The Amplitude Games: homework and references for String Amplitudes

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1 Monodromy relations among open-string tree amplitudes

In this exercise, you will be guided to derive the monodromy relations among color-ordered open-string amplitudes $A^{\rm tree}_{\rm open}(1,2,\ldots,n;\alpha')$ that are universally valid for bosonic strings and superstrings [1, 2]. The external states are taken to be massless such that $s_{ij} = 2\alpha' k_i \cdot k_j$ according to the conventions in the lecture. The goal is to relate different color orderings or integration regions for the punctures $z_2, z_3, \ldots, z_{n-2}$ in an SL_2-frame where $(z_1, z_{n-1}, z_n) = (0, 1, \infty)$. This will be achieved by exploiting that the underlying correlator among vertex operators is meromorphic except for the universal Koba–Nielsen factor $\prod_{1\leq a<b}^n |z_{ab}|^{s_{ab}}$ with $z_{ab} = z_a - z_b$.

At 4 points with $s = 2\alpha' k_1 \cdot k_2$ and $t = 2\alpha' k_2 \cdot k_3$, the above SL_2-frame yields color-ordered amplitudes

$$
\begin{aligned}
A^{\rm tree}_{\rm open}(2,1,3,4;\alpha') &\longleftrightarrow \int_{-\infty}^0 \mathrm{d}z_2\, |z_2|^s\, |1-z_2|^t\, f(z_2) \\
A^{\rm tree}_{\rm open}(1,2,3,4;\alpha') &\longleftrightarrow \int_{0}^1 \mathrm{d}z_2\, |z_2|^s\, |1-z_2|^t\, f(z_2) \\
A^{\rm tree}_{\rm open}(1,3,2,4;\alpha') &\longleftrightarrow \int_{1}^\infty \mathrm{d}z_2\, |z_2|^s\, |1-z_2|^t\, f(z_2)\, ,
\end{aligned}
\tag{1}
$$

where the polarization dependent rational function $f(z_2)$ is determined by the contractions among the vertex operators, and its detailed form will not be needed in the following.

The proof of the monodromy relations is based on applying Cauchy's theorem to the meromorphic function $F(z_2) = (z_2)^s(1-z_2)^t f(z_2)$ with rational $f(z_2)$:

$$
\oint_{\mathcal{C}} \mathrm{d}z_2\, F(z_2) = 0\, . \tag{2}
$$

The integration contour $\mathcal{C}$ is depicted in figure 1 below and consists of a semi-circle $\mathcal{C}_\infty$ at infinity as well as the real axis with the following shorthands

$$
\mathcal{C}_{2134} = (-\infty, 0)\, , \qquad \mathcal{C}_{1234} = (0,1)\, , \qquad \mathcal{C}_{1324} = (1,\infty)\, . \tag{3}
$$

(Strictly speaking, the $\mathcal{C}_{ijkl}$ should be infinitesimally displaced from the real axis by some positive imaginary part $i\epsilon$ to avoid tentative poles of $f(z_2)$ at $z_2 = 0$ and $z_2 = 1$.)

(i) Assuming that the semi-circle $\mathcal{C}_\infty$ does not contribute to $\oint_{\mathcal{C}} \mathrm{d}z_2\, F(z_2)$ and using the representation

$$
F(z_2) = \begin{cases} e^{i\pi s}\, |z_2|^s\, |1-z_2|^t\, f(z_2) & : \ z_2 \in \mathcal{C}_{2134} \\ |z_2|^s\, |1-z_2|^t\, f(z_2) & : \ z_2 \in \mathcal{C}_{1234} \\ e^{-i\pi t}\, |z_2|^s\, |1-z_2|^t\, f(z_2) & : \ z_2 \in \mathcal{C}_{1324} \end{cases}
$$

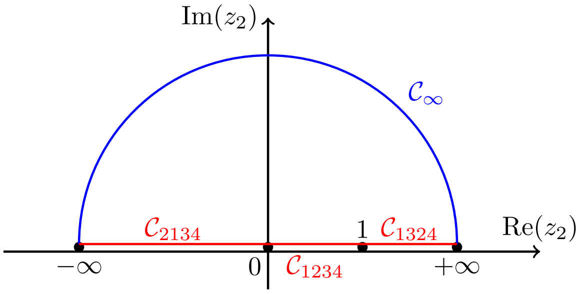

Figure 1: The contour $\mathcal{C}$ consists of the semicircle $\mathcal{C}_\infty$ drawn in blue and subsets $\mathcal{C}_{2134}$, $\mathcal{C}_{1234}$, $\mathcal{C}_{1324}$ of the real line drawn in red.

