# Solution to homework "String Amplitudes"

## 1) Monodromy relations

(i) Start with Cauchy

fig. 1

assumed to not contribute

$e^{i\pi s}|z_2|^s|1-z_2|^t f(z_2)$

$$0 = \oint_C dz_2\, F(z_2) = \int_{C_\infty} dz_2\, F(z_2) + \int_{C_{2134}} dz_2\, F(z_2)$$

$$+ \int_{C_{1234}} dz_2\, F(z_2) + \int_{C_{1324}} dz_2\, F(z_2)$$

$f(z_2)\,|z_2|^s\,|1-z_2|^t$

$e^{-i\pi t}|z_2|^s|1-z_2|^t f(z_2)$

$$= \left\{ e^{i\pi s} \int_{-\infty}^{0} dz_2 + \int_0^1 dz_2 + e^{-i\pi t} \int_1^\infty dz_2 \right\} |z_2|^s |1-z_2|^t f(z_2)$$

from $C_{2134}$ from $C_{1234}$ from $C_{1324}$

(ii) given that each term in the result of (i) is some

$$A^{\text{tree}}_{\text{open}}(i,j,2,k;\alpha') = \int_{z_j}^{z_k} dz_2\, |z_2|^s |1-z_2|^t f(z_2)$$

and $(z_1, z_3, z_4) = (0, 1, \infty)$, can take ...

... real part of (i) $\Rightarrow$

$$0 = \cos(\pi s)\, A^{\text{tree}}_{\text{open}}(2134) + A^{\text{tree}}_{\text{open}}(1234) + \cos(\pi t)\, A^{\text{tree}}_{\text{open}}(1324)$$

... imaginary part of (i)

$$\Rightarrow 0 = \sin(\pi s)\, A^{\text{tree}}_{\text{open}}(2134) - \sin(\pi t)\, A^{\text{tree}}_{\text{open}}(1324)$$

using $A^{\text{tree}}_{\text{open}}(\ldots) \in \mathbb{R}$ in both cases

(iii) isolate the leading $\alpha'$-orders via

$$A^{\text{tree}}_{\text{open}}(i,j,k,l;\alpha') = A^{\text{tree}}_{\text{YM}}(i,j,k,l) + \mathcal{O}(\alpha')$$

$$\cos(\pi s_{ij}) = 1 + \mathcal{O}(\alpha'^2)$$

carries one order of $\alpha'$

$$\sin(\pi s_{ij}) = \pi s_{ij} + \mathcal{O}(\alpha'^3)$$

$\Rightarrow$ recover KK relations from $\mathrm{Re}(\dots)$

$$0 = \left(1+\mathcal{O}(\alpha'^2)\right)\left(A^{\text{tree}}_{\text{YM}}(2134)+\mathcal{O}(\alpha')\right) + \left(A^{\text{tree}}_{\text{YM}}(1234)+\mathcal{O}(\alpha')\right)$$
$$+ \left(1+\mathcal{O}(\alpha'^2)\right)\left(A^{\text{tree}}_{\text{YM}}(1324)+\mathcal{O}(\alpha')\right)$$

$\Rightarrow$ recover BCJ relations from $\mathrm{Im}(\dots)$

$$0 = \pi s\left(1+\mathcal{O}(\alpha'^2)\right)\left(A^{\text{tree}}_{\text{YM}}(2134)+\mathcal{O}(\alpha')\right)$$
$$- \pi t\left(1+\mathcal{O}(\alpha'^2)\right)\left(A^{\text{tree}}_{\text{YM}}(1324)+\mathcal{O}(\alpha')\right)$$

(iv) start by introducing meromorphic aux. function

$$F(\vec{z}) = \prod_{1\le a<b}^{n-1} (z_b - z_a)^{s_{ab}} f(\vec{z})$$

with $z_n \to \infty$, $f(\vec{z})$ rational, $\vec{z} = (z_1, z_2, \dots, z_{n-1})$;

relate to actual string integrand (non-mero)

