# String amplitudes

Plan: I) 4pt tree-level warmup

II) Why string amplitudes

III) Double copies @ string tree level

IV) Aspects of string loop amplitudes

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open strings: 10 dim SYM

closed strings: Type II supergrav.

not in external states throughout these lectures

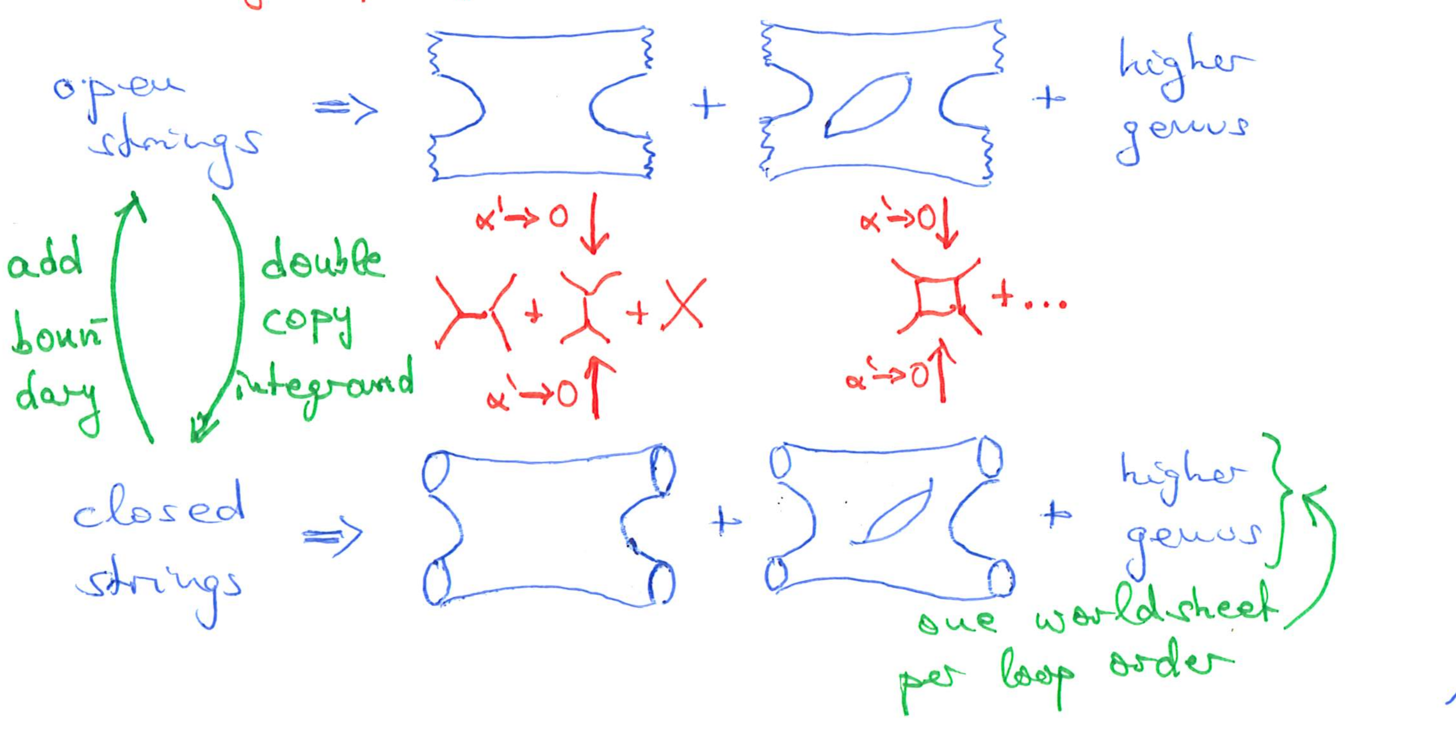
These lectures: $\begin{pmatrix} \text{super-} \\ \text{strings} \end{pmatrix} = \begin{pmatrix} \text{susy} \\ \text{field} \\ \text{thy} \end{pmatrix} \oplus \sum_{n=1}^{\infty} \begin{pmatrix} \text{higher} \\ \text{spin @} \\ m^2 \in \frac{\mathbb{N}}{\alpha'} \end{pmatrix}$

inverse string tension $\Leftrightarrow$ length scale $l_s = \sqrt{\alpha'}$

Expand string amplitudes

$$\sum_{h=0}^{\infty} \underbrace{g_{\text{string}}^{2h-2}}_{\text{string coupling}} \left( \text{worldsheet @ genus } h \right)$$

$\uparrow$ fattened out worldline



Conformal symmetry on 2dim worldsheet $\Sigma$

$\Rightarrow$ map asymptotic state $j$ to puncture $z_j$

$\Rightarrow$ worldsheet $\Sigma \Big|^{n\text{ pt}}_{\text{genus } h}$ contributes via

$$\int_{\mathcal{M}_{h;n}} \langle V_1(z_1) V_2(z_2) \ldots V_n(z_n) \underbrace{\begin{pmatrix}\text{ghosts}\\ \text{or}\\ \text{currents}\end{pmatrix}}_{} \rangle_\Sigma$$

correlator on $\Sigma$

$\mathcal{M}_{h;n}$: mod. space of $n$-punctured genus-$h$ surfaces

$V_1(z_1)$: "vertex operator", field on $\Sigma$ with info $\{k^\mu, \epsilon^\mu, \text{color}\}$

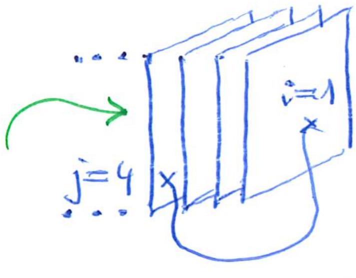
(ghosts or currents): depend on loop order and formulation (RNS or pure spinors)

Open-string endpoints $i, j$ $\Rightarrow$ $U(N)$ d.o.f.

$$|\text{open}(i,j)\rangle = \sum_{a=1}^{N^2} \underbrace{(T^a)_i{}^j}_{} |\text{open}; a\rangle$$

$(T^a)_i{}^j$: generator of $U(N)$, "Chan Paton factor"

N coincident D branes




Jargon: $T^a \leftrightarrow$ "color" & massless $|\text{open}; a\rangle \leftrightarrow$ "gluon"

