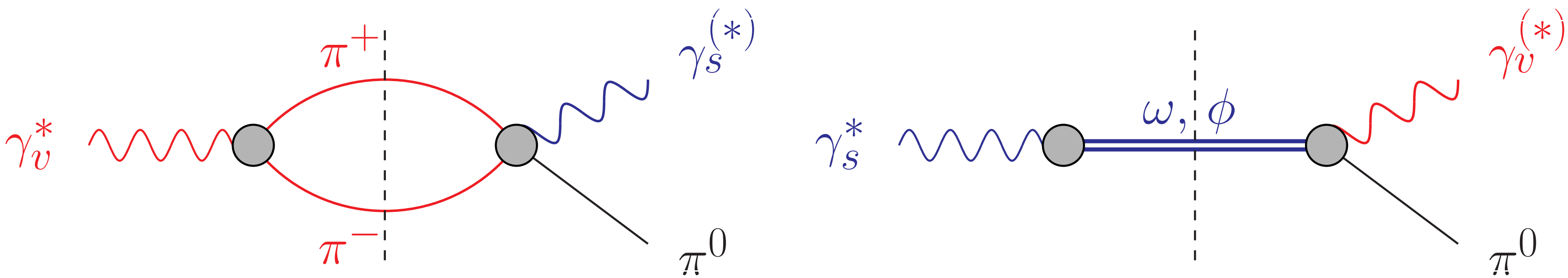

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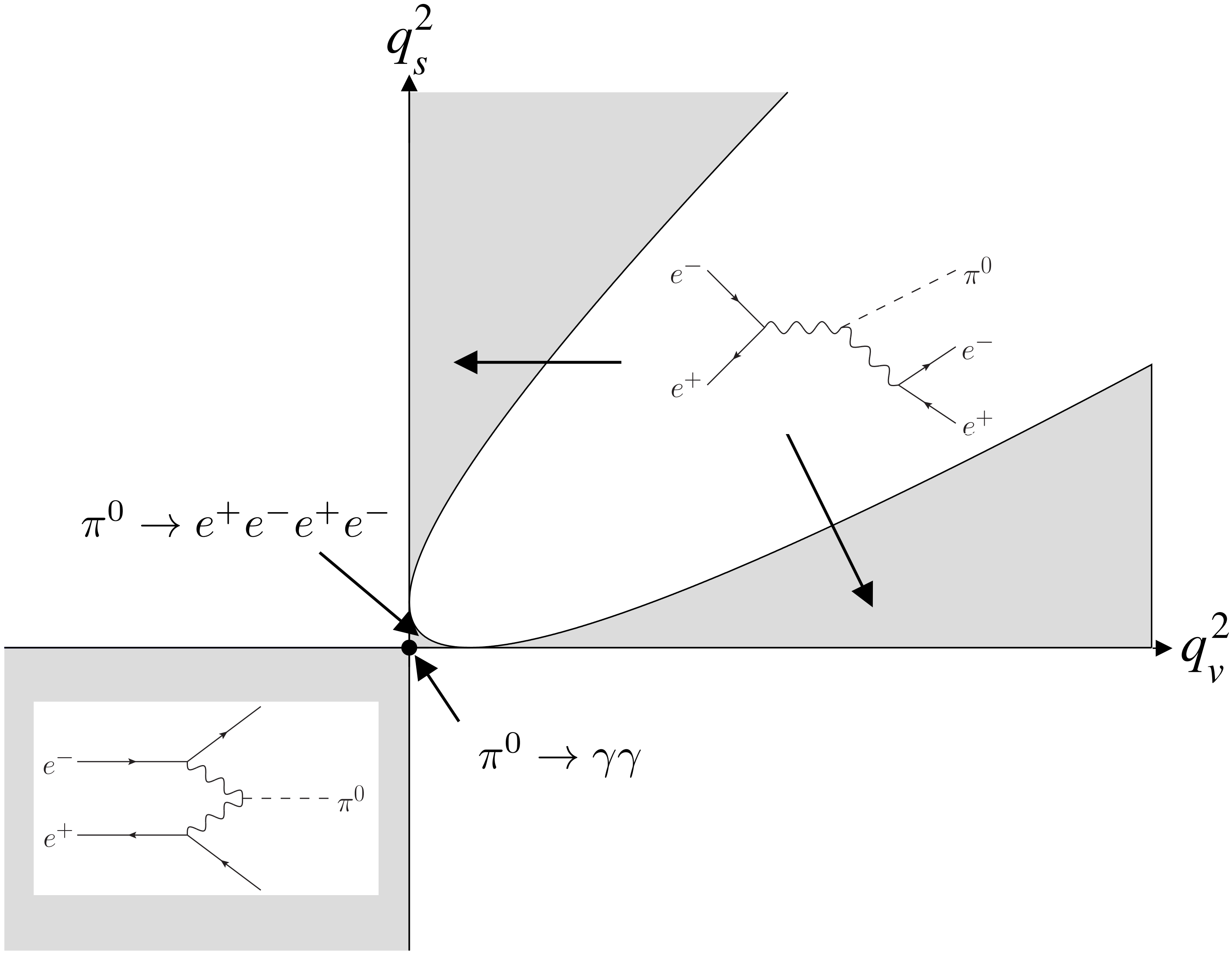
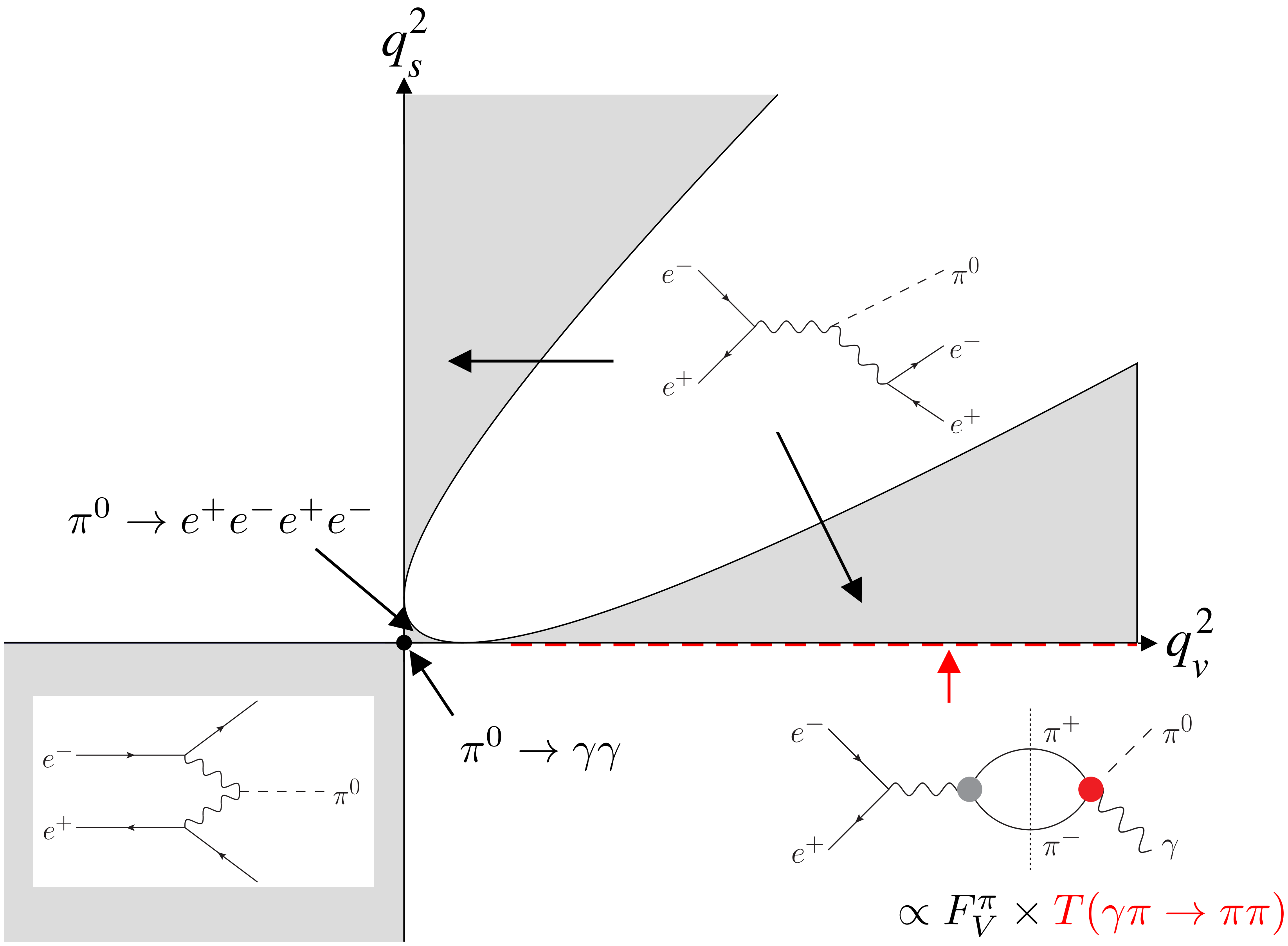
▷ isoscalar photon: 3 pions