of the above integrand $F(z_2)$, explain why Cauchy's theorem implies that

$$e^{i\pi s}\int_{-\infty}^{0} \mathrm{d}z_2\, |z_2|^s\, |1-z_2|^t\, f(z_2) + \int_0^1 \mathrm{d}z_2\, |z_2|^s\, |1-z_2|^t\, f(z_2) + e^{-i\pi t}\int_1^\infty \mathrm{d}z_2\, |z_2|^s\, |1-z_2|^t\, f(z_2) = 0\,.$$

(ii) Assuming that color-ordered amplitudes $A^{\rm tree}_{\rm open}(i,j,k,l;\alpha')$ are real, deduce that

$$\cos(\pi s)A^{\rm tree}_{\rm open}(2,1,3,4;\alpha') + A^{\rm tree}_{\rm open}(1,2,3,4;\alpha') + \cos(\pi t)A^{\rm tree}_{\rm open}(1,3,2,4;\alpha') = 0$$
$$\sin(\pi s)A^{\rm tree}_{\rm open}(2,1,3,4;\alpha') - \sin(\pi t)A^{\rm tree}_{\rm open}(1,3,2,4;\alpha') = 0\,.$$

(iii) By isolating the leading orders in α', conclude that color-ordered tree amplitudes among non-abelian gauge bosons satisfy the Kleiss–Kuijf relations

$$A^{\rm tree}_{\rm YM}(2,1,3,4) + A^{\rm tree}_{\rm YM}(1,2,3,4) + A^{\rm tree}_{\rm YM}(1,3,2,4) = 0$$

and the Bern–Carrasco–Johansson (BCJ) relations

$$(k_1+k_2)^2 A^{\rm tree}_{\rm YM}(2,1,3,4) - (k_2+k_3)^2 A^{\rm tree}_{\rm YM}(1,3,2,4) = 0\,.$$

(iv) Generalize the integration contour $\mathcal{C}$ and the integrand $F_{\rm open}(z_2) \to F_{\rm open}(z_2,z_3,\ldots,z_{n-2})$ to an n-point setting to derive the monodromy relation

$$\begin{aligned} 0 &= e^{2\pi i\alpha' k_1\cdot k_2} A^{\rm tree}_{\rm open}(2,1,3,\ldots,n;\alpha') + A^{\rm tree}_{\rm open}(1,2,3,\ldots,n;\alpha') \\ &\quad + e^{-2\pi i\alpha' k_2\cdot k_3} A^{\rm tree}_{\rm open}(1,3,2,4,\ldots,n;\alpha') + e^{-2\pi i\alpha' k_2\cdot(k_3+k_4)} A^{\rm tree}_{\rm open}(1,3,4,2,\ldots,n;\alpha') \\ &\quad + \ldots + e^{-2\pi i\alpha' k_2\cdot(k_3+k_4+\ldots+k_{n-1})} A^{\rm tree}_{\rm open}(1,3,4,\ldots,n-1,2,n;\alpha') \end{aligned}$$

and deduce the n-point BCJ relations

$$k_1\cdot k_2 A^{\rm tree}_{\rm YM}(2,1,3,\ldots,n) = \sum_{j=3}^{n-1} k_2\cdot(k_3+k_4+\ldots+k_j) A^{\rm tree}_{\rm YM}(1,3,\ldots,j,2,j+1,\ldots,n)\,.$$

Note: By combining all permutations of the monodromy- and BCJ relations, color-ordered n-point tree amplitudes in string and gauge theory can be reduced to a basis of dimension $(n-3)!$.

(v) In a formal α'-expansion $A^{\rm tree}_{\rm open}(1,2,\ldots,n;\alpha') = \sum_{m=0}^\infty (\alpha')^m A^{\rm tree}_{(m)}(1,2,\ldots,n)$ of the open-superstring tree amplitude with $A^{\rm tree}_{(1)} = 0$, explain why the α'^3-order $A^{\rm tree}_{(3)}$ obeys the BCJ- and KK-relations of field-theory amplitudes $A^{\rm tree}_{\rm SYM}$. There is no need to know about the explicit form of $A^{\rm tree}_{(3)}$. Since the property $A^{\rm tree}_{(1)} = 0$ of superstring amplitudes does not hold for the open bosonic string, only a subsector of the α'^3 order in tree amplitudes of open bosonic strings obey field-theory relations.