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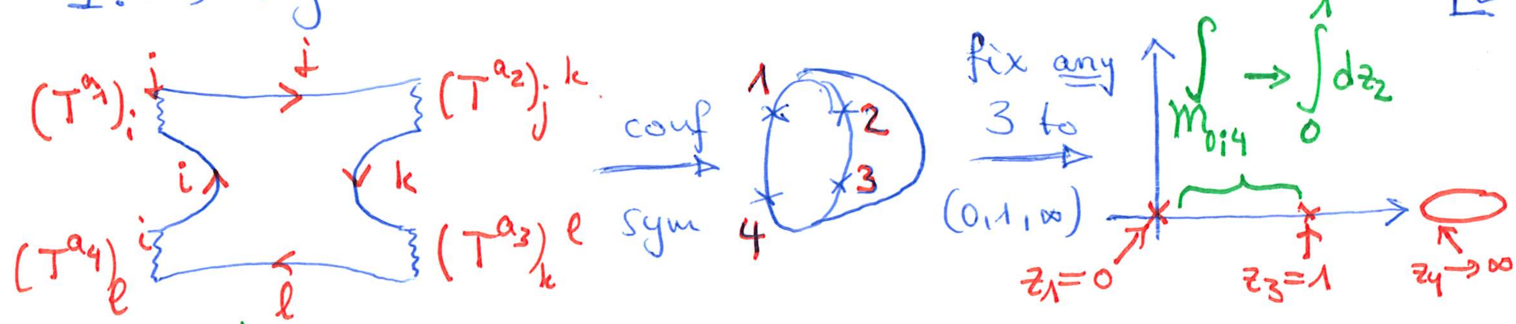
I) 4pt tree-lv warmup

I.1) 4 gluons on disk

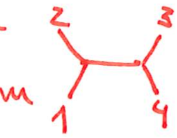



$\hookrightarrow$ color factor $\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$ interlocked with $z_1 < z_2 < z_3 < z_4$

integrand $\left\langle \prod_{j=1}^{4} V_j^{\text{gluon}}(z_j, \epsilon_j, k_j) \right\rangle_{\text{disk}} \xrightarrow{z_4 \to \infty} |z_2|^{s_{12}} |1-z_2|^{s_{23}} \left( \frac{n_s}{z_2} + \frac{n_t}{1-z_2} \right)$

$s_{ij} = \alpha'(k_i+k_j)^2$

fct. of $\epsilon_j, k_j$ "numerator" of SYM diagram

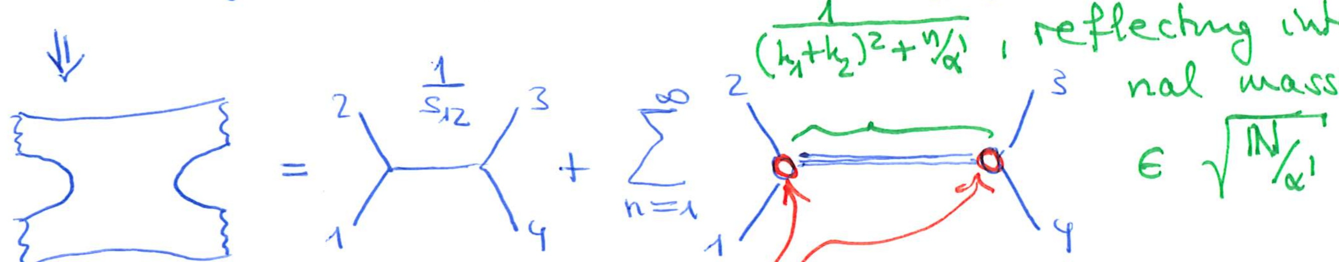



$$\Rightarrow A_{\text{open}}^{\text{tree}}(\underbrace{1,2,3,4}_{\#\,\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})}; \alpha') = n_s \underbrace{\int_0^1 \frac{dz_2}{z_2} z_2^{s_{12}} (1-z_2)^{s_{23}}}_{Z_s(s_{12}, s_{23})} + (s \leftrightarrow t)$$

- pole expansion

$$Z_s = \int_0^1 \frac{dz_2}{z_2} z_2^{s_{12}} \sum_{n=0}^{\infty} \binom{s_{23}}{n} (-z_2)^n = \sum_{n=0}^{\infty} \frac{(-)^n \binom{s_{23}}{n}}{s_{12}+n}$$

$\Downarrow$



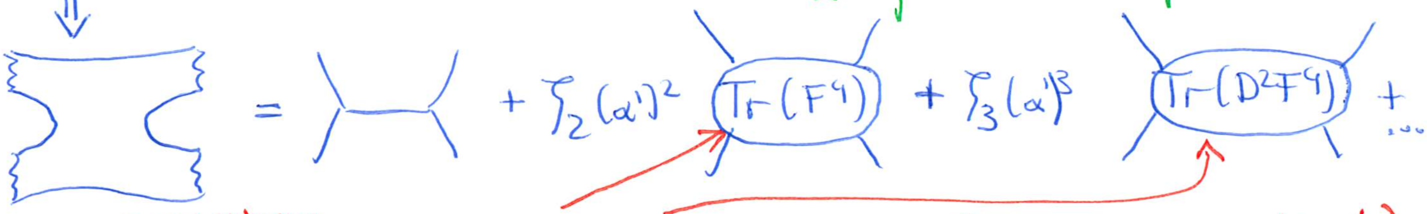

from $\binom{s_{23}}{n}$, up to $\partial^n$ @ mass level $n$ $\Rightarrow$ higher spin

- low-energy expansion

Riemann zeta $\zeta_n = \sum_{k=1}^{\infty} k^{-n}$ @ $n \geq 2$

$$Z_s = \frac{\Gamma(s_{12})\Gamma(s_{23}+1)}{\Gamma(s_{12}+s_{23}+1)} = \frac{1}{s_{12}} \underbrace{- \zeta_2 s_{23} - \zeta_3 s_{23} s_{13}}_{\text{no poles} \Rightarrow \text{4pt vertex}} + O(s_{ij}^3)$$

$\Downarrow$




$\int D\begin{bmatrix}\text{massive} \\ \text{string} \\ \text{modes}\end{bmatrix} \Rightarrow$ effective interactions (gauge covariantized)

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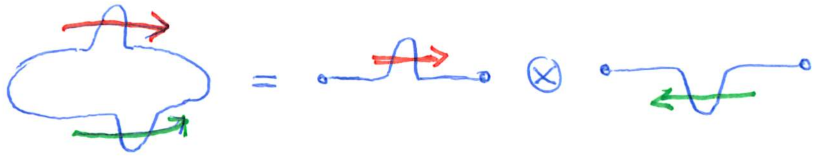
I.2) 4 gravitons on sphere

non-interacting on sphere / torus / etc.

