Amplitudes from the Nodal Sphere

Exercise 1 *CHY amplitudes*

The first few exercises are a warm-up to get you familiar with the CHY representation.

(a) *Holomorphic δ-functions:* Let $z \in \mathbb{C}$ be our complex coordinate, and define

$$\bar{\delta}(z) := \frac{1}{2\pi i}\bar{\partial}\left(\frac{1}{z}\right) = \frac{1}{2\pi i} d\bar{z}\,\partial_{\bar{z}}\left(\frac{1}{z}\right) .$$

Show that $\bar{\delta}(z)$ acts like a holomorphic version of the standard Dirac δ-function by integrating it against a holomorphic test function $f(z)$, i.e. for a region Ω containing the origin, show that

$$\int_\Omega \bar{\delta}(z) \wedge f(z) dz = f(0) .$$

(b) *Scattering equations and the measure:* Show that the n scattering equations

$$\mathcal{E}_i := \sum_{j\neq i} \frac{k_i \cdot k_j}{\sigma_{ij}} , \tag{1}$$

transform covariantly under Möbius transformations on the support of momentum conservation. From this, we conclude that only $n-3$ of these constraints are independent, and you should calculate the Jacobian J^{SE} of fixing this symmetry,

$$\prod{}'\bar{\delta}(\mathcal{E}_j) := J^{\text{SE}}_{j_1 j_2 j_3} \prod_{j\neq j_{1,2,3}} \bar{\delta}(\mathcal{E}_j) ,$$

as well as the Jacobian J from the measure. Putting this all together, we find the following gauge-fixed form for the full CHY-measure;

$$d\mu_n^{\text{CHY}} := \frac{\prod_{i=1}^n \sigma_i}{\text{vol}\,\text{SL}(2,\mathbb{C})} \prod{}'\bar{\delta}(\mathcal{E}_j) = J_{i_1 i_2 i_3}\, J^{\text{SE}}_{j_1 j_2 j_3} \prod_{i\neq i_1,i_2,i_3} d\sigma_i \prod_{j\neq j_{1,2,3}} \bar{\delta}(\mathcal{E}_j) .$$

(c) *Three-particle amplitudes:* We can now easily verify the 3-particle amplitudes. Recall the CHY formula

$$\mathcal{A}_n = \int d\mu_n^{\text{CHY}}\, \mathcal{I}_n , \tag{2}$$

with

$$\mathcal{I}_n^{\text{BAS}} = \mathcal{C}_n\,\tilde{\mathcal{C}}_n , \qquad \mathcal{I}_n^{\text{YM}} = \mathcal{C}_n\,\text{Pf}'M , \qquad \mathcal{I}_n^{\text{grav}} = \text{Pf}'M\,\text{Pf}'\tilde{M} .$$

Check that this gives the correct 3-particle amplitudes in all three theories.

We're now going to do something that may look ambitious for an exercise: we'll *prove* the CHY amplitude representation. Of course we will not be able to fill in all the details here, but we'll be able to follow the main idea of the proof. Even more importantly, it'll allow us to draw conclusions about extensions of this worldsheet formula to loop representation in Exercise 2.

How do you go about proving a formula like (2)? Our main tool will be the BCFW recursion

$$\mathcal{A}_n = \sum_{L,\text{states}} \frac{1}{K_L^2}\, \mathcal{A}_{n_L+1}(z_L)\mathcal{A}_{n_R+1}(z_L)\,, \tag{3}$$

where the sum runs over factorization channels L (with $1 \in L$) such that $\hat{K}_L$ is on-shell,

$$\hat{K}_L = \sum_{i\in L} k_i + z_L q\,, \qquad z_L = -\frac{K_L^2}{2K_L \cdot q}\,.$$

We have chosen the deformation $\hat{k}_1 = k_1 + zq$, $\hat{k}_n = k_n - zq$, and the reference vector q has to satisfy $q^2 = k_1 \cdot q = k_n \cdot q = 0$.[1] Moreover, we used the notation $n_L = |L|$, and $n_R = n - n_L$. You have met this recursion already in both Henriette's and Jacob's lectures, so its usefulness here will not come as a surprise. However, instead of using it as a constructive tool for calculating a given amplitude, here we will use it to prove the CHY representation inductively. In part (c) of the exercise, you already checked that it reproduces the correct 3-particle 'seed' amplitudes; all that remains to show is that the CHY formula satisfies (3).

(d) *BCFW recursion in the CHY language:* Instead of proving that the CHY formulæ satisfy (3) directly, it will be easier to check the two properties used the derivation of the BCFW recursion: (i) factorization, and (ii) ... What is this second property? [2]

For factorization, argue that it is sufficient to investigate the behaviour of the CHY formulæ close to a boundary of the moduli space, where a subset of punctures collide, see fig. 1.