Note: Based on standard transcendentality conjectures on multiple zeta values, the argument in (v) can be extended and refined to deduce that the entire single-valued open-superstring amplitude $\text{sv}\, A^{\text{tree}}_{\text{open}}$ obeys BCJ- and KK-relations.

2 Chiral splitting and double periodicity on a torus

In this exercise, you will explore the double-periodicity properties of the (meromorphic) chiral amplitudes $\mathcal{F}$ in the chiral-splitting approach to closed-string genus-one amplitudes. The underlying torus worldsheet is parametrized through the parallelogram with corners $0, 1, \tau{+}1, \tau$, where τ is the modular parameter with $\text{Im}\,\tau > 0$.

(i) Use one of the representations of the odd Jacobi theta function

$$\theta_1(z,\tau) = 2iq^{1/8}\sin(\pi z)\prod_{n=1}^{\infty}(1-q^n)(1-e^{2\pi i z}q^n)(1-e^{-2\pi i z}q^n) = \sum_{r\in\mathbb{Z}-\frac{1}{2}}(-1)^{r-1/2}e^{2\pi i r z}q^{r^2/2} \tag{4}$$

with $q = e^{2\pi i\tau}$ to derive the (pseudo-)periodicity properties

$$\theta_1(z{+}1,\tau) = -\theta_1(z,\tau)\,, \qquad \theta_1(z{+}\tau,\tau) = -e^{-i\pi\tau-2\pi i z}\theta_1(z,\tau)\,. \tag{5}$$

(ii) Consider the chiral Koba–Nielsen factor for the integrand of n-point genus-one amplitudes

$$\mathcal{I}_n(z,\tau,\ell,k) = \exp\Bigg(i\pi\tau\ell^2 + 2\pi i\sum_{j=1}^{n}(k_j\cdot\ell)\,z_j + \sum_{1\leq a<b}^{n}k_a\cdot k_b\log\theta_1(z_{ab},\tau)\Bigg)\,, \tag{6}$$

where we are setting $\alpha' = 1/2$ for simplicity. Check that the B-cycle monodromies of $\mathcal{I}_n$ in one puncture, say $z_1 \to z_1{+}\tau$, can be compensated by a shift of the loop momentum ℓ. Explain why the loop integral $\int \mathrm{d}^D\ell\,\big|\mathcal{I}_n(z,\tau,\ell,k)\big|^2$ is well-defined on a torus, i.e. that the integral gives a doubly-periodic function under $z_1 \to z_1{+}1$ and $z_1 \to z_1{+}\tau$.

Note: Momentum conservation $\sum_{j=1}^{n}k_j = 0$ and the on-shell condition $k_j^2 = 0$ are understood to hold throughout the exercise, and ℓ, k_j are all treated as real in the complex conjugate of $\mathcal{I}_n$.

(iii) The chiral genus-one five-point amplitude takes the schematic form

$$\mathcal{F}_5(z,\tau,\ell,k) = \mathcal{I}_5(z,\tau,\ell,k)\Bigg(2\pi i t_\mu\ell^\mu + \sum_{1\leq a<b}^{5}t_{ab}\,\partial_z\log\theta_1(z_{ab},\tau)\Bigg) \tag{7}$$

with kinematic factors t_μ and $t_{ab} = -t_{ba}$ multilinear in the polarizations $\epsilon_1,\ldots,\epsilon_5$. Which relation between $k_j^\mu t_\mu$ and t_{ab} has to be required such that $\int \mathrm{d}^D\ell\,\big|\mathcal{F}_5(z,\tau,\ell,k)\big|^2$ is well-defined on a torus?