vertex op's factorize $V^{\text{closed}}(\epsilon, \bar{\epsilon}, z, \bar{z}) = V^{\text{open}}(\epsilon, z) \otimes V^{\text{open}}(\bar{\epsilon}, \bar{z})$

closed-string spectrum is double copy



$$M^{\text{tree}}_{\text{closed}}(\{1,2,3,4\};\alpha') = \int_{\mathcal{M}_{0,4}} \Big\langle \prod_{j=1}^{4} V^{\text{graviton}}(z_j;\epsilon_j,\bar\epsilon_j,k_j)\Big\rangle_{\text{sphere}}$$

$(z_1,z_3,z_4)$
$\to (0,1,\infty)$

$$= \int_{\mathbb{C}\setminus\{0,1,\infty\}} d^2z_2\, \underbrace{|z_2|^{2s_{12}}|1-z_2|^{2s_{23}}\left(\frac{n_s}{z_2}+\frac{n_t}{1-z_2}\right)\left(\frac{\overline{n_s}}{\bar z_2}+\frac{\overline{n_t}}{1-\bar z_2}\right)}_{|\langle\prod_j V^{\text{gluon}}(z_j,\epsilon_j,k_j)\rangle_{\text{disk}}|^2}$$

Kawai-Lewellen-Tye (KLT) '86

$$= \underbrace{\sin(\pi s_{12})}_{\hookrightarrow \text{ string thy KLT kernel } S^{(\alpha')}_{4pt} \text{ [Henriette]}}\, A^{\text{tree}}_{\text{open}}(1,2,3,4;\alpha')\, \overline{A^{\text{tree}}_{\text{open}}}(1,2,4,3;\alpha')$$

$$= \frac{\Gamma(1+s_{12})\Gamma(1+s_{23})\Gamma(1+s_{13})}{\Gamma(1-s_{12})\Gamma(1-s_{23})\Gamma(1-s_{13})}\underbrace{\left(\frac{|n_s|^2}{s_{12}}+\frac{|n_t|^2}{s_{23}}+\frac{|n_s+n_t|^2}{s_{13}}\right)}_{M^{\text{tree}}_{\text{SUGRA}}(\{1,2,3,4\})}$$

$= n_u = -n_s - n_t$, "color-kinematics duality"

- low-energy expansion

$$\prod_{j=2}^{4}\frac{\Gamma(1+s_{1j})}{\Gamma(1-s_{1j})} = 1 \underbrace{- 2\zeta_3 s_{12}s_{13}s_{23} + \mathcal{O}(s_{ij}^5)}$$

cancels poles of $M^{\text{tree}}_{\text{SUGRA}}$

$\Rightarrow$ 4 vertices $\alpha'^3\zeta_3 R^4$, $\alpha'^5\zeta_5 D^4R^4$, etc.

- compare with open string

$$\begin{cases} A^{\text{tree}}_{\text{open}}(1,2,3,4;\alpha') = F(s_{ij})\, A^{\text{tree}}_{\text{SYM}}(1,2,3,4) \\ F(s_{ij}) = \dfrac{\Gamma(1+s_{12})\Gamma(1+s_{23})}{\Gamma(1+s_{12}+s_{23})} = \exp\left(\sum_{n=2}^{\infty}\dfrac{\zeta_n}{n}(-)^n\left[s_{12}^n+s_{23}^n-(s_{12}+s_{23})^n\right]\right)\end{cases}$$

$$\begin{cases} M^{\text{tree}}_{\text{closed}}(4pt) = \text{sv}\,F(s_{ij})\, M^{\text{tree}}_{\text{SUGRA}}(4pt) \\ \text{sv}\,F(s_{ij}) = \exp\left(-2\sum_{k=1}^{\infty}\dfrac{\zeta_{2k+1}}{2k+1}\left[s_{12}^{2k+1}+s_{23}^{2k+1}+s_{13}^{2k+1}\right]\right)\end{cases}$$

$$= F(s_{ij})\Big|_{\substack{\zeta_{2k}\to 0 \\ \zeta_{2k+1}\to 2\zeta_{2k+1}}}$$

refer to these observed rules as "sv"

comes from single-valued polylog's [see later]

# II) Why string amplitudes?

## II.1) Synergy with field-thy double copy

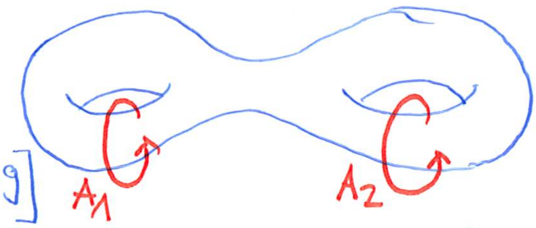
- gravity $\Longleftrightarrow$ (gauge thy)$^{\otimes 2}$ @ given loop order

  from $V^{\text{closed}} = V^{\text{open}} \otimes \overline{V^{\text{open}}}$ & $\alpha' \to 0$

  non-interacting on sphere / torus / etc.

- gravitational double copy @ fixed loop momentum

  $$\ell_I^\mu = \frac{1}{2\pi} \oint_{A_I} \partial_z X^\mu = -\frac{1}{2\pi} \oint_{A_I} \partial_{\bar z} X^\mu$$




  chiral splitting [D'Hoker, Phong '89]

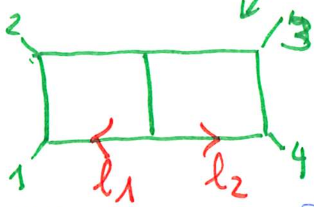
  double-copy $\mathcal{M}_{h,n}$ integrand in h-loop

  amplitudes @ fixed loop momenta $\ell_{I=1,2,\ldots,h}$

  $$M^{h\text{-loop}}_{\text{closed}} = \int_{\mathcal{M}_{h,n}} \int d^{h \cdot D}\ell \; \underbrace{F(z,\epsilon,k,\ell)}_{} \; \tilde{F}(\bar z, \bar\epsilon, k, \ell)$$

  $\alpha' \to 0$

  (meromorphic) chiral amplitude from $V^{\text{open}}(z,\epsilon,k)$




- even <u>open</u> strings are field-thy double copies!