dominated by narrow resonances ω, ϕ

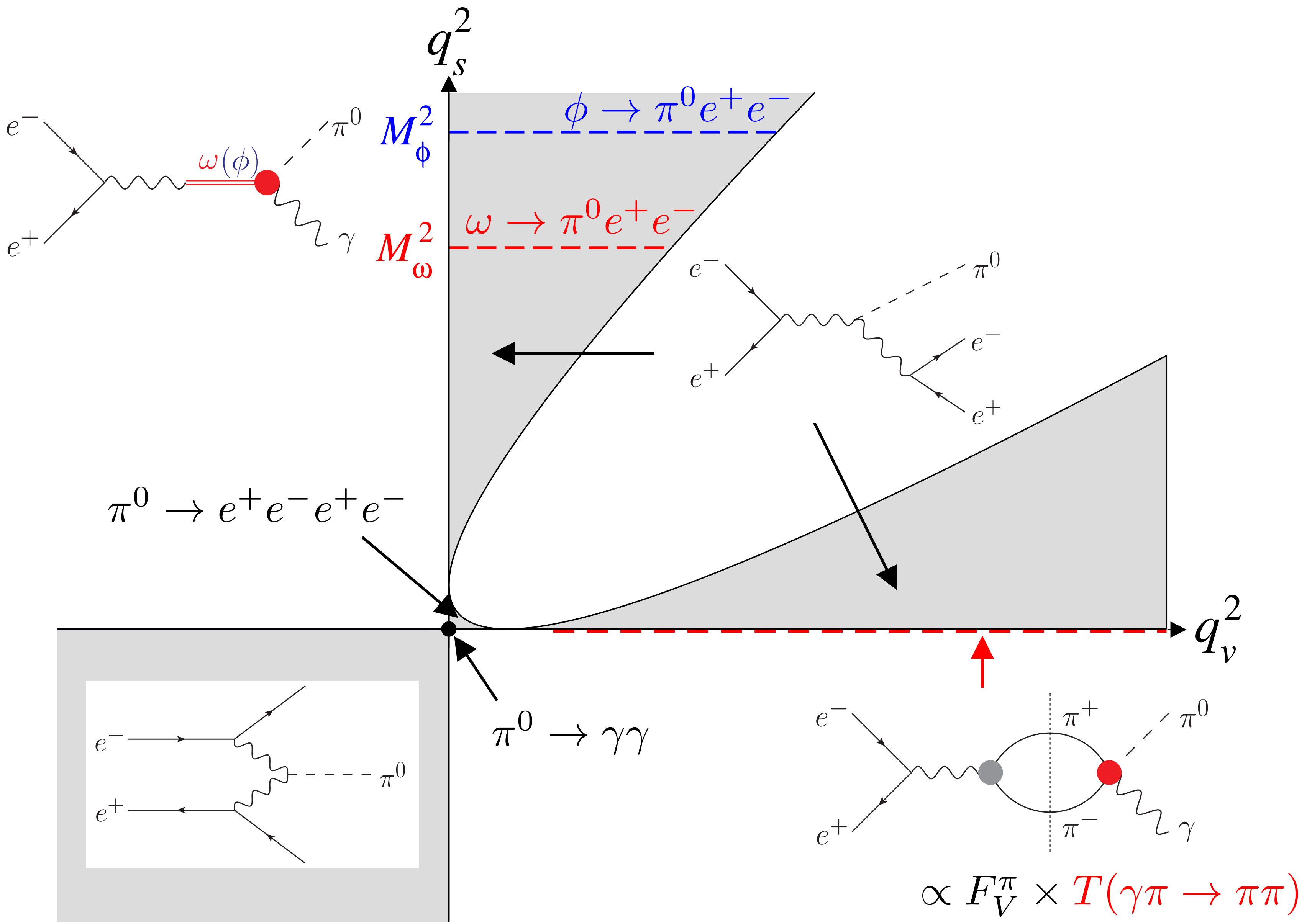
$\pi^0 \to \gamma^*\gamma^*$ transition form factor



$\pi^0 \to \gamma^*\gamma^*$ transition form factor



$\pi^0 \to \gamma^*\gamma^*$ transition form factor



Khuri–Treiman representation $\gamma^* \to 3\pi$

- $\gamma^*(q) \to \pi^+(p_+)\pi^-(p_-)\pi^0(p_0)$ amplitude:

$$\langle 0|j_\mu(0)|\pi^+(p_+)\pi^-(p_-)\pi^0(p_0)\rangle = -\epsilon_{\mu\nu\rho\sigma}\, p_+^\nu p_-^\rho p_0^\sigma\, \mathcal{F}(s,t,u;q^2)$$

  s,t,u: pion–pion invariant masses, $s+t+u = q^2 + 3M_\pi^2$

- "reconstruction theorem": neglect discontinuities in F-waves...
  $\longrightarrow$ decomposition into crossing-symmetric isobars

$$\mathcal{F}(s,t,u;q^2) = \mathcal{F}(s,q^2) + \mathcal{F}(t,q^2) + \mathcal{F}(u,q^2)$$

- normalisation fixed from Wess–Zumino–Witten anomaly:

$$\mathcal{F}(0,0,0;0) = F_{3\pi} = \frac{1}{4\pi^2 F_\pi^3}$$

- (s-channel) P-wave projection: $f_1(s,q^2) = \mathcal{F}(s,q^2) + \hat{\mathcal{F}}(s,q^2)$
  $\hat{\mathcal{F}}(s,q^2)$: contribution from crossed channels $\mathcal{F}(t/u,q^2)$

Khuri–Treiman representation $\gamma^* \to 3\pi$

Unitarity relation for $\mathcal{F}(s, q^2 = \text{fixed})$:

$$\text{disc}\,\mathcal{F}(s,q^2) = 2i\{\underbrace{\mathcal{F}(s,q^2)}_{\text{right-hand cut}} + \underbrace{\hat{\mathcal{F}}(s,q^2)}_{\text{left-hand cut}}\} \times \theta(s-4\,M_\pi^2) \times \sin\,\delta_1^1(s)\,e^{-i\delta_1^1(s)}$$

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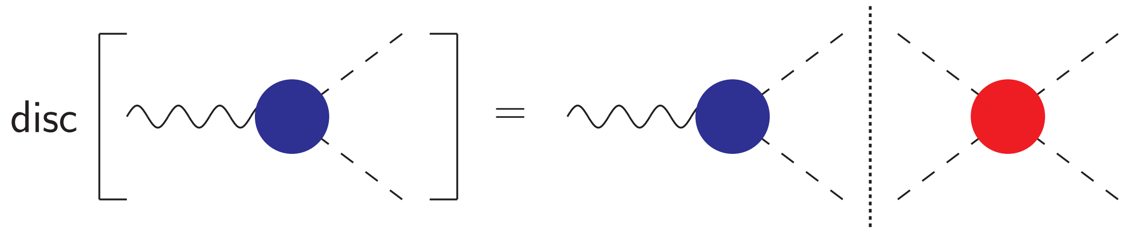
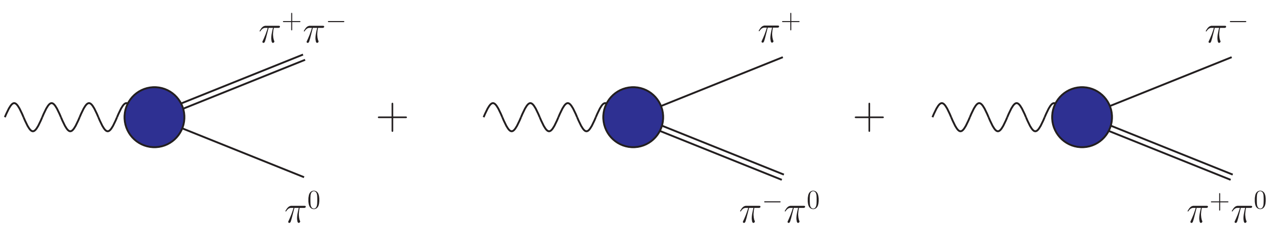


- right-hand cut only $\longrightarrow$ Omnès problem

$$\mathcal{F}(s,q^2) = \Omega(s)\, a(q^2)\,, \qquad \Omega(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\delta_1^1(s')}{s'-s}\right\}$$

$\longrightarrow$ amplitude given in terms of pion vector form factor

$\mathcal{F}(s,t,u;q^2) =$



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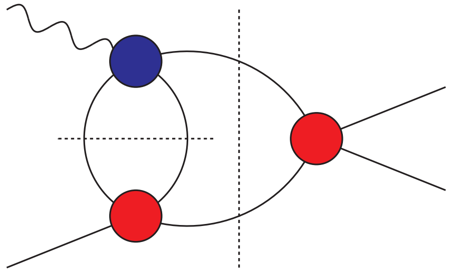
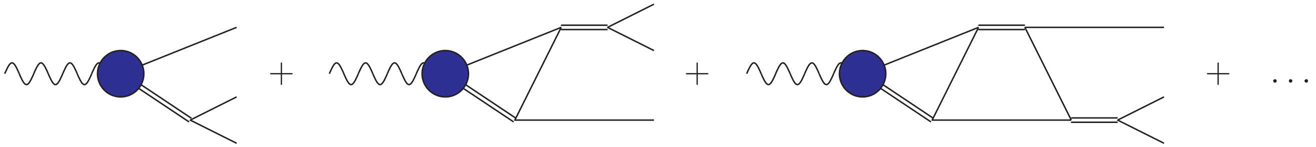


- inhomogeneities $\hat{\mathcal{F}}(s,q^2)$: angular averages over the $\mathcal{F}(t)$, $\mathcal{F}(u)$

$$\mathcal{F}(s,q^2) = \Omega(s)\left\{a(q^2) + \frac{s^2}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta_1^1(s')\hat{\mathcal{F}}(s',q^2)}{|\Omega(s')|(s'-s)}\right\}$$

$$\hat{\mathcal{F}}(s,q^2) = \frac{3}{2}\int_{-1}^{1} dz\,(1-z^2)\mathcal{F}\big(t(s,z),q^2\big)$$

$\mathcal{F}(s,q^2) =$ 

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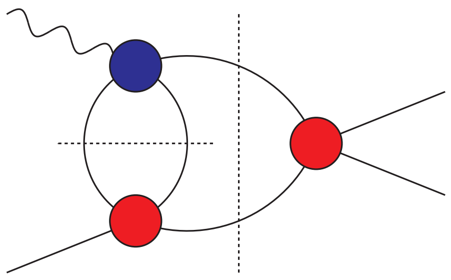

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$$\hat{\mathcal{F}}(s,q^2) = \frac{3}{2}\int_{-1}^1 dz\,(1-z^2)\mathcal{F}\big(t(s,z),q^2\big)$$

- crossed-channel scatt. between s-, t-, u-channel (left-hand cuts)

Dispersive representation $\gamma^* \to 3\pi$

- parameterisation of subtraction function $a(q^2)$

  $\longrightarrow$ to be fitted to $e^+e^- \to 3\pi$ cross section data:

$$a(q^2) = \frac{F_{3\pi}}{3} + \frac{q^2}{\pi}\int_{\rm thr}^{\infty} ds' \frac{\operatorname{Im}\mathcal{A}(s')}{s'(s'-q^2)} + C_n(q^2)$$