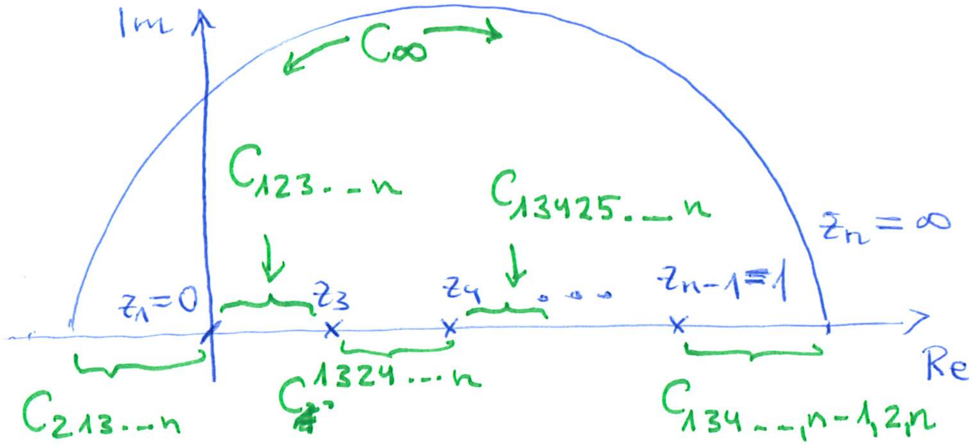
$$G(\vec{z}) = \prod_{1\le a<b}^{n-1} |z_b - z_a|^{s_{ab}} f(\vec{z})$$

such that ($z_1 = 0$ & $z_{n-1} = 1$)

$$A^{\text{tree}}_{\text{open}}(i,j,\dots,k;\alpha') = \int_{C_{ij\dots k}} G(\vec{z}) \prod_{b=2}^{n-2} dz_b$$

$-\infty < z_i < z_j < \dots < z_k < \infty$

$L^{z_2}$




on different segments $C_{\ldots}$ of depicted contour $C$

$$F(\vec{z}) = G(\vec{z}) \times \begin{cases} e^{i\pi s_{12}} & : \ z_2 \in C_{213\ldots n} \\ 1 & : \ z_2 \in C_{123\ldots n} \\ e^{-i\pi s_{23}} & : \ z_2 \in C_{1324\ldots n} \\ e^{-i\pi(s_{23}+s_{24})} & : \ z_2 \in C_{13425\ldots n} \\ \vdots & \\ \exp\left(-i\pi \sum_{j=3}^{n-1} s_{2j}\right) & : \ z_2 \in C_{134\ldots n-1,2,n} \end{cases}$$

then, by Cauchy & non-contribution of $C_\infty$

$$0 = \oint_C dz_2 \int_{0<z_3<z_4<\ldots<z_{n-2}<1} dz_3 \ldots dz_{n-2} \, F(\vec{z})$$

$$= \Big\{ e^{i\pi s_{12}} \int_{C_{213\ldots n}} + \int_{C_{123\ldots n}} + e^{-i\pi s_{23}} \int_{C_{1324\ldots n}} + e^{-i\pi(s_{23}+s_{24})} \int_{C_{13425\ldots n}}$$

$$+ \ldots + \int_{C_{134\ldots n-1,2,n}} e^{-i\pi \sum_{j=3}^{n-1} s_{2j}} \Big\} dz_2 \, dz_3 \ldots dz_{n-2} \, G(\vec{z})$$

$$= e^{i\pi s_{12}} A^{\text{tree}}_{\text{open}}(213\ldots n) + A^{\text{tree}}_{\text{open}}(123\ldots n)$$

$$+ e^{-i\pi s_{23}} A^{\text{tree}}_{\text{open}}(1324\ldots n) + e^{-i\pi(s_{23}+s_{24})} A^{\text{tree}}_{\text{open}}(1342\ldots n)$$

$$+ \ldots + \exp\left(-i\pi \sum_{j=3}^{n-1} s_{2j}\right) A^{\text{tree}}_{\text{open}}(134\ldots n-1,2,n)$$

BCJ relations then follow from Im$(\ldots)$, i.e. $e^{\pm i\pi x} \to \pm \sin(\pi x)$, after isolating leading orders in $\alpha'$ via $\sin(\pi x) \to \pi x$ and

$$A^{\text{tree}}_{\text{open}}(\ldots) \to A^{\text{tree}}_{\text{SYM}}(\ldots)$$

(v) Exploit the absence of odd $\alpha'$-powers in the $(\ldots)$ of the trigonometric expansions

$$\cos(\pi s_{ij}) = 1 - \frac{(\pi s_{ij})^2}{2} + \frac{(\pi s_{ij})^4}{24} + \mathcal{O}(\alpha'^6)$$

$$\sin(\pi s_{ij}) = \pi s_{ij}\left(1 - \frac{(\pi s_{ij})^2}{6} + \frac{(\pi s_{ij})^4}{120} + \mathcal{O}(\alpha'^6)\right)$$