## II.2) Exploring periods of moduli / config. spaces

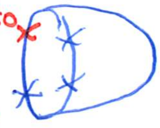
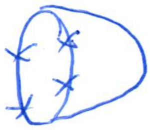
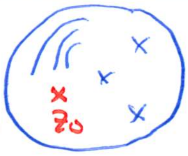
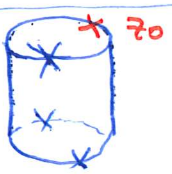
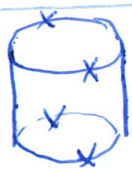
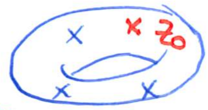
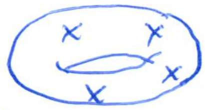
$\rightarrow$ which function spaces do we get when integrating one puncture after the other, order by order in $s_{ij}$ (i.e. $\alpha'$)

- MZVs $\zeta_{n_1,n_2,\ldots,n_r} = \sum_{0<k_1<k_2<\ldots<k_r} k_1^{-n_1} k_2^{-n_2} \ldots k_r^{-n_r}$

  with $n_j \in \mathbb{N}$ and $n_r \geq 2$

• MPLs $G(a_1,a_2,\ldots,a_w;z) = \int_0^z \frac{dt}{t-a_1}\, G(a_2,\ldots,a_w;t)$

$\Rightarrow \zeta_{n_1,\ldots,n_r} = (-1)^r\, G(\underbrace{00\ldots01}_{n_r}\ \ldots\ \underbrace{00\ldots01}_{n_1};z=1)$

| | open strings | | closed strings | |
|---|---|---|---|---|
| tree level | $\int$ not $z_0$  $\Downarrow$ multiple polylogarithms (MPL) | $\int$ all $z_j$  $\Downarrow$ multiple zeta values (MZVs) | $\int$ not $z_0$  $\Downarrow$ single-valued MPL [Brown '04] | $\int$ all $z_j$  $\Downarrow$ single-valued MZVs [Brown, Schnetz '13] |
| one loop | $\int$ not $z_0$  $\Downarrow$ elliptic polylogs [Brown, Levin '11] | $\int$ all $z_j$  $\Downarrow$ elliptic MZVs [Enriquez '13] | $\int$ not $z_0$  $\Downarrow$ elliptic MGFs, sv elliptic MPLs [DGP '18, DKS '20] | $\int$ all $z_j$  $\Downarrow$ modular graph forms (MGFs) [DGGV '15] |
| higher loops | | | higher-genus MGFs, modular graph tensors | |

elliptic MPLs : $\tilde{\Gamma}\begin{pmatrix} n_1 & n_2 & \ldots & n_r \\ z_1 & z_2 & \ldots & z_r \end{pmatrix}; z, \tau)$ in Claude's lecture

$\hookrightarrow$ elliptic MZVs $\omega(n_1 \ldots n_r; \tau)$ from $z=1$

MGFs : non-holomorphic modular forms

given $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

weight $(A,B)$

$\rightsquigarrow f(\gamma\cdot\tau) = (c\tau+d)^A (c\bar{\tau}+d)^B f(\tau)$

II.3) Explore & exploit string dualities

## III) Double copies @ string tree level

Recall $n$-point field-thy KLT [Henriette]

$$M^{\text{tree}}_{L\otimes R} = \sum_{\sigma,\rho}^{(n-3)!} A^{\text{tree}}_L(\sigma)\, \underbrace{S_0(\sigma|\rho)}\, A^{\text{tree}}_R(\rho)$$

no $\alpha'$, obtained via $\sin(\pi s_{ij}) \to k_i \cdot k_j$ from stringy KLT

$\Downarrow$

web of field-thy double copy

| R \ L | SYM | $\chi$PT | etc. |
|---|---|---|---|
| SYM | SUGRA | BI | -- |
| $\chi$PT | BI | special Galileon | -- |

now add $\alpha'$-dependent choices for $A_L$ / $A_R$
while preserving $S_0 \Leftrightarrow \alpha'$-independent kernel

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### III.1) Open superstring @ n points

Recall 4pt amplitude in form

$$A^{\text{tree}}_{\text{open}}(1,2,3,4;\alpha') = F(s_{ij})\, A^{\text{tree}}_{\text{SYM}}(1,2,3,4)$$

$$F(s_{ij}) = \int_0^1 dz_2 \, \frac{s_{12}}{z_{21}} \, |z_{21}|^{s_{12}} \, |z_{32}|^{s_{23}}$$

$z_{ab} = z_a - z_b$

$$= \frac{\Gamma(1+s_{12})\,\Gamma(1+s_{23})}{\Gamma(1+s_{12}+s_{23})} = 1 + \mathcal{O}(\alpha'^2)$$

- generalizes to $(n-3)!$ vector $A_{\text{SYM}}$ & matrix $F$

$$A^{\text{tree}}_{\text{open}}(1,\rho,n-1,n;\alpha') = \sum_{\sigma\in S_{n-3}} F_\rho{}^\sigma(s_{ij})\, \underbrace{A^{\text{tree}}_{\text{SYM}}(1,\sigma,n-1,n)}_{}$$

$\sigma(2,3,\ldots,n-2)$

$(n-3)!\times(n-3)!$ basis of disk int's: all $\alpha'$-dependence

$(n-3)!$ BCJ basis: all polarization dep.

$$F_\rho{}^\sigma = \int\limits_{0<z_{\rho(2)}<z_{\rho(3)}<\ldots<z_{\rho(n-2)}<1} dz_2\, dz_3 \ldots dz_{n-2} \prod_{1\leq a<b}^{n-1} |z_{ab}|^{s_{ab}}\ \sigma\left\{\prod_{m=2}^{n-2}\sum_{j=1}^{m-1}\frac{s_{mj}}{z_{mj}}\right\}$$

$$= \delta_\rho{}^\sigma + \mathcal{O}(\alpha'^2)$$

MZVs $\zeta_{n_1,\ldots,n_r}$ of weight $w=\sum_j n_j$ at order $\alpha'^w$, e.g. $\zeta_{3,5}$ @ $(\alpha')^8$ of $(n\geq5)$pt $F$'s