(e) *Scattering equations and $\partial\widehat{\mathfrak{M}}_{0,n}$:* Let's look at one specific divisor on the boundary of the moduli space where a subset L of the punctures 'bubbles off', $\mathfrak{M}_{0,n_L+1} \times \mathfrak{M}_{0,n_R+1} \subset \partial\widehat{\mathfrak{M}}_{0,n}$. We can parametrize Riemann surfaces close to this boundary component by

$$\sigma_i = \sigma_L + \varepsilon x_i\,, \qquad i \in L\,. \tag{4}$$

Here σ_L labels the 'junction point' between the two spheres, and x_i serve as coordinates on the new sphere, see fig. 1. Show that the scattering equations then relate this boundary component to the singular kinematic configuration $K_L^2 = 0$, i.e. show

$$K_L^2 \sim \varepsilon \qquad \text{with } K_L = \sum_{i\in L} k_i\,.$$

Bonus: *One loop:* It is now straightforward to extend this argument to one loop: show that the one-loop scattering equations $E_A^{(1)}$ on the nodal sphere relate boundaries of the moduli space (parametrized as above) to singular kinematic configurations

$$\begin{aligned} &\varepsilon \sim K_L^2 && \text{for } \sigma_+, \sigma_- \in \Sigma_R \\ &\varepsilon \sim 2\ell \cdot K_L + K_L^2 && \text{for } \sigma_+ \in \Sigma_L\,,\ \sigma_- \in \Sigma_R\,. \end{aligned}$$

[1] For theories with spin, we need to impose further conditions on q. Of course, this plays an important role in proving the CHY formulæ, but we won't need the details in this exercise.

[2] Just *identify* the property, we will not prove it here. The calculation is straightforward but lengthy.

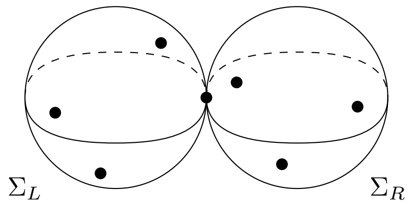


Figure 1: *The boundary divisor* $\mathfrak{M}_{0,n_L+1} \times \mathfrak{M}_{0,n_R+1} \subset \partial\widehat{\mathfrak{M}}_{0,n}$. A subset I of the punctures colliding as in (4) is equivalent to them 'bubbling off' on a separate sphere, here labeled Σ_L. Our notation conventions are $x \in \Sigma_L$, and $\sigma \in \Sigma_R$. In particular, the 'junction point' is σ_L from the perspective of Σ_R on the RHS, and $x = \infty$ from Σ_L.

(f) Bonus: *Measure and integrand:* Use your result of (e) to show that on the boundary

$$d\mu_n^{\mathrm{CHY}} = \varepsilon^{2(n_L-1)} \frac{d\varepsilon}{\varepsilon}\, \bar{\delta}\left(K_L^2 + \varepsilon \mathcal{F}\right)\, d\mu_{n_L+1}^{\mathrm{CHY}}\, d\mu_{n_R+1}^{\mathrm{CHY}}\,,$$

where $\mathcal{F}$ is of order one in ε, and we won't need its precise form. This tells us that the amplitude vanishes on the boundary, unless we find a compensating factor

$$\mathcal{I}_n = \varepsilon^{-2(n_L-1)} \sum_{\text{states}} \mathcal{I}_{n_L+1} \mathcal{I}_{n_R+1}\,, \tag{5}$$

from the CHY integrand. Show that this relation holds for bi-adjoint scalar amplitudes, and conclude that they therefore satisfy the BCFW recursion.[3] In the course of this, we also learn which poles contribute for partial amplitudes — can you give a simple criterion?

And a final bonus exercise: *Scattering equations again:* What we've seen above is that the scattering equations relate boundaries of the moduli space to singular kinematics. But what about the other way round? Is every solution to the scattering equations of the form (4) for $K_L^2 \sim \varepsilon$? How does this relate to the factorization argument in (e)?

[3] It is straightforward if lengthy to show that (5) also holds for integrands involving the reduced Pfaffian, which completes the proof also for Yang-Mills theory and gravity.

Exercise 2 *Scattering equations beyond one loop*

For two loops, the scattering equations on the nodal sphere can be derived systematically: first calculate the RNS ambitwistor string correlator on a genus-two Riemann surface, and then use residue theorems to derive a formula on a bi-nodal sphere. However, we can also understand their form from a more physical perspective: they have to encode the correct pole structure for the linearized integrand representation. In this exercise, we work this out in more detail.

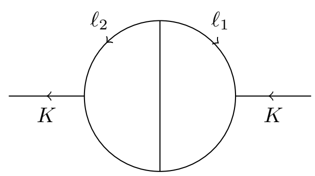
(a) *Poles in the linear integrand representation:* Of course we first have to understand what poles actually appear in the linear integrand representation. For this, let us repeat the residue theorem we discussed in the lectures, but now at two loops (and with an eye towards a general formulation): consider first the following shift of the loop momenta

$$\ell_I \to \ell_I + \eta_I \qquad \text{for } \ell_I \cdot \eta_J = k_i \cdot \eta_J = 0\,.$$

Here $I = 1, 2$, and we take η to be a $(d+2)$-dimensional vector. Now, whereas at one loop, there was only one natural variable, $\eta^2 = z$, here we have three,

$$z_1 = \eta_1^2\,, \qquad z_2 = \eta_2^2\,, \qquad z_{12} = \eta_1 \cdot \eta_2\,.$$

Nice integrand representations however will in general not come from this choice of variables, but rather from linear combinations of them that are suited for the parametrization of the Feynman diagram we started out with. To make this clear, consider the following diagram



With loop momenta assigned as given, a *good* choice of variables for the residue theorem is

$$z_1 = \eta_1^2\,, \qquad z_2 = \eta_2^2\,, \qquad z_3 = \big(\eta_1 + \eta_2\big)^2 = z_1 + z_2 + 2z_{12}\,.$$

Good here means that we can easily classify the types of poles that appear. For the scattering equations at two loops, we will