- $\mathcal{A}(q^2)$ includes resonance poles:

$$\mathcal{A}(q^2) = \sum_V \frac{c_V}{M_V^2 - q^2 - i\sqrt{q^2}\Gamma_V(q^2)} \qquad V = \omega, \phi, \omega', \omega''$$

  c_V real

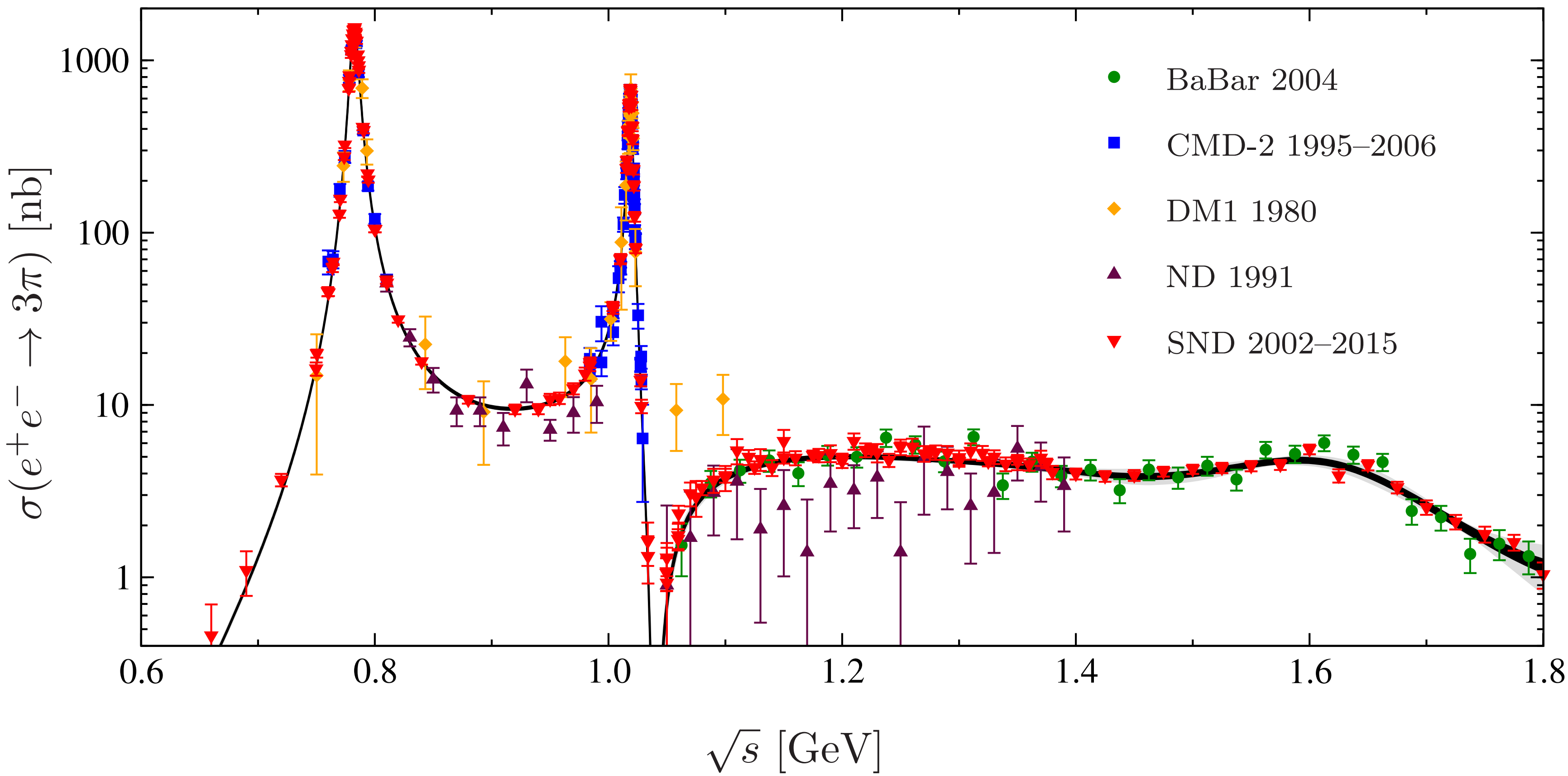
- conformal polynomial (inelasticities)

$$C_n(q^2) = \sum_{i=1}^{n} c_i \Big(z(q^2)^i - z(0)^i\Big), \qquad z(q^2) = \frac{\sqrt{s_{\text{inel}} - s_1} - \sqrt{s_{\text{inel}} - q^2}}{\sqrt{s_{\text{inel}} - s_1} + \sqrt{s_{\text{inel}} - q^2}}$$

- exact implementation of $\gamma^* \to 3\pi$ anomaly:

$$\frac{F_{3\pi}}{3} = \frac{1}{\pi}\int_{s_{\text{thr}}}^{\infty} ds' \frac{\operatorname{Im} a(s')}{s'}$$

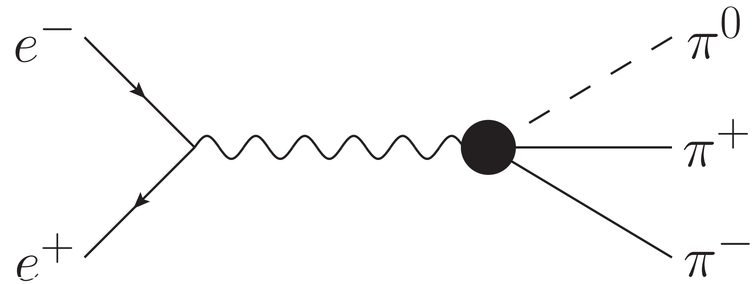
Fit results $e^+e^- \rightarrow 3\pi$ data up to 1.8 GeV



Hoferichter, Hoid, BK 2019

- black / gray bands represent fit and total uncertainties
- vacuum polarisation removed from the cross section

From $e^+e^- \to 3\pi$ to $e^+e^- \to \pi^0\gamma^*$

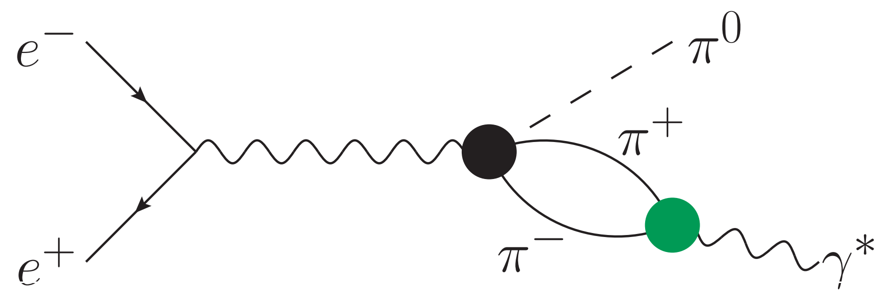


- amplitude for $e^+e^- \to 3\pi \quad \propto \; \mathcal{F}(s,q^2) + \mathcal{F}(t,q^2) + \mathcal{F}(u,q^2)$

$$\mathcal{F}(s,q^2) = \Omega(s)\left\{a(q^2) + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\sin\delta_1^1(s')\hat{\mathcal{F}}(s',q^2)}{|\Omega(s')|(s'-s)}\right\}$$

subtraction function $a(q^2)$ adjusted to reproduce $e^+e^- \to 3\pi$

From $e^+e^- \to 3\pi$ to $e^+e^- \to \pi^0\gamma^*$



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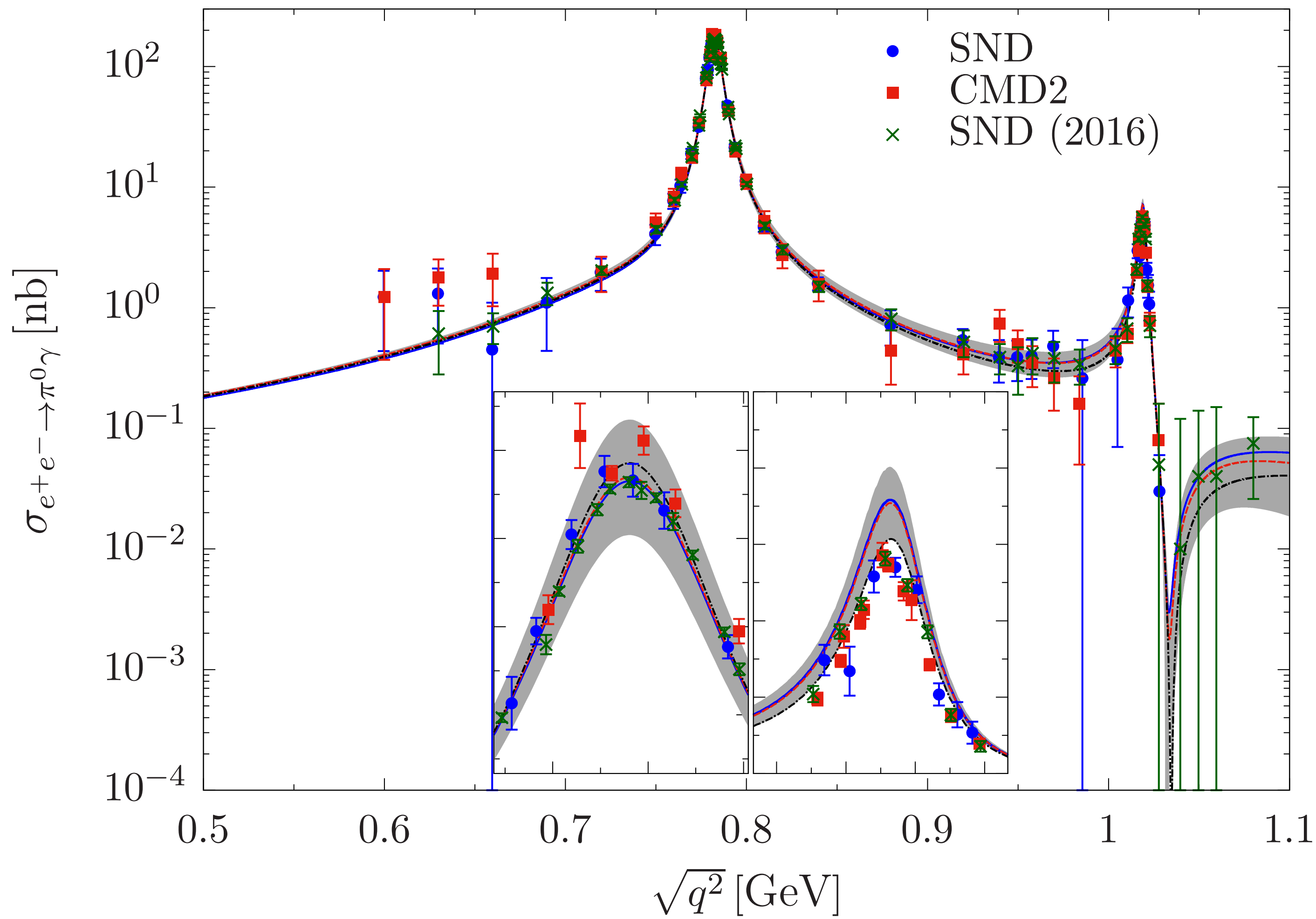
  subtraction function $a(q^2)$ adjusted to reproduce $e^+e^- \to 3\pi$