Note: An explicit solution to the above requirements that integrates to the correct (parity-even part of the) five-point amplitude can be constructed from the permutation-invariant t_8-tensor

$$t_8(f_1,f_2,f_3,f_4) = \text{tr}(f_1f_2f_3f_4) - \frac{1}{4}\text{tr}(f_1f_2)\text{tr}(f_3f_4) + \text{cyc}(f_2,f_3,f_4)\,, \tag{8}$$

where the traces are over the Lorentz indices of the linearized field strength $f_j^{\mu\nu} = k_j^\mu\epsilon_j^\nu - k_j^\nu\epsilon_j^\mu$. We furthermore introduce a two-particle field-strength $f_{ij}^{\mu\nu} = -f_{ji}^{\mu\nu} = -f_{ij}^{\nu\mu}$ via

$$f_{12}^{\mu\nu} = (k_2\cdot\epsilon_1)f_2^{\mu\nu} - (k_1\cdot\epsilon_2)f_1^{\mu\nu} + f_1^{\mu}{}_{\lambda}f_2^{\lambda\nu} - f_2^{\mu}{}_{\lambda}f_1^{\lambda\nu} \tag{9}$$

which determines the kinematic factors in (7) to be

$$\begin{aligned} t^\mu &= \epsilon_1^\mu t_8(f_2,f_3,f_4,f_5) + (1\leftrightarrow 2,3,4,5) \\ t_{12} &= t_8(f_{12},f_3,f_4,f_5) \end{aligned} \tag{10}$$

3 Further reading [not part of the homework assignment]

This section gathers suggestions for further reading, where all page numbers refer to the scan of the lecture notes.

- page 4: the Kawai-Lewellen-Tye formula between open- and closed-string tree-level amplitudes was firstly derived in [3], related to monodromy relations in [4] and studied from the mathematical perspective of twisted de Rham theory in [5]
- page 5: the original reference for chiral splitting is [6]
- page 5: the field-theory limit of string loop amplitudes can be elegantly analyzed in terms of tropical geometry [7], also see [8] for a nice account on the role of the loop momentum
- page 5: an upcoming textbook on multiple zeta values can be accessed via [9], also see [10] for a datamine of their $\mathbb{Q}$-relations
- page 8: the $A^{\rm tree}_{\rm SYM}$-representation of the n-point tree-level amplitude of the open superstring was constructed in [11] and brought into KLT form in [12]
- page 8: an introductory reference on the motivic coaction of the disk integrals $F_\rho{}^\sigma$ is [13], also see [14] for generalizations to unintegrated punctures as in the examples of Ruth's lecture
- page 9: the interpretation of disk integrals as "Z-theory" amplitudes was discussed in [15, 16]
- page 10: different approaches to proving the relation $J = {\rm sv}\, Z$ between disk and sphere integrals can for instance be found in [17, 18]
- page 11: single-valued polylogarithms in one variable and several ones are constructed in [19] and [20], respectively
- page 12: the web of double-copy relations involving tree amplitudes in different string theories can be found in [21], also see [22] for a recent extension to tree amplitudes with one massive state
- page 13: at genus 3, the four-point closed-string amplitude has been computed in the low-energy limit in [23], and a very recent proposal for the complete chiral amplitude based on ambitwistor-string techniques can be found in [24]
- page 14: the appearance of D-dimensional box integrals from one-loop four-point string amplitudes was already shown in the seminal paper [25] from 1982
- page 15: see [26] for a pedagogical account on elliptic multiple zeta values and iterated Eisenstein integrals in string amplitudes; more recent techniques to perform all-order α'-expansions are based on differential equations, either in τ [27] or in z [28]
- page 16: even though the prehistory reaches back to 1999, the notion of modular graph functions or forms was introduced in 2015 [29]; an overview of the state of the art on modular graph forms as of fall 2020 can be obtained from [30]
- homework problem 1: the loop-level analogue of the monodromy relations among $A^{\rm tree}_{\rm open}$ has been studied from a variety of perspectives [31, 32, 33, 34, 35]

There are further lecture series on string amplitudes on the web:

- emphasis on multiloop amplitudes, supermoduli and modular graph forms (E. D'Hoker 2018)

  `https://qmap.ucdavis.edu/events/events-past-events/amplitudes-summer-school`

- with emphasis on conformal-field-theory techniques (O. Schlotterer 2019)

  `https://indico.desy.de/event/22450/overview`

For string theory in general, the lecture notes [36] are particularly suitable for introductory reading, and useful textbooks include [37, 38, 39, 40, 41, 42, 43].