- under motivic coaction [Claude, Ruth]

$$\Delta(\zeta_{2k+1}) = \zeta_{2k+1}\otimes 1 + 1\otimes\zeta_{2k+1}$$

$$\Delta(\zeta_{3,5}) = \zeta_{3,5}\otimes 1 + 1\otimes\zeta_{3,5} - 5\,\zeta_3\otimes\zeta_5$$

the F-matrix lines up with $\Delta\left(\int_\gamma\omega\right) = \sum_j \int_\gamma \omega_j \otimes \int_{\gamma_j}\omega$

$$\Delta(F_\rho{}^\sigma) = \sum_{\tau\in S_{n-3}} F_\rho{}^\tau \otimes F_\tau{}^\sigma$$

no $\zeta_{2k}$ in 2nd entry

- open string in KLT form: rewrite $F$ via

$$Z(\rho|\sigma) = \int_{\rho(z_i<z_{i+1})} \frac{dz_1\, dz_2 \ldots dz_n}{\text{vol}\, SL_2(\mathbb{R})}\ \frac{\prod_{1\leq a<b}^{n}|z_{ab}|^{s_{ab}}}{\sigma(z_{12}z_{23}\ldots z_{n-1,n}z_{n1})}$$

cycle

form

↳ fix any $(z_i,z_j,z_k)=(0,1,\infty)$

$$\Rightarrow F_\rho{}^\sigma = \sum_{\tau\in S_{n-3}} S_0(\sigma|\tau)\, Z(\rho|1,\tau,n,n-1)$$

field-thy KLT kernel

$$\Rightarrow A^{\text{tree}}_{\text{open}}(\sigma) = \sum_{\tau,\rho\in S_{n-3}} Z(\sigma|\tau)\, S_0(\tau|\rho)\, A^{\text{tree}}_{\text{SYM}}(\rho)$$

↳ open superstring $\cong$ SYM $\otimes$ "Z theory"

- interpret $Z$ as doubly-partial amplitudes in eff. theory of bi-colored scalars $\phi = \phi_{a,\tilde{b}} T^a \otimes \tilde{T}^{\tilde{b}}$

$$\Box \phi_{a,\tilde{a}} = \underbrace{\tfrac{1}{2} f_{abc} \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}} \phi_{b,\tilde{b}} \phi_{c,\tilde{c}}}_{} + \underbrace{\zeta_2 (\alpha')^2 (\partial^2 \phi^3 + \phi^4)_{a,\tilde{a}}}_{}$$
$$+ \zeta_3 (\alpha')^3 (\partial^4 \phi^3 + \partial^2 \phi^4 + \phi^5)_{a,\tilde{a}} + \mathcal{O}(\alpha'^4)$$

e.o.m. of bi-adjoint scalars encoding vertices $\partial^2\phi^4, \phi^5$

$\Rightarrow$ field-thy limit $Z(\rho|\sigma) = \underbrace{m_{\phi^3}(\rho|\sigma)} + \mathcal{O}(\alpha'^2)$ doubly partial amplitude

- abelian limit $T^a \to \mathbb{1}$ ($\to$ subscript "ab")

open-string photons = Born-Infeld $+ \mathcal{O}(\alpha'^2)$

$$M^{\text{tree}}_{ab} = \sum_{\rho \in S_{n-1}} A^{\text{tree}}_{\text{open}}(1,\rho;\alpha')$$

do not sum over Parke-Taylor ordering

$$Z_{ab}(\sigma) = \sum_{\rho \in S_{n-1}} Z(1,\rho|\sigma)$$

NLSM of Goldstone bosons

- abelian Z-theory $\Rightarrow \chi PT + \mathcal{O}(\alpha'^2)$

$$Z_{ab}(\sigma) = A^{\text{tree}}_{\chi PT}(\sigma) + \mathcal{O}(\alpha'^2)$$

double copies with SYM to UV-completion of supersymmetrized Born-Infeld

$$M^{\text{tree}}_{ab} = \sum_{\sigma,\rho \in S_{n-3}} Z_{ab}(\rho) S_0(\rho|\sigma) A^{\text{tree}}_{\text{SYM}}(\sigma)$$

$\alpha' \to 0$ recovers double copy $BI = \chi PT \otimes SYM$

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## III.2) Closed superstring from sv(open superstring)

In passing from disk to sphere worldsheet

- no more cyclic ordering "$z_j < z_{j+1}$"
- independent Parke-Taylor factors in $z_{ij}$ & $\bar{z}_{ab}$

$\Rightarrow$ closed-string analogue of $Z(\rho|\sigma)$ is

$$J(\rho|\sigma) = \frac{1}{\pi^{n-3}} \int_{\mathbb{C}^n \setminus \{z_i = z_j\}} \frac{d^2 z_1 \ldots d^2 z_n}{\mathrm{vol}\, SL_2(\mathbb{C})} \frac{\prod_{1\le a<b}^{n} |z_{ab}|^{2s_{ab}}}{\rho(\bar z_{12} \bar z_{23} \ldots \bar z_{n1})\, \sigma(z_{12} z_{23} \ldots z_{n1})}$$

another $\alpha'$-uplift of bi-adj. $\phi^3$ since

$$J(\rho|\sigma) = m_{\phi^3}(\rho|\sigma) + \mathcal{O}(\alpha'^3)$$

- at 4pt, traditional string KLT is

$$J(1234|1243) = -\frac{\sin(\pi s_{12})}{\pi} Z(1234|1243)\, Z(1243|1234)$$

<u>but</u>: all $\zeta_{2k} \in \mathbb{Q}\pi^{2k}$ cancel between $Z$'s and

$$\sin(\pi x) = \pi x \exp\Big(-\sum_{n=1}^{\infty} \frac{\zeta_{2n}}{n} x^{2n}\Big)$$

- cleaned-up open-/closed-string relation

$$J(\rho|\sigma) = \mathrm{sv}\, Z(\rho|\sigma) \qquad \forall\ \rho,\sigma \in S_n$$

with "single-valued MZVs"

$$\mathrm{sv}\,\zeta_{2k} = 0\,, \qquad \mathrm{sv}\,\zeta_{2k+1} = 2\zeta_{2k+1}\,, \qquad \mathrm{sv}\,\zeta_{3,5} = -10\zeta_3\zeta_5$$

- implies field-thy KLT

$$M^{\mathrm{tree}}_{\mathrm{closed}}(\{1,2,\ldots,n\};\alpha') = \sum_{\sigma,\rho,\tau,\beta \in S_{n-3}} \overline{A^{\mathrm{tree}}_{\mathrm{SYM}}(\sigma)}$$

$$\times\, S_0(\sigma|\rho)\, \underbrace{J(\rho|\tau)\, S_0(\tau|\beta)\, A^{\mathrm{tree}}_{\mathrm{SYM}}(\beta)}$$

$$= \sum_{\sigma,\rho \in S_{n-3}} \overline{A^{\mathrm{tree}}_{\mathrm{SYM}}(\sigma)}\, S_0(\sigma|\rho)\, \mathrm{sv}\, A^{\mathrm{tree}}_{\mathrm{open}}(\rho)$$