- fit to $e^+e^- \to 3\pi$ data

  combine with $e^+e^- \to \pi^+\pi^-$ form factor

  $\longrightarrow$ prediction for $e^+e^- \to \pi^0\gamma^{(*)}$

Comparison to $e^+e^- \to \pi^0\gamma$ data



Hoferichter, Hoid, BK, Leupold, Schneider 2018

- "prediction"—no further parameters adjusted
- timelike π^0 transition form factor data very well reproduced

Asymptotics and pQCD constraints (1)

- so far: dispersion relation based on (dominant) 2π, 3π
  $\longrightarrow$ high precision at low energies
- double-spectral-function representation:

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{1}{\pi^2}\int_{4M_\pi^2}^{\infty} dx \int_{s_{\rm thr}}^{\infty} dy \frac{\rho^{\rm disp}(x,y)}{(x-q_1^2)(y-q_2^2)}$$

$$\rho^{\rm disp}(x,y) = \frac{q_\pi^3(x)}{12\pi\sqrt{x}} {\rm Im}\left[F_\pi^{V*}(x) f_1(x,y)\right] + [x \leftrightarrow y]$$

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- asymptotically: pion wave function $\phi_\pi(x) = 6x(1-x) + \ldots$

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = -\frac{2F_\pi}{3}\int_0^1 dx \frac{\phi_\pi(x)}{xq_1^2+(1-x)q_2^2} + \mathcal{O}\big(Q^{-4}\big)$$

  implies asymptotically — Brodsky, Lepage 1979–1981

$$F_{\pi^0\gamma^*\gamma^*}\big(-Q^2,-Q^2\big) \sim \frac{2F_\pi}{3Q^2}\,, \qquad F_{\pi^0\gamma^*\gamma^*}\big(-Q^2,0\big) \sim \frac{2F_\pi}{Q^2}$$

  $\longrightarrow$ rewrite this as double-spectral representation $\rho^{\rm pQCD}(x,y)$

  Khodjamirian 1999; Hoferichter et al. 2018

Asymptotics and pQCD constraints (2)

- dispersion-theoretical $\rho^{\rm disp}(x,y)$ at low energies $x,y \leq s_m$
- doubly-asymptotic $\rho^{\rm pQCD}(x,y)$ for $x,y > s_m$
  $\longrightarrow$ does not contribute to singly-virtual TFF

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = \frac{1}{\pi^2}\int_0^{s_{\rm m}} dx \int_0^{s_{\rm m}} dy \frac{\rho^{\rm disp}(x,y)}{(x-q_1^2)(y-q_2^2)} + \frac{1}{\pi^2}\int_{s_{\rm m}}^{\infty} dx \int_{s_{\rm m}}^{\infty} dy \frac{\rho^{\rm pQCD}(x,y)}{(x-q_1^2)(y-q_2^2)}$$

- pQCD piece alone: $\quad F_{\pi^0\gamma^*\gamma^*}\big(-Q^2,-Q^2\big) = \dfrac{2F_\pi}{3Q^2} + \mathcal{O}\big(Q^{-4}\big)$

  dispersive part: $\quad \dfrac{1}{\pi^2}\displaystyle\int_0^{s_{\rm m}} dx \int_0^{s_{\rm m}} dy \frac{\rho^{\rm disp}(x,y)}{(x+Q^2)(y+Q^2)} = \mathcal{O}\big(Q^{-4}\big)$

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- anomaly and Brodsky–Lepage: $\rho^{\text{disp}}(x,y)$ fulfils two sum rules

  $\longrightarrow$ add effective pole: $\rho^{\text{eff}} = \dfrac{g_{\text{eff}}}{4\pi^2 F_\pi}\pi^2 M_{\text{eff}}^4 \delta\big(x-M_{\text{eff}}^2\big)\delta\big(y-M_{\text{eff}}^2\big)$

  find $g_{\text{eff}} \sim 10\%$ (small), $M_{\text{eff}} \sim 1.5\ldots 2.0\,\text{GeV}$ (reasonable)

Uncertainties in the π^0 pole contribution

Normalisation

- uncertainty on $\pi^0 \to \gamma\gamma$ $\pm 1.5\%$ — PrimEx 2020

Dispersive input

- different $\pi\pi$ phase shift inputs:
  - ▷ Bern vs. Madrid — Colangelo et al. 2011, García-Martín et al. 2011
  - ▷ effective form factor phase (incl. ρ', ρ'') — Schneider et al. 2012
- cutoff in Khuri–Treiman integrals $1.8 \ldots 2.5\,\mathrm{GeV}$