References

[1] N. E. J. Bjerrum-Bohr, P. H. Damgaard and P. Vanhove, "Minimal Basis for Gauge Theory Amplitudes," Phys. Rev. Lett. **103** (2009), 161602 [arXiv:0907.1425 [hep-th]].

[2] S. Stieberger, "Open & Closed vs. Pure Open String Disk Amplitudes," [arXiv:0907.2211 [hep-th]].

[3] H. Kawai, D. C. Lewellen and S. H. H. Tye, "A Relation Between Tree Amplitudes of Closed and Open Strings," Nucl. Phys. B **269** (1986), 1-23

[4] N. E. J. Bjerrum-Bohr, P. H. Damgaard, T. Sondergaard and P. Vanhove, "The Momentum Kernel of Gauge and Gravity Theories," JHEP **01** (2011), 001 [arXiv:1010.3933 [hep-th]].

[5] S. Mizera, "Combinatorics and Topology of Kawai-Lewellen-Tye Relations," JHEP **08** (2017), 097 [arXiv:1706.08527 [hep-th]].

[6] E. D'Hoker and D. H. Phong, "Conformal Scalar Fields and Chiral Splitting on Superriemann Surfaces," Commun. Math. Phys. **125** (1989), 469

[7] P. Tourkine, "Tropical Amplitudes," Annales Henri Poincare **18** (2017) no.6, 2199-2249 [arXiv:1309.3551 [hep-th]].

[8] P. Tourkine, "Integrands and loop momentum in string and field theory," Phys. Rev. D **102** (2020) no.2, 026006 [arXiv:1901.02432 [hep-th]].

[9] J. I. Burgos Gil and J. Fresan, "Multiple zeta values, from numbers to motives," `http://javier.fresan.perso.math.cnrs.fr/mzv.pdf`

[10] J. Blümlein, D. J. Broadhurst and J. A. M. Vermaseren, "The Multiple Zeta Value Data Mine," Comput. Phys. Commun. **181** (2010) 582 [arXiv:0907.2557 [math-ph]].

[11] C. R. Mafra, O. Schlotterer and S. Stieberger, "Complete N-Point Superstring Disk Amplitude I. Pure Spinor Computation," Nucl. Phys. B **873** (2013), 419-460 [arXiv:1106.2645 [hep-th]].

[12] J. Broedel, O. Schlotterer and S. Stieberger, "Polylogarithms, Multiple Zeta Values and Superstring Amplitudes," Fortsch. Phys. **61** (2013), 812-870 [arXiv:1304.7267 [hep-th]].

[13] O. Schlotterer, "The number theory of superstring amplitudes," in proceedings of the conference "Numbers and Physics," ICMAT, Madrid, Spain, September 2014,

`https://www.physics.uu.se/digitalAssets/741/c_741148-l_3-k_madrid_osrevised.pdf`

[14] R. Britto, S. Mizera, C. Rodriguez and O. Schlotterer, "Coaction and double-copy properties of configuration-space integrals at genus zero," JHEP **05** (2021), 053 [arXiv:2102.06206 [hep-th]].

[15] J. J. M. Carrasco, C. R. Mafra and O. Schlotterer, "Abelian Z-theory: NLSM amplitudes and α'-corrections from the open string," JHEP **06** (2017), 093 [arXiv:1608.02569 [hep-th]].

[16] C. R. Mafra and O. Schlotterer, "Non-abelian Z-theory: Berends-Giele recursion for the α'-expansion of disk integrals," JHEP **1701** (2017) 031 [arXiv:1609.07078 [hep-th]].

[17] O. Schlotterer and O. Schnetz, "Closed strings as single-valued open strings: A genus-zero derivation," J. Phys. A **52** (2019) no.4, 045401 [arXiv:1808.00713 [hep-th]].

[18] F. Brown and C. Dupont, "Single-valued integration and superstring amplitudes in genus zero," Commun. Math. Phys. **382** (2021) no.2, 815-874 [arXiv:1910.01107 [math.NT]].

[19] F. Brown, "Single-valued multiple polylogarithms in one variable", C. R. Acad. Sci. Paris **338**, 527 (2004).

[20] V. Del Duca, S. Druc, J. Drummond, C. Duhr, F. Dulat, R. Marzucca, G. Papathanasiou and B. Verbeek, "Multi-Regge kinematics and the moduli space of Riemann spheres with marked points," JHEP **08** (2016), 152 [arXiv:1606.08807 [hep-th]].