$\Downarrow$

field-thy double copy SYM $\otimes$ sv(open superstring)

• origin of single-valued MZVs

$$\zeta_{n_1,\ldots,n_r} = G(\underbrace{00\ldots01}_{n_r}\ldots\ldots\underbrace{00\ldots01}_{n_1};z=1)\,(-1)^r$$

$$\mathrm{sv}\,\zeta_{n_1,\ldots,n_r} = G^{sv}(00\ldots01\ldots\ldots00\ldots01;z=1)\,(-1)^r$$

where single-valued polylogs $G^{sv}$

* have no monodromies around $z=0$ & $z=1$
* obey same $\partial_z$-eq. as $G(\ldots;z)$ [not $\partial_{\bar z}$]
* are built from $G(\ldots;z)$ & $G(\ldots;\bar z)$ & MZVs

• explicitly, for instance

$$G^{sv}(1;z) = \mathrm{sv}\log(1-z) = \log|1-z|^2$$

$$G^{sv}(0,1;z) = -\,\mathrm{sv}\,\mathrm{Li}_2(z) = -\mathrm{Li}_2(z) + \overline{\mathrm{Li}_2(z)} + \log(1-z)\log|z|^2$$

and more generally

$$G^{sv}(a_1,\ldots,a_w;z) = \frac{\bar z}{(-\pi)^w}\int_{\mathbb{C}^w\setminus\left\{\substack{z_i=z_j\\ z_i=a_j}\right\}} \frac{d^2z_1}{z_1-a_1}\frac{d^2z_2}{z_2-a_2}\ldots\ldots\frac{d^2z_w}{z_w-a_w} \times\frac{1}{(0-\bar z_1)\,\bar z_{12}\,\bar z_{23}\cdots\bar z_{w-1,w}(\bar z_w-\bar z)}$$

$\exists$ alternative constructions of $G^{sv}$ via coaction or via generating series of $G$ & $\bar G$

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## III. 3) Web of stringy double copies

Open bosonic strings have (scalar) tachyon @ $m^2 = -\frac{1}{\alpha'}$

Heterotic strings from (open bos) $\otimes$ (open superstrings)

$\hookrightarrow$ tachyon free

- bos / het tree amplitudes : replace $A^{\text{tree}}_{\text{SYM}}$ by new kin. factors with tachyon poles $\frac{1}{s_{ij}-1}$
  ⇝ produced by massive gauge thy

$$\mathcal{L}_{(DF)^2+YM} = \frac{1}{4\alpha'} F^a_{\mu\nu} F^{\mu\nu a} + \frac{1}{2}(D_\mu F^{\mu\nu}_a)^2 - \frac{f_{abc}}{3} F^\mu_{a\nu} F^\nu_{b\lambda} F^\lambda_{c\mu}$$
$$+ \frac{1}{2\alpha'}(\varphi^\alpha)^2 + \frac{1}{2}(D_\mu \varphi^\alpha)^2 + \frac{1}{2} C^{\alpha ab} \varphi^\alpha F^{\mu\nu}_a F_{b\mu\nu} + \frac{d^{\alpha\beta\gamma}}{6} \varphi^\alpha \varphi^\beta \varphi^\gamma$$

scalars in non-adjoint rep. of gauge group; they are not placed onto external legs

- web of double copies (field thy) ⊗ (stringy object)

coupling to bi-adj scalars $\phi$

| $\otimes$ | SYM | $(DF)^2+YM$ | $(DF)^2+YM+\phi^3$ |
|---|---|---|---|
| Z theory | open superstring | open bos string | comp. open bos string |
| sv (open superstring) | closed superstring | het string (gravity) | het. string (gauge & gravity) |
| sv (open bos string) | het string (gravity) | closed bos string | comp. closed bos string |

compactified spacetime dim's ⇒ additional color d.o.f.

## IV) Aspects of string loop amplitudes

next page: status report on massless chiral amplitudes $\mathcal{F}$ available in simplified form

$$\Rightarrow M^{h\text{-loop}}_{\text{closed}} = \int_{\mathcal{M}_{hin}} \int d^{h\cdot D}\ell \; \mathcal{F}(z,\epsilon,k,\ell)\, \tilde{\mathcal{F}}(\bar{z},\bar{\epsilon},k,\ell)$$

incl. 3h−3 cplx.-structure moduli $\tau_j$ for shape of genus-($h\geq 2$) surface (just one $\tau$ @ $h=1$)

| genus $h$ | 4 | 5 | 6 | 7 | $\geq 8$ |
|---|---|---|---|---|---|
| 3 | $\alpha'\to 0$ : '13, any $\alpha'$ : '21 | | | | |
| 2 | 2005 | '20 / '21 | | | |
| 1 | 1982 | bos : '89, SUSY : '13 | bos : '89, SUSY : '16 to '18 | | bos : '89 & '12 & '14 |
| 0 | late 60's | 2002 to 2009 | | | 2011 |

$\#(\text{legs})$

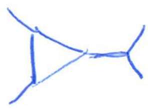
from ambi-twistor strings [Yvonne]

manifest colour-kinematics duality in SYM loop integrand in any $D \leq 10$

- @ 4pt genus 1 & $\alpha' \to 0$ : first computation of

$$A^{\text{1-loop}}_{\text{SYM}}(1,2,3,4) = s_{12}s_{23}\, A^{\text{tree}}_{\text{SYM}}(1,2,3,4) \int \frac{d^D\ell}{\ell^2(\ell+k_1)^2(\ell+k_1+k_2)^2(\ell-k_4)^2}$$

$$M^{\text{1-loop}}_{\text{SUGRA}}(\{1,2,3,4\}) = \left| s_{12}s_{23}\, A^{\text{tree}}_{\text{SYM}}(1,2,3,4)\right|^2 \times \left\{ \int \frac{d^D\ell}{\ell^2(\ell+k_1)^2(\ell+k_1+k_2)^2(\ell-k_4)^2} + \text{cyc}(2,3,4)\right\}$$

$\Rightarrow$ only  , no  or  ,

also @ n pt : string thy $\Rightarrow$ no-triangle property of max. SUSY field theories