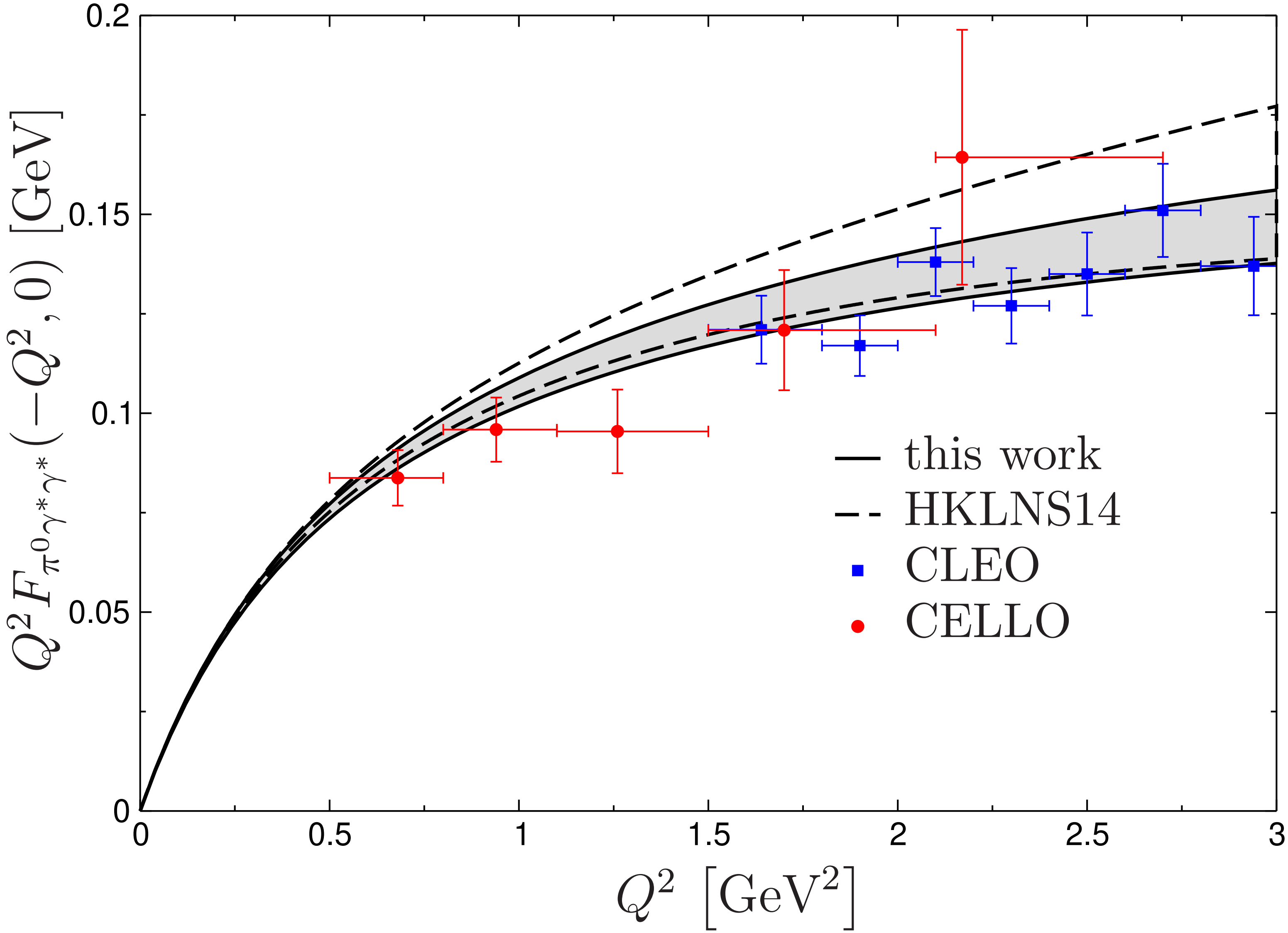
Brodsky–Lepage limit uncertainty

- allow for ${}^{+20\%}_{-10\%}$, 3σ band around data — BaBar 2009, Belle 2012

Onset of pQCD asymptotics

- vary $s_m = 1.7(3)$GeV^2

Results: singly-virtual



Hoferichter, Hoid, BK, Leupold, Schneider 2018

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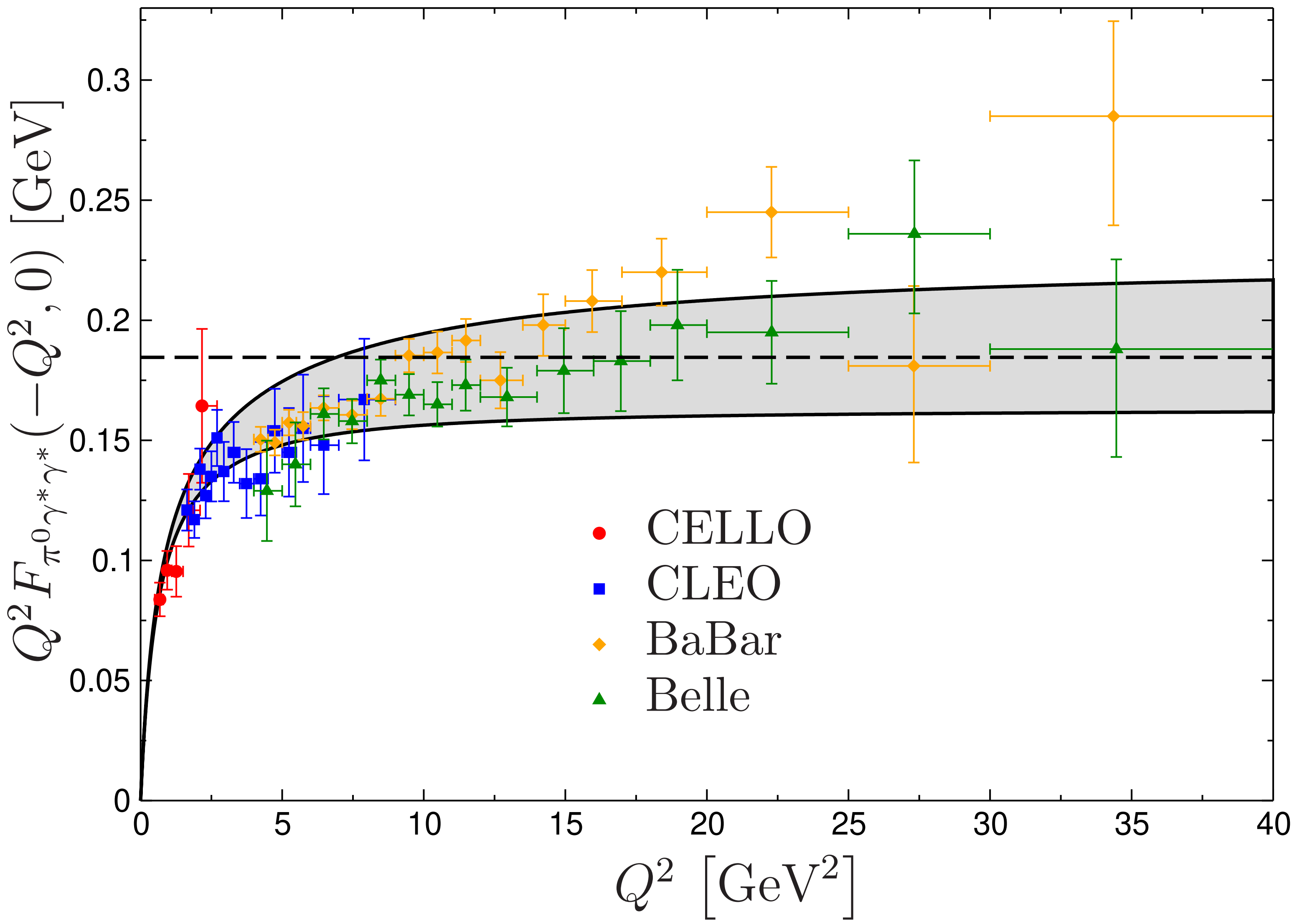
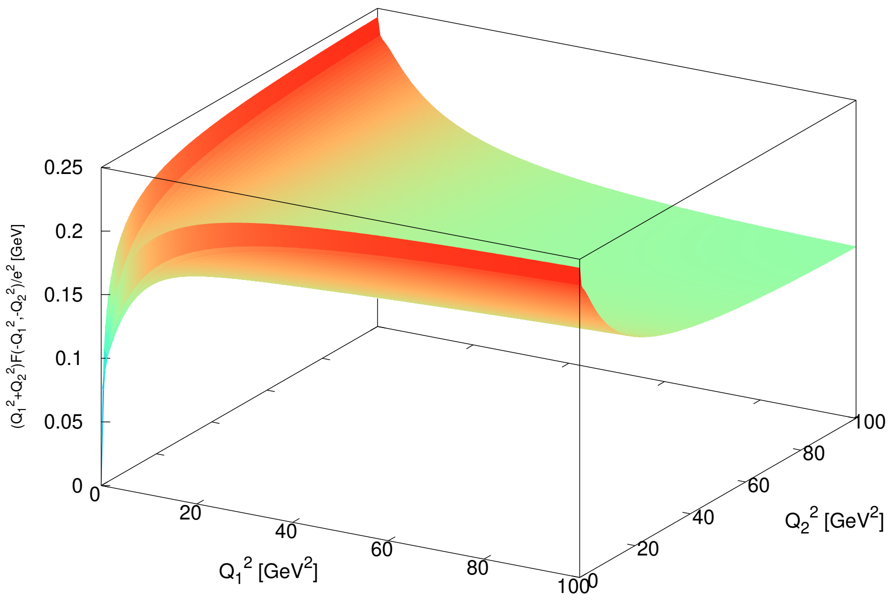


Hoferichter, Hoid, BK, Leupold, Schneider 2018

Comparison dispersive vs. pole models

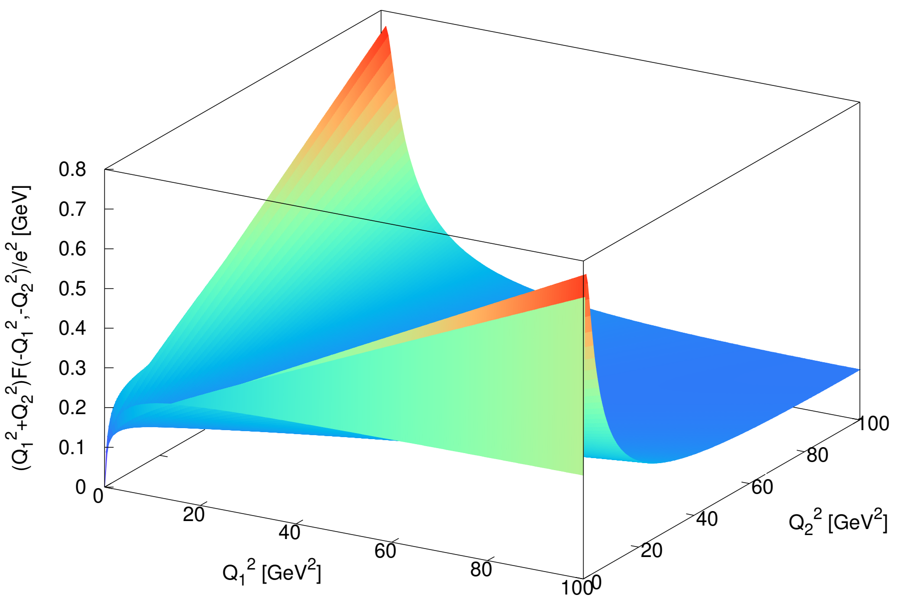
- plot $\big(Q_1^2+Q_2^2\big)F_{\pi^0\gamma^*\gamma^*}\big(-Q_1^2,-Q_2^2\big)$:

dispersive



Hoferichter, Hoid, BK,
Leupold, Schneider 2018

LMD+V fit to lattice



Gérardin, Meyer, Nyffeler 2016

Result: $(g-2)_\mu$ from π^0 pole

Final result for the π^0 pole contribution $[10^{-11}]$

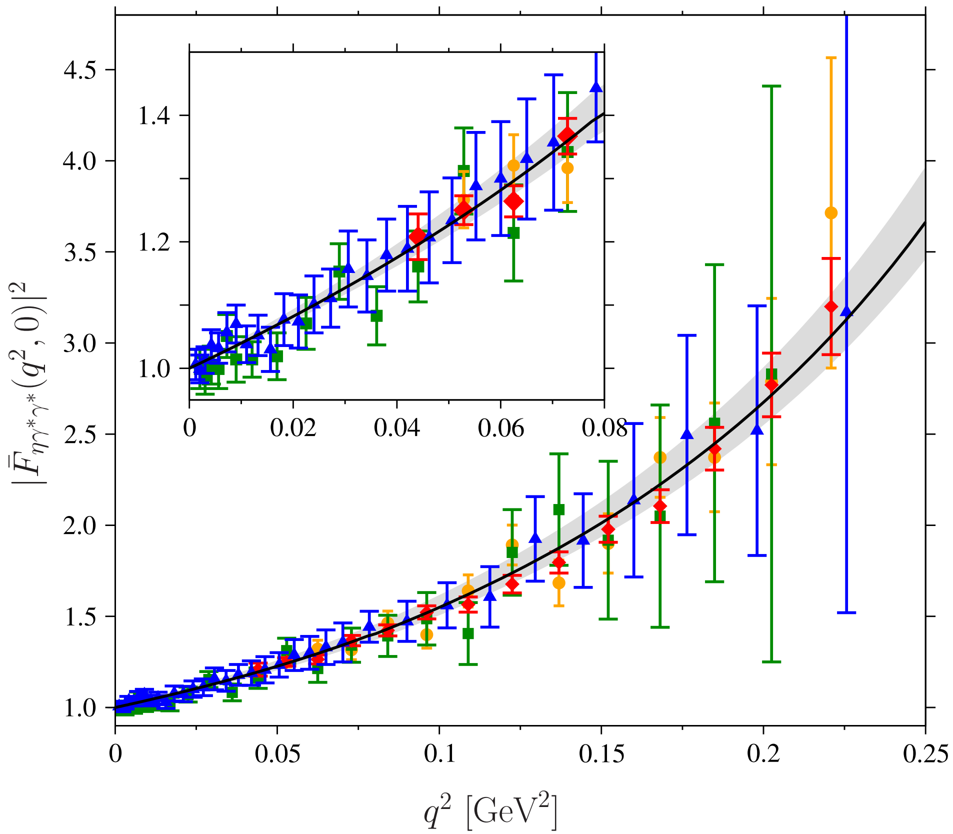
63.0 ± 0.9	chiral anomaly / $\pi^0 \to \gamma\gamma$
± 1.1	dispersive input
${}^{+2.2}_{-1.4}$	Brodsky–Lepage
± 0.6	onset of pQCD contribution s_m
$= 63.0\,{}^{+2.7}_{-2.1}$	Hoferichter, Hoid, BK, Leupold, Schneider 2018

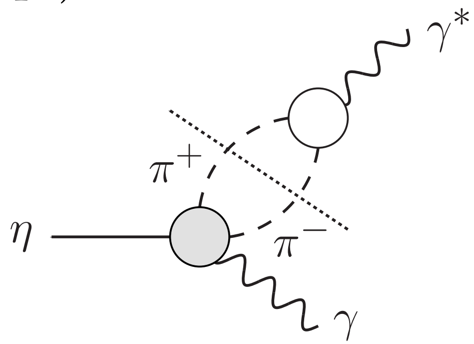
- model-independent, data-driven determination with all physical low- and high-energy constraints implemented
- perfectly consistent with
  - ▷ Padé approxim. $63.6(2.7) \times 10^{-11}$ Masjuan, Sánchez-Puertas 2017
  - ▷ lattice $62.3(2.3) \times 10^{-11}$ Gérardin et al. 2019

Transition form factor $\eta \to \gamma^*\gamma$

Hanhart et al. 2013, BK, Plenter 2015

$$F_{\eta\gamma^*\gamma}(q^2,0) = F_{\eta\gamma\gamma} + \frac{q^2}{12\pi^2}\int_{4M_\pi^2}^{\infty} dt\, \frac{q_\pi^3(t)[F_\pi^V(t)]^* F_{\eta\pi\pi\gamma}(t)}{t^{3/2}(t-q^2)}$$
$$+ \Delta F^{I=0}_{\eta\gamma^*\gamma}(q^2,0) \ [\longrightarrow \text{VMD}]$$