[21] T. Azevedo, M. Chiodaroli, H. Johansson and O. Schlotterer, "Heterotic and bosonic string amplitudes via field theory," JHEP **10** (2018), 012 [arXiv:1803.05452 [hep-th]].

[22] M. Guillen, H. Johansson, R. L. Jusinskas and O. Schlotterer, "Scattering Massive String Resonances through Field-Theory Methods," [arXiv:2104.03314 [hep-th]].

[23] H. Gomez and C. R. Mafra, "The closed-string 3-loop amplitude and S-duality," JHEP **10** (2013), 217 [arXiv:1308.6567 [hep-th]].

[24] Y. Geyer, R. Monteiro and R. Stark-Muchão, "Superstring loop amplitudes from the field theory limit," [arXiv:2106.03968 [hep-th]].

[25] M. B. Green, J. H. Schwarz and L. Brink, "N=4 Yang-Mills and N=8 Supergravity as Limits of String Theories," Nucl. Phys. B **198** (1982), 474-492

[26] J. Broedel and O. Schlotterer, "One-Loop String Scattering Amplitudes as Iterated Eisenstein Integrals," draft version of a proceedings article for the KMPB Conference "Elliptic Integrals, Elliptic Functions and Modular Forms in Quantum Field Theory", DESY Zeuthen, Germany, October 2017; available for download on

`https://www.physics.uu.se/research/theoretical-physics/people/oliver-schlotterer`

[27] C. R. Mafra and O. Schlotterer, "All Order α' Expansion of One-Loop Open-String Integrals," Phys. Rev. Lett. **124** (2020) no.10, 101603 [arXiv:1908.09848 [hep-th]].

[28] J. Broedel and A. Kaderli, "Amplitude recursions with an extra marked point," [arXiv:1912.09927 [hep-th]].

[29] E. D'Hoker, M. B. Green, Ö. Gürdogan and P. Vanhove, "Modular Graph Functions," Commun. Num. Theor. Phys. **11** (2017), 165-218 [arXiv:1512.06779 [hep-th]].

[30] J. E. Gerken, "Modular Graph Forms and Scattering Amplitudes in String Theory," [arXiv:2011.08647 [hep-th]].

[31] P. Tourkine and P. Vanhove, "Higher-loop amplitude monodromy relations in string and gauge theory," Phys. Rev. Lett. **117** (2016) no.21, 211601 [arXiv:1608.01665 [hep-th]].

[32] S. Hohenegger and S. Stieberger, "Monodromy Relations in Higher-Loop String Amplitudes," Nucl. Phys. B **925** (2017), 63-134 [arXiv:1702.04963 [hep-th]].

[33] E. Casali, S. Mizera and P. Tourkine, "Monodromy relations from twisted homology," JHEP **12** (2019), 087 [arXiv:1910.08514 [hep-th]].

[34] E. Casali, S. Mizera and P. Tourkine, "Loop amplitudes monodromy relations and color-kinematics duality," JHEP **03** (2021), 048 [arXiv:2005.05329 [hep-th]].

[35] S. Stieberger, "Open & Closed vs. Pure Open String One-Loop Amplitudes," [arXiv:2105.06888 [hep-th]].

[36] D. Tong, "String Theory," [arXiv:0908.0333 [hep-th]].

[37] R. Blumenhagen, D. Lüst and S. Theisen, "Basic Concepts of String Theory", Springer 2013

[38] J. Polchinski, "String theory I. An introduction to the bosonic string" and "String theory II. Superstring theory and beyond", Cambridge University Press 2005

[39] M. Green, J. Schwarz and E. Witten, "Superstring theory I. Introduction" and "Superstring theory II. Loop amplitudes, anomalies & phenomenology", Cambridge University Press 1987

[40] K. Becker, M. Becker and J. Schwarz, "String theory and M-theory, a modern introduction", Cambridge University Press 2006

[41] E. Kiritsis, "String theory in a nutshell", Princeton University Press 2019

[42] P. Di Francesco, P. Mathieu and D. Sénéchal, "Conformal field theory", Springer 1997

[43] M. Ammon and J. Erdmenger, "Gauge / Gravity duality, foundations and applications", Cambridge University Press 2015