expand Im(monodromy rel) via

$$\begin{aligned} 0 = 2\pi\alpha' \sum_{m=0}^{\infty} (\alpha')^m \Big\{ & k_1\cdot k_2 \left(1 + \alpha'^2(\ldots) + \alpha'^4(\ldots)\right) A_{(m)}(213\ldots n) \\ & - k_2\cdot k_3 \left(1 + \alpha'^2(\ldots) + \alpha'^4(\ldots)\right) A_{(m)}(1324\ldots n) + \ldots \\ & - k_2\cdot(k_3+k_4+\ldots+k_{n-1}) \left(1 + \alpha'^2(\ldots) + \alpha'^4(\ldots)\right) A_{(m)}(134\ldots n-1,2,n) \Big\} \end{aligned}$$

pick up the 3rd order in $\alpha'$ beyond the prefactor $2\pi\alpha'$: since $A_{(1)} = 0$ for superstrings, there is no contribution from $\alpha'^2$ in the trigonometric expansions and we have

$$k_1\cdot k_2\, A_{(3)}(213\ldots n) = \sum_{j=3}^{n-1} k_2\cdot(k_3+k_4+\ldots+k_j)\, A_{(3)}(13\ldots j,2,j+1,\ldots n)$$

in Re (monodromy rel), can argue in the same way that the $\alpha'^2$ & $\alpha'^4$ terms of the $\cos(\pi\alpha' \dots)$ do not contribute to the $\alpha'^3$-order

$$\Rightarrow 0 = A_{(3)}(213 \dots n) + A_{(3)}(123 \dots n) + \sum_{j=3}^{n-1} A_{(3)}(134 \dots j, 2, j+1, \dots, n)$$

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2) Chiral splitting & double periodicity

(i) * $z \to z+1$ in sum representation:

trivial to see since $e^{2\pi i r z} \to e^{2\pi i r(z+1)} = -e^{2\pi i r z}$

with $r \in \mathbb{Z} - 1/2$

* $z \to z+1$ in product representation:

follows from $\sin(\pi z) \to \sin(\pi(z+1)) = -\sin(\pi z)$

and invariance of $e^{\pm 2\pi i z}$ along with $q^n$

* $z \to z+\tau$ in product representation:

$$2i\sin(\pi z) \to \left(e^{i\pi z} q^{1/2} - e^{-i\pi z} q^{-1/2}\right) = \frac{-1}{q^{1/2} e^{i\pi z}} \left(1 - e^{2\pi i z} q\right)$$

$$\prod_{n=1}^{\infty} \left(1 - e^{2\pi i z} q^n\right) \Rightarrow \prod_{n=1}^{\infty} \left(1 - e^{2\pi i z} q^{n+1}\right)$$

compensate

$$= \frac{1}{1 - e^{2\pi i z} q} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i z} q^n\right)$$

$$\prod_{n=1}^{\infty} \left(1 - e^{-2\pi i z} q^n\right) \to \left(1 - e^{-2\pi i z}\right) \prod_{n=1}^{\infty} \left(1 - e^{-2\pi i z} q^n\right)$$

$$= \frac{2i\sin(\pi z)}{e^{i\pi z}} \prod_{n=1}^{\infty} \left(1 - e^{-2\pi i z} q^n\right)$$

putting everything together:

$$\theta_1(z+\tau,\tau) = q^{1/8}\prod_{n=1}^{\infty}(1-q^n) \times \left(\frac{-1}{q^{1/2}e^{i\pi z}}\right)\prod_{n=1}^{\infty}(1-e^{2\pi i z}q^n)$$

$$\times \frac{2i\sin(\pi z)}{e^{i\pi z}}\prod_{n=1}^{\infty}(1-e^{-2\pi i z}q^n)$$

$$= \frac{-1}{q^{1/2}e^{2\pi i z}}\,\theta_1(z,\tau)$$

* $z \to z+\tau$ in sum representation

$q^{-\frac{1}{2}}q^{\frac{1}{2}(r+1)^2}$

$$\theta_1(z+\tau,\tau) = \sum_{r\in\mathbb{Z}-1/2}(-1)^{r-1/2}\,e^{2\pi i r z}\,q^{\frac{r^2}{2}+r}$$

$s = r+1$

$$= q^{-1/2}\sum_{s\in\mathbb{Z}-1/2}(-)^{s-3/2}\,e^{2\pi i(s-1)z}\,q^{s^2/2}$$

$$= \frac{-1}{q^{1/2}e^{2\pi i z}}\,\theta_1(z,\tau)$$

(ii) * Under $z_1 \to z_1+1$, momentum conservation cancels transformation of

$$\sum_{1\le a<b}^{n} k_a\cdot k_b \log(\theta_1(z_{ab},\tau))\Big|_{z_1\to z_1+1} = \sum_{2\le a<b}^{n} k_a\cdot k_b\log\theta_1(z_{ab},\tau)$$

$$+\sum_{b=2}^{n} k_1\cdot k_b \log(-\theta_1(z_{1b},\tau))$$

$i\pi + \log(\theta_1(z_{1b},\tau))$

$$= \sum_{1\le a<b}^{n} k_a\cdot k_b \log\theta_1(z_{ab},\tau)$$

so only transformation of the Koba-Nielsen exponent comes from $(k_j\cdot\ell)\,z_j$ @ $j=1$

$$\Rightarrow I_n\Big|_{z_1\to z_1+1} = e^{2\pi i k_1\cdot \ell}\,I_n$$

For real $k_1\cdot l$, this is just a phase which cancels from the transformation of $|I_n|^2$.