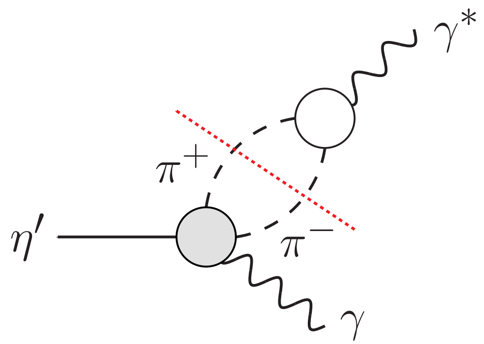


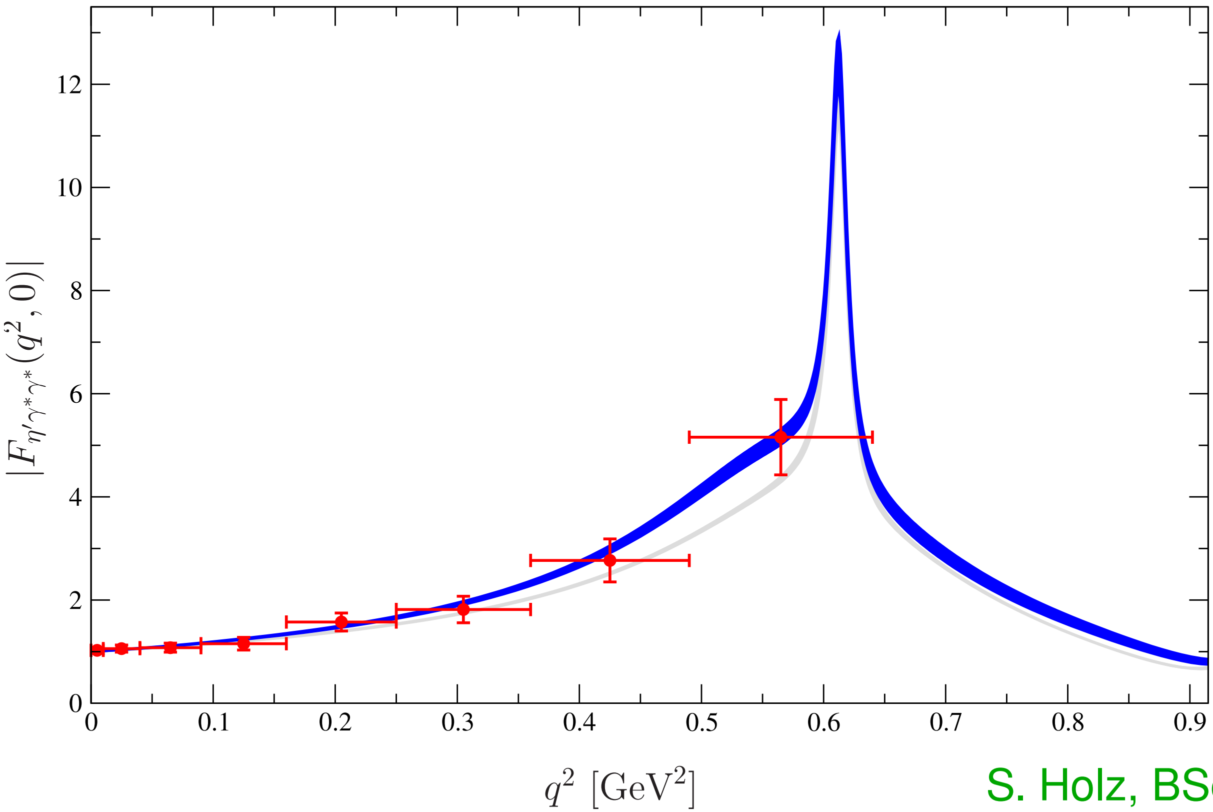
$\longrightarrow$ statistical advantage of hadronic $\eta \to \pi^+\pi^-\gamma$ over direct $\eta \to \ell^+\ell^-\gamma$ (rate suppressed $\propto \alpha^2_{\text{QED}}$)

data: NA60 2009, 2016
A2 2014, 2017

Transition form factor $\eta' \to \gamma^*\gamma$

- isovector: combine high-precision data on $\eta' \to \pi^+\pi^-\gamma$ and $e^+e^- \to \pi^+\pi^-$
- isoscalar: VMD, couplings fixed from $\eta' \to \omega\gamma$ and $\phi \to \eta'\gamma$

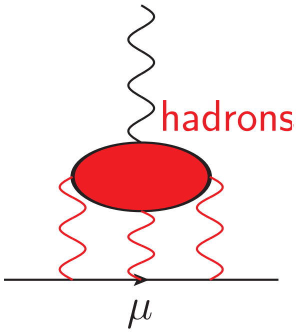




S. Holz, BSc thesis 2016
data: BESIII 2015

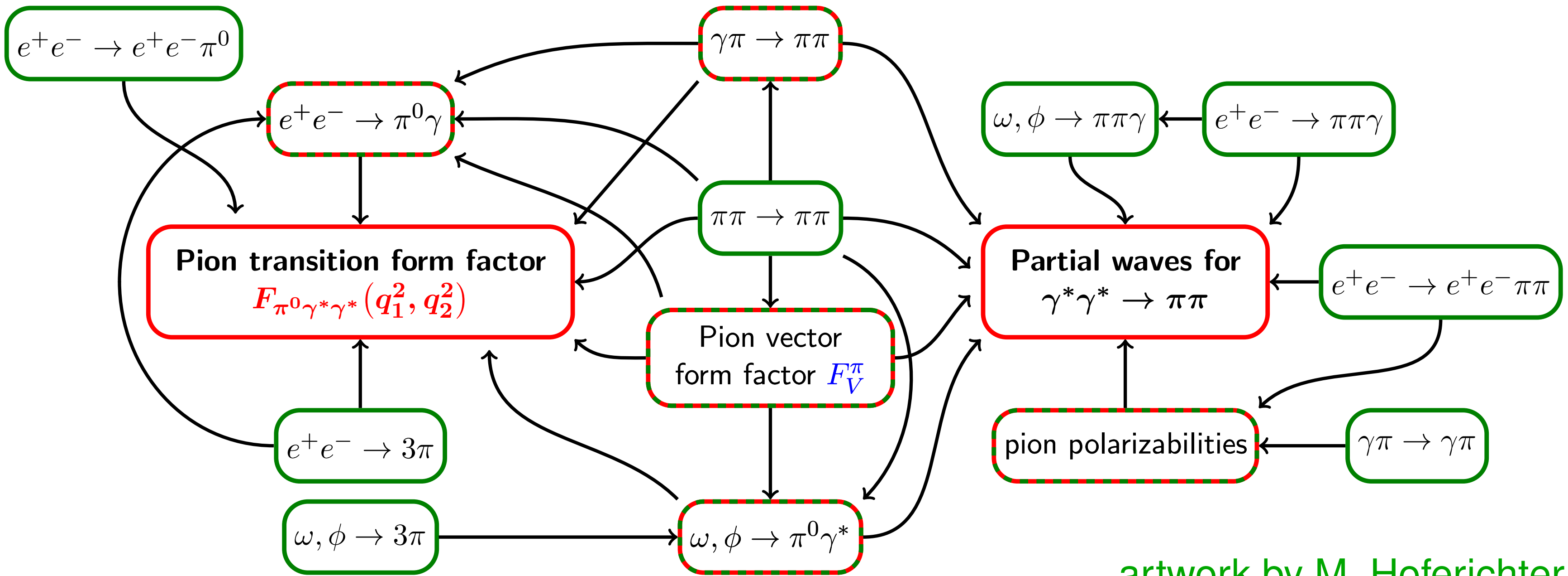
“White Paper” summary HLbL

hadronic state	$a_\mu^{\mathrm{HLbL}}\,[10^{-11}]$	
pseudoscalar poles	$93.8^{+4.0}_{-3.6}$	η, η': Masjuan, Sánchez-Puertas 2017
pion box	$-15.9(2)$	Colangelo et al. 2017
S-wave $\pi\pi$ rescatt.	$-8(1)$	Colangelo et al. 2017
kaon box	$-0.5(1)$	
scalars+tensors $\gtrsim 1\,\mathrm{GeV}$	$\sim -1(3)$	
axial vectors	$\sim 6(6)$	
short distance	$\sim 15(10)$	
heavy quarks	$\sim 3(1)$	
total	$92(19)$	Aoyama et al. 2020



$\longrightarrow$ further need for improvement to reach 10% accuracy for a_μ^{HLbL}

Summary: dispersion relations for HLbL



artwork by M. Hoferichter

Dispersive analyses of π^0, $\eta^{(\prime)}$ transition form factors:

- QCD constraints + high-precision data on
  $e^+e^- \to \pi^+\pi^-(\pi^0)$ var. / $\eta \to \pi^+\pi^-\gamma$ KLOE / $\eta' \to \pi^+\pi^-\gamma$ BESIII
  allow for high-precision dispersive predictions of π^0, $\eta^{(\prime)} \to \gamma^*\gamma^{(*)}$

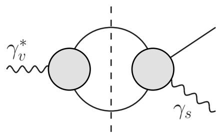
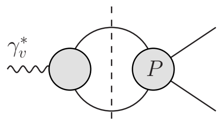
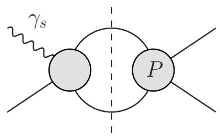
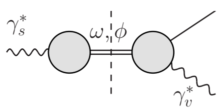
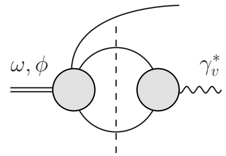
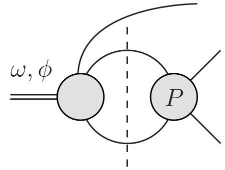
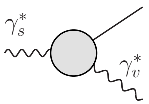
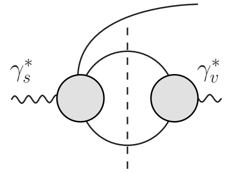
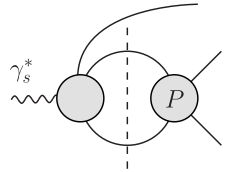
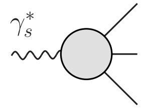
Main challenges for HLbL at 10% accuracy:

- axial vectors & short-distance constraints various

Spares

Summary: processes and unitarity relations for $\pi^0 \to \gamma^*\gamma^*$

Colangelo, Hoferichter, BK, Procura, Stoffer 2014

process	unitarity relations	SC 1	SC 2
			$F_{\pi^0\gamma\gamma}$
		$F_{3\pi}$	$\sigma(\gamma\pi \to \pi\pi)$
			$\Gamma_{\pi^0\gamma}$
		$\Gamma_{3\pi}$	$\frac{d^2\Gamma}{ds\,dt}(\omega,\phi \to 3\pi)$
			$\sigma(e^+e^- \to \pi^0\gamma)$
		$\sigma(e^+e^- \to 3\pi)$	$\sigma(\gamma\pi \to \pi\pi)$ $\frac{d^2\Gamma}{ds\,dt}(\omega,\phi \to 3\pi)$
		$F_{3\pi}$	$\sigma(e^+e^- \to 3\pi)$