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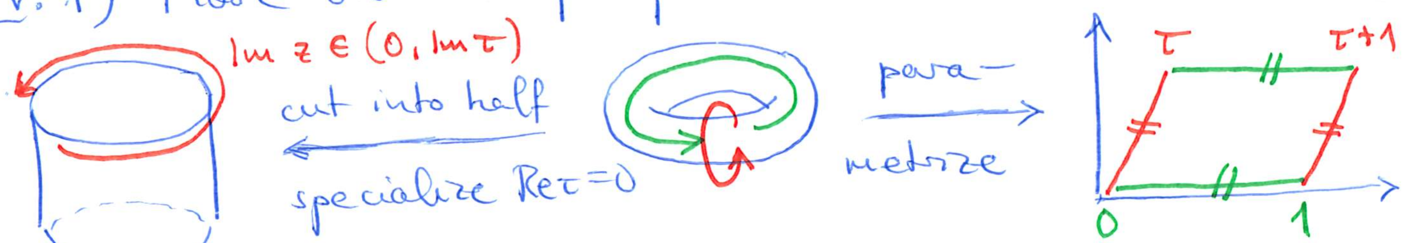
## IV. 1) More on 1-loop 4pt




$\mathrm{Tr}(T^1T^2T^3T^4)$ : $\underbrace{\text{planar cyl. \& Möbius strip}}_{\text{UV } \infty\text{-cancellation only for gauge group } SO(32)}$,

$\mathrm{Tr}(T^1T^2)\,\mathrm{Tr}(T^3T^4)$ : non-planar cylinder, UV-finite via closed-string exchange

$$A^{1\text{-loop}}_{\text{open}}(1,2,3,4) = \int_0^\infty dt \int_{\substack{0<\mathrm{Im}\, z_2<\mathrm{Im}\, z_3 \\ <\mathrm{Im}\, z_4<t}} dz_2\, dz_3\, dz_4 \int d^D\ell \sum_\nu Z_\nu(it) \times \Big\langle \prod_{j=1}^4 V^{\text{gluon}}_\nu(z_j,\epsilon_j,k_j)\Big\rangle^\nu_{\text{torus}}$$

fixed $z_1=0$

"spin structure", boundary conditions $\nu \in \{(\pm,\pm),(\pm,\mp)\}$ of worldsheet spinors (RNS) as $\psi^\mu(z+1)=\pm\psi^\mu(z)$ & $\psi^\mu(z+\tau)=\pm\psi^\mu(z)$

partition function, fingerprints of entire string spectrum in loop

- spin sum $\Longleftrightarrow$ spacetime SUSY (in RNS formalism)

$\Rightarrow$ eliminate 4 factors of $\frac{1}{z_{ij}}$ from $\langle \prod_j V(\ldots)\rangle$

$$\Rightarrow A^{1\text{-loop}}_{\text{open}}(1,2,3,4;\alpha') = s_{12}s_{23}A^{\text{tree}}_{\text{sym}}(1,2,3,4)\int_0^\infty dt \int_{0<\mathrm{Im}\, z_j<\mathrm{Im}\, z_{j+1}<t} dz_2\, dz_3\, dz_4$$

$$\boxed{\int d^D\ell \exp\Big(-2\pi\alpha' t\,\ell^2 - 4\pi\alpha' \sum_{j=1}^4 \ell\cdot k_j\, \mathrm{Im}\, z_j + \sum_{1\le a<b}^4 s_{ab}\log|\theta_1(z_{ab},it)|\Big)}$$

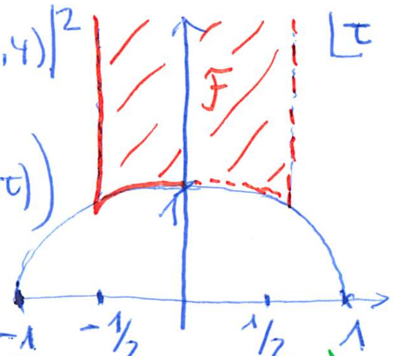
$\hookrightarrow$ Gaussian, produce $G_{\text{open}}(z,\tau) = \log|\theta_1(z,\tau)| - \frac{\pi(\mathrm{Im}\, z)^2}{\mathrm{Im}\,\tau}$ upon completing the square, eventually get

$$\sim \frac{1}{t^5}\exp\Big(\sum_{1\le a<b}^4 s_{ab}\, G_{\text{open}}(z_{ab},it)\Big)$$

- closed string: $\tau$ in fund domain $\mathcal{F}$ of $SL_2(\mathbb{Z})$

$$M^{1\text{-loop}}_{\text{closed}}(\{1,2,3,4\};\alpha') = |s_{12}s_{23}A^{\text{tree}}_{\text{sym}}(1,2,3,4)|^2$$

$$\times \underbrace{\int_{\mathcal{F}} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\Big(\prod_{j=2}^4 \int_{\text{torus}(\tau)} \frac{d^2 z_j}{\mathrm{Im}\,\tau}\Big)}_{\text{mod. invariant measures}} \exp\Big(\sum_{1\le a<b}^4 s_{ab}\, G_{\text{closed}}(z_{ab},\tau)\Big)$$

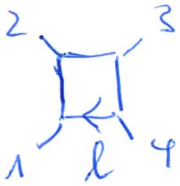



mod. invariant Green fct ($z=u\tau+v$ with $u,v\in(0,1)$)

$$G_{\text{closed}}(z,\tau) = \log\Big|\frac{\theta_1(z,\tau)}{\eta(\tau)}\Big|^2 - \frac{2\pi(\mathrm{Im}\, z)^2}{\mathrm{Im}\,\tau} = \frac{-\mathrm{Im}\,\tau}{\pi}\sum_{\substack{(m,n)\in\mathbb{Z}^2\\ \neq(0,0)}} \frac{\exp(2\pi i(mv-nu))}{|m\tau+n|^2}$$

- UV-finiteness of $M^{1\text{-loop}}_{\text{closed}}$ : $\ell$-integrand
  $\sim \exp\left(-4\pi\alpha' \,\mathrm{Im}\,\tau\, \ell^2\right)$ and $\mathrm{Im}\,\tau \geq \frac{\sqrt{3}}{2}$ everywhere in $\mathcal{F}$
  $\Rightarrow$ suppression in UV region $|\ell| \to \infty$
- $\alpha' \to 0$ : recover box integrals of SYM & SUGRA in the Schwinger parametrization

worldline length, inherited from $\alpha'\,\mathrm{Im}\,\tau$




$$\sim \int_0^\infty dT \, T^{3-D/2} \int_{0<u_2<u_3<u_4<1} du_2\, du_3\, du_4 \exp\Big(-T \sum_{1\leq a<b} k_a\cdot k_b \left(u_{ab}^2 - |u_{ab}|\right)\Big)$$