* Under $z_1 \to z_1+\tau$, the exponent of $I_n(l)$ becomes

$$i\pi\tau l^2 + 2\pi i \left\{ k_1\cdot l\,(z_1+\tau) + \sum_{j=2}^{n} k_j\cdot l\, z_j\right\} + \sum_{2\le a<b}^{n} k_a\cdot k_b \log\theta_1(z_{ab})$$

$$+ \sum_{b=2}^{n} k_1\cdot k_b \log\theta_1(z_{1b}+\tau,\tau) \leftarrow \log\left(\frac{-1}{q^{1/2}e^{2\pi i z_{1b}}}\,\theta_1(z_{1b},\tau)\right)$$

$$= \underbrace{i\pi - i\pi\tau - 2\pi i z_{1b}}_{\text{reduces to } +2\pi i z_b \text{ by mom. conservation}} + \log\left(\theta_1(z_{1b},\tau)\right)$$

$$= \underbrace{i\pi\tau l^2 + 2\pi i k_1\cdot l\,\tau}_{i\pi\tau(l+k_1)^2} + \underbrace{2\pi i \sum_{j=1}^{n}(k_j\cdot l)\, z_j + 2\pi i \sum_{b=2}^{n} k_1\cdot k_b\, z_b}_{2\pi i \sum_{j=1}^{n}(k_j\cdot(l+k_1))\, z_j}$$

$$+ \sum_{1\le a<b}^{n} k_a\cdot k_b \log\theta_1(z_{ab},\tau)$$

which is the old exponent @ $l \to l+k_1$

$$\Rightarrow I_n(l)\Big|_{z_1\to z_1+\tau} = I_n(l+k_1)$$

Upon integration over $l$, can shift back from $l+k_1$ to $l$ by translation inv. of $\mathbb{R}^D$

$$\int_{\mathbb{R}^D} d^D l\, |I_n(l)|^2 \Big|_{z_1\to z_1+\tau} = \int_{\mathbb{R}^D} d^D l\, |I_n(l+k_1)|^2$$

(iii) Need to show that $F_5$ transforms in the same way as $I_5$ under $z_1 \to z_1+\tau$, i.e.

$$F_5(l)\Big|_{z_1\to z_1+\tau} = F_5(l+k_1)$$

use $\theta_1$ – function identities following from (i)

$$\partial_z \log \theta_1(z,\tau)\Big|_{z\to z+1} = \partial_z \log \theta_1(z,\tau)$$

$$\partial_z \log \theta_1(z,\tau)\Big|_{z\to z+\tau} = \partial_z \log \theta_1(z,\tau) - 2\pi i$$

such that

$$F_5(\ell)\Big|_{z_1\to z_1+\tau} = \overbrace{I_5(\ell+k_1)}^{\text{by (ii)}} \Big\{ \overbrace{2\pi i\, t_\mu(\ell^\mu+k_1^\mu) - 2\pi i\, t_\mu k_1^\mu}^{2\pi i\, t_\mu \ell^\mu}$$

$$+ \sum_{2\le a<b}^{n} t_{ab}\,\partial_z \log\theta_1(z_{ab},\tau) + \sum_{b=2}^{n} t_{1b}\Big(\partial_z \log \theta_1(z_{1b}) - 2\pi i\Big)\Big\}$$

these two types of terms will depart from the desired $F_5(\ell+k_1)$

$$= F_5(\ell+k_1) + I_5(\ell+k_1)(-2\pi i)$$

$$\times \Big\{ \underbrace{t_\mu k_1^\mu + t_{12} + t_{13} + t_{14} + t_{15}}_{\text{has to vanish}} \Big\}$$

Under the kinematic condition $0 = t_\mu k_1^\mu + \sum_{b=2}^{5} t_{1b}$, $\int d^D\ell\, |F_5(\ell)|^2$ is invariant under $z_1 \to z_1+\tau$ for the same reason as $\int d^D\ell\, |I_5(\ell)|^2$ is.

Under $z_1 \to z_1 + 1$, the kinematic part $F_5/I_5$ is invariant term by term.