$\gamma\pi \to \pi\pi$

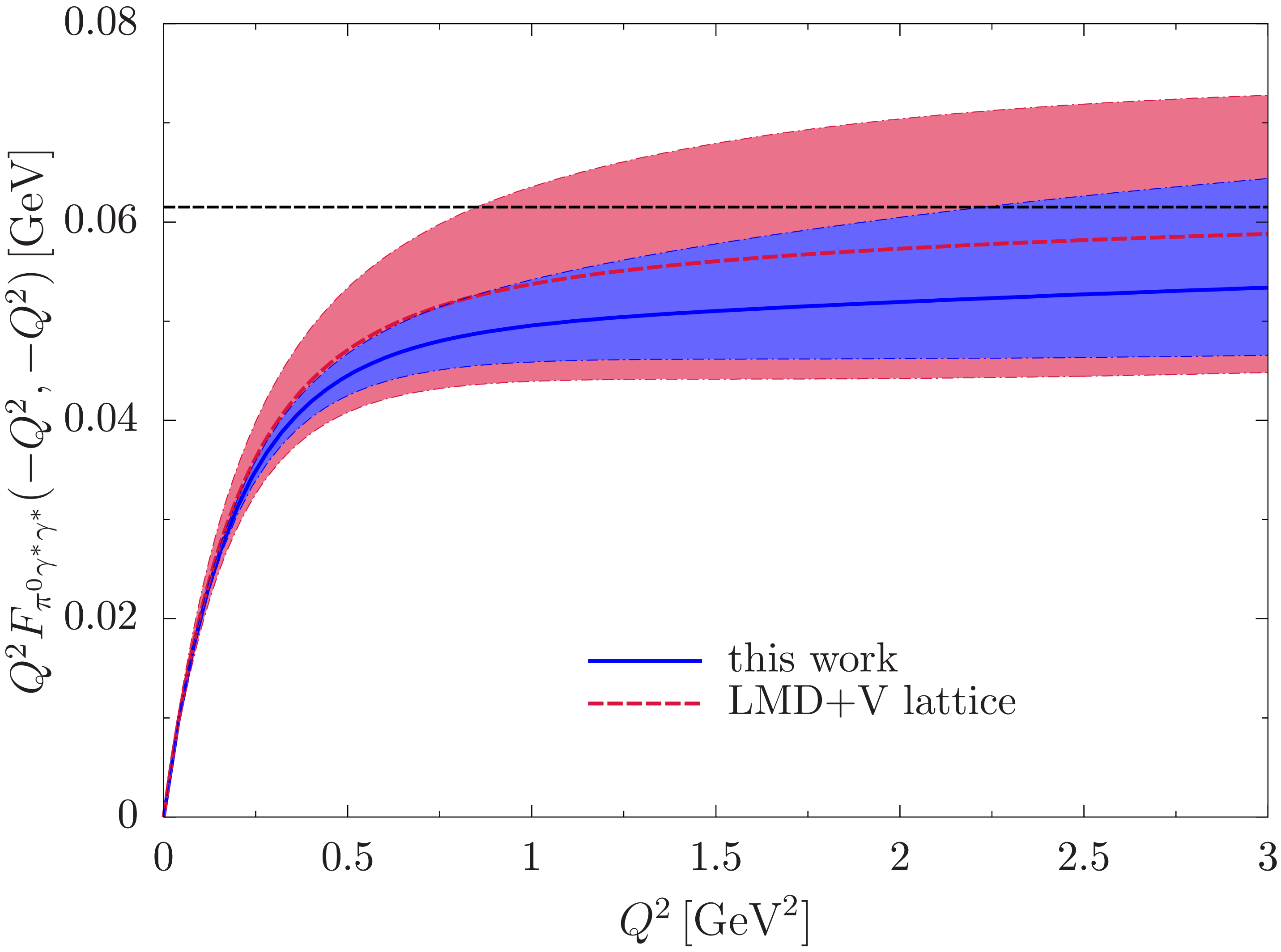
$\omega \to 3\pi$, $\phi \to 3\pi$

$\gamma^* \to 3\pi$

common theme:
resum $\pi\pi$ rescattering

Results π^0 TFF: doubly-virtual (diagonal)

in comparison to Gérardin, Meyer, Nyffeler 2016



Hoferichter, Hoid, BK, Leupold, Schneider 2018

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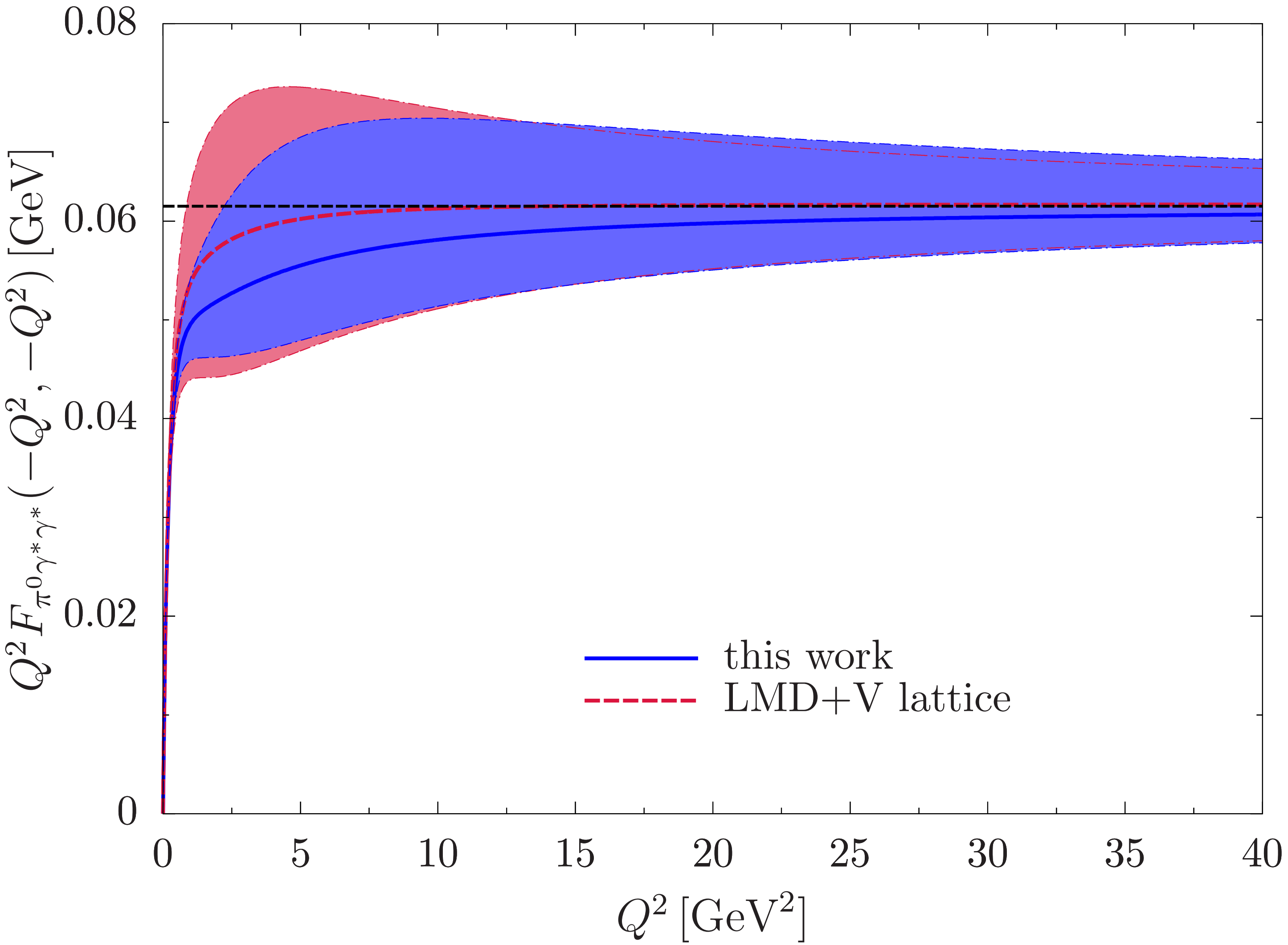
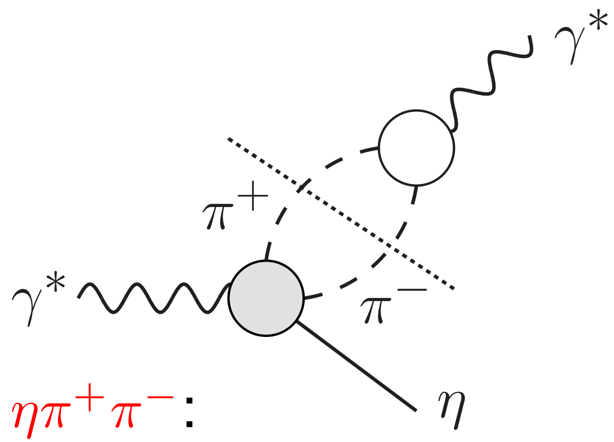


Hoferichter, Hoid, BK, Leupold, Schneider 2018

How to go *doubly* virtual? — $e^+e^- \to \eta\pi^+\pi^-$

- idea (again): beat α_{QED}^2 suppression of $e^+e^- \to \eta e^+e^-$ by measuring $e^+e^- \to \eta\pi^+\pi^-$ instead



- test factorisation hypothesis in $e^+e^- \to \eta\pi^+\pi^-$:

$$F_{\eta\pi\pi\gamma^*}(t,k^2) \overset{!?}{=} F_{\eta\pi\pi\gamma^*}(t,0) \times \tilde{F}_{\eta\gamma\gamma^*}(k^2)$$

How to go *doubly* virtual? — $e^+e^- \to \eta\pi^+\pi^-$

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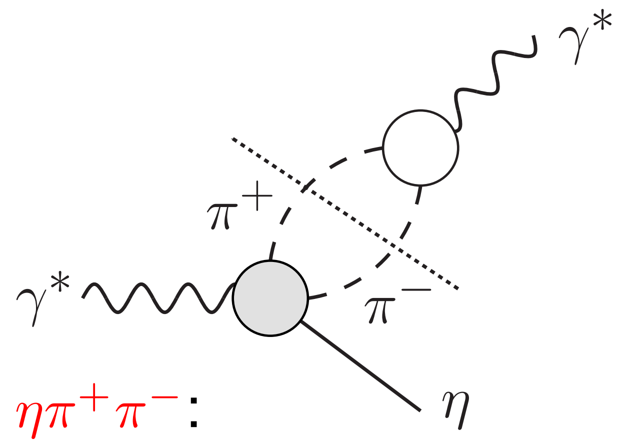