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## IV. 2) One-loop string amplitudes & elliptic polylogs

At $n \geq 5$ pt, chiral amplitudes $\rightsquigarrow$ Kronecker–Eisenstein coeff's $g^{(k)}(z,\tau)$ in [Claude]

$$F(z,\alpha,\tau) = \frac{\theta_1'(0,\tau)\,\theta_1(z+\alpha,\tau)}{\theta_1(z,\tau)\,\theta_1(\alpha,\tau)} = \sum_{k=0}^{\infty} \alpha^{k-1} g^{(k)}(z,\tau)$$

and holo Eisenstein series

$$G_k(\tau) = -g^{(k)}(0,\tau) = \sum_{\substack{(m,n)\in\mathbb{Z}^2\\ \neq (0,0)}} \frac{1}{(m\tau+n)^k}, \quad k \geq 4$$

e.g. $g^{(1)}(z_{ab},\tau)$ @ 5pt, $g^{(2)}(z_{ab})$ & $g^{(1)}(z_{ab})\,g^{(1)}(z_{cd})$ @ 6pt

- $\ell$-integral $\Rightarrow$ doubly periodic completion $f^{(k)}$ in

$$\exp\left(2\pi i \alpha \frac{\mathrm{Im}\, z}{\mathrm{Im}\,\tau}\right) F(z,\alpha,\tau) = \sum_{k=0}^{\infty} \alpha^{k-1} f^{(k)}(z,\tau)$$

e.g. $f^{(1)}(z,\tau) = \partial_z \log \theta_1(z,\tau) + 2\pi i \frac{\mathrm{Im}\, z}{\mathrm{Im}\,\tau} = \partial_z G_{\text{open/closed}}(z,\tau)$

• open-string $z_j$-integrals via elliptic polylogs :

map cylinder bdy to $(0,1)$, then use

$$\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \ldots & n_r \\ a_1 & a_2 & \ldots & a_r \end{smallmatrix} ; z,\tau\right) = \int_0^z dx\, f^{(n_1)}(x-a_1,\tau)\, \Gamma\left(\begin{smallmatrix} n_2 & \ldots & n_r \\ a_2 & \ldots & a_r \end{smallmatrix} ; x,\tau\right)$$

and $G_{\mathrm{open}}(z,\tau) = \Gamma\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} ; z,\tau\right)$ to expand 4pt t-integrand

$$\int_{0<z_2<z_3<z_4<1} dz_2\, dz_3\, dz_4 \exp\Big(\sum_{1\le a<b}^{4} s_{ab}\, G_{\mathrm{open}}(z_{ab},\tau)\Big) = \frac{1}{6}$$

$$+\, 2 s_{13}\, \omega(0,1,0,0;\tau) + 2(s_{12}^2+s_{23}^2)\,\omega(0,1,1,0,0;\tau)$$

$$-\, 2 s_{12} s_{23}\, \omega(0,1,0,1,0;\tau) + \mathcal{O}(\alpha'^3)$$

with eMZVs $\omega(n_1,\ldots,n_r;\tau) = \Gamma\left(\begin{smallmatrix} n_r & \ldots & n_1 \\ 0 & \ldots & 0 \end{smallmatrix} ; z=1,\tau\right)$

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## IV.3) Modular graph forms from closed strings

@ 1-loop n-pt, all $z_j$-integrands are lattice sums

$$f^{(k)}(z,\tau) = (-)^{k-1} \sum_{(m,n)\neq(0,0)} \frac{e^{2\pi i(mv-nu)}}{(m\tau+n)^k}$$

$$G_{\mathrm{closed}}(z,\tau) = -\frac{\mathrm{Im}\,\tau}{\pi} \sum_{(m,n)\neq(0,0)} \frac{e^{2\pi i(mv-nu)}}{|m\tau+n|^2}$$

$\Rightarrow$ straightforward Fourier integrals $\displaystyle\int_{\mathrm{torus}(\tau)} \frac{d^2z}{\mathrm{Im}\,\tau} = \int_0^1 du \int_0^1 dv$

$\Rightarrow$ (nested) lattice sum incl. non-holo Eisenstein ser.

$$E_k(\tau) = \Big(\frac{\mathrm{Im}\,\tau}{\pi}\Big)^k \sum_{(m,n)\neq(0,0)} \frac{1}{|m\tau+n|^{2k}} \quad , \quad k\ge 2$$

• multiple sums often simplify

$$\Big(\frac{\mathrm{Im}\,\tau}{\pi}\Big)^3 \sum_{\substack{(m_1,n_1),(m_2,n_2),\\(m_3,n_3)\neq(0,0)}} \frac{\delta(m_1+m_2+m_3)\,\delta(n_1+n_2+n_3)}{\prod_{j=1}^3 |m_j\tau+n_j|^2} \overset{\text{Zagier}}{=} E_3(\tau)+\zeta_3$$

- back to 4pt:

$$\left(\prod_{j=2}^{4}\int_{\text{torus}(\tau)}\frac{d^2 z_j}{\operatorname{Im}\tau}\right)\exp\left(\sum_{1\le a<b}^{4} s_{ab}\, G_{\text{closed}}(z_{ab},\tau)\right)$$

already @ $(\alpha')^4 \Rightarrow$ beyond $E_k(\tau)$

$$= 1 + E_2(\tau)(s_{12}^2+s_{13}^2+s_{23}^2) - (5E_3(\tau)+\zeta_3)\, s_{12}s_{13}s_{23} + \mathcal{O}(\alpha'^4)$$

more generally: "modular graph forms" (MGF)

- relating eMZVs & MGFs: can express both via iterated Eisenstein integrals to expose their rel's

$$\int_{\tau}^{i\infty} d\tau_1\, G_{k_1}(\tau_1)\,\tau_1^{j_1}\int_{\tau_1}^{i\infty} d\tau_2\, G_{k_2}(\tau_2)\,\tau_2^{j_2}\int_{\tau_2}^{i\infty}\dots$$

$0 \le j_i \le k_i - 2$

single-valued map open $\rightarrow$ closed strings proposed in 2010.10558, relying on $\int d\tau\, G_k$