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$$F_{\eta\pi\pi\gamma^*}(t,k^2) \stackrel{!?}{=} F_{\eta\pi\pi\gamma^*}(t,0) \times \tilde{F}_{\eta\gamma\gamma^*}(k^2)$$

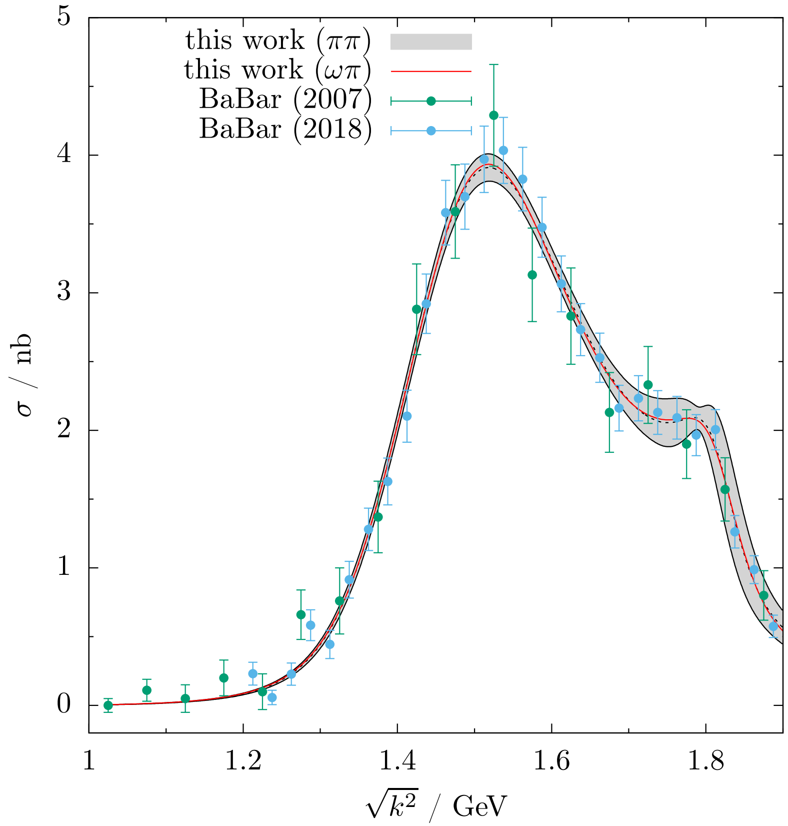
  ▷ allow same form for $F_{\eta\pi\pi\gamma^*}(t,0)$ as in $\eta \to \pi^+\pi^-\gamma$; 3 models:

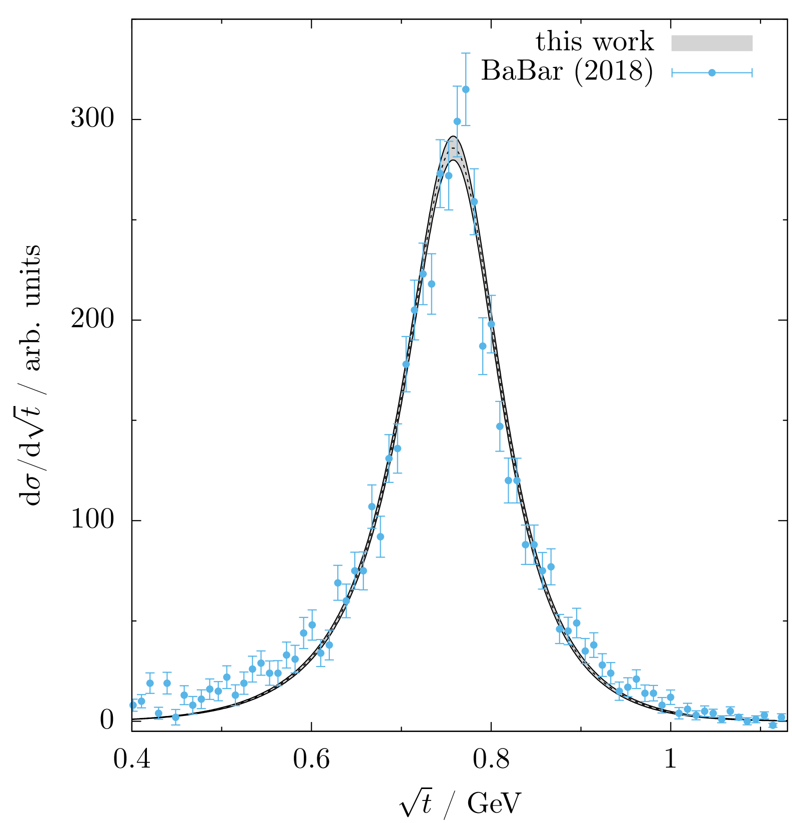
  1. $P^{(1)}(t,0) \times \Omega(t)$, linear function $P^{(1)}(t,0)$
  2. $P^{(2)}(t,0) \times \Omega(t)$, quadratic function $P^{(2)}(t,0)$
  3. $P^{(a_2)}(t,k^2) \times \Omega(t)$, a_2 left-hand cut
     $\longrightarrow$ induces "natural" factorisation breaking

  ▷ fit subtractions to $\pi^+\pi^-$ distribution in $e^+e^- \to \eta\pi^+\pi^-$
  $\longrightarrow$ are they compatible with the ones in $\eta \to \pi^+\pi^-\gamma$?

Holz, Plenter et al. 2021

How to go *doubly* virtual? — $e^+e^- \to \eta\pi^+\pi^-$

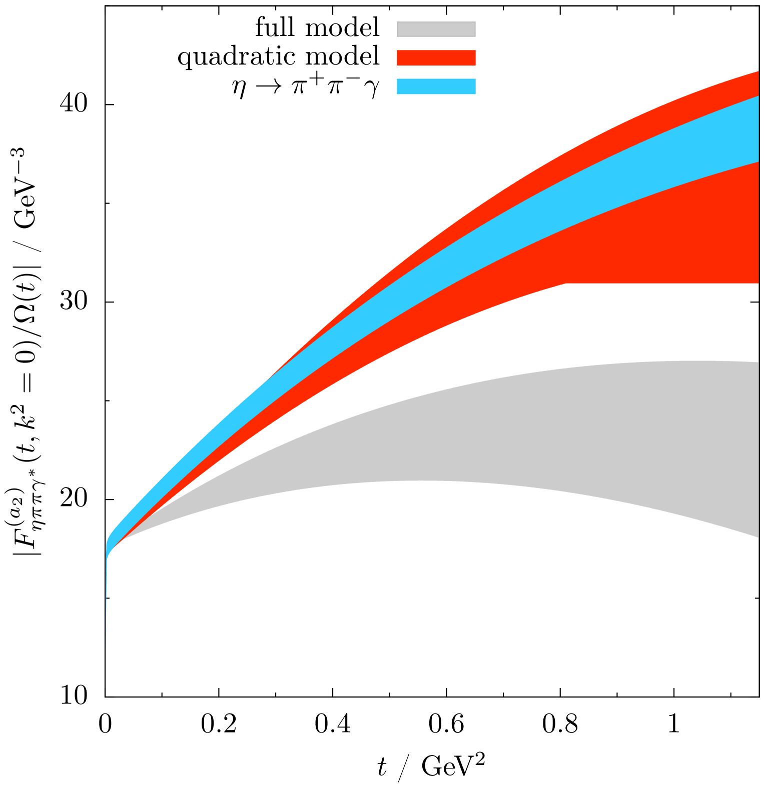


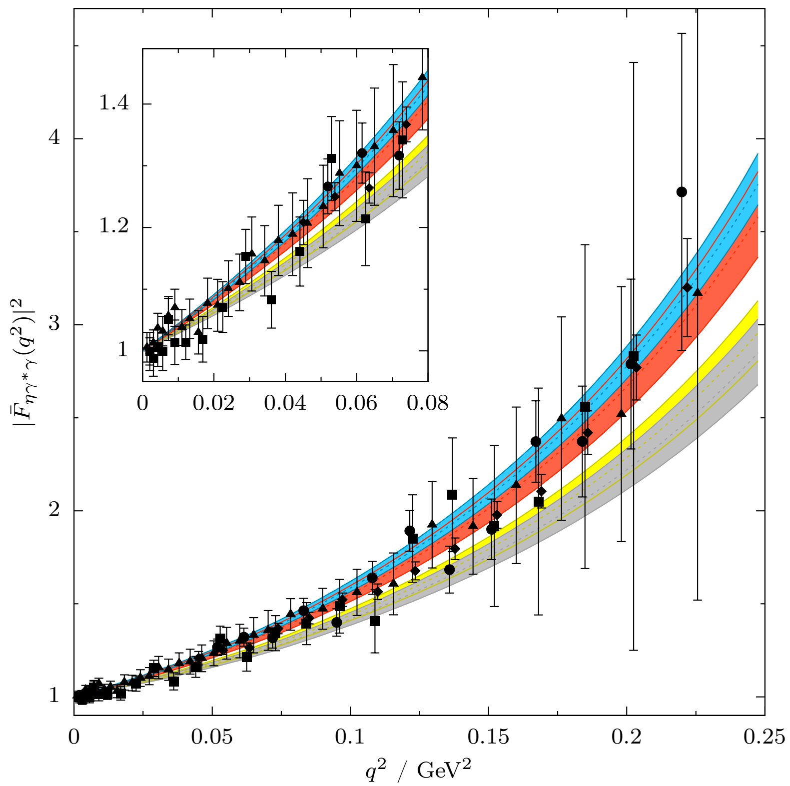


Holz, Plenter et al. 2021; data: BaBar 2007, 2018

- $\tilde{F}_{\eta\gamma\gamma^*}(k^2)$ parameterised by sum of Breit–Wigners (ρ, ρ', ρ'')
- differential spectra $d\sigma/d\sqrt{t}$ integrated over large k^2 range
- $\pi\pi$ spectrum imperfectly described below (?!) the $\rho(770)$ peak

Extrapolation from $e^+e^- \to \eta\pi^+\pi^-$ to $\eta \to \pi^+\pi^-\gamma$





- subtractions fixed from k^2-integrated $\pi\pi$ spectra — compatible with $\eta \to \pi^+\pi^-\gamma$? Holz, Plenter et al. 2021
  - ▷ yes with the naïve, factorising, quadratic model
  - ▷ no with the physically motivated a_2 model
- extrapolated form factor prediction too low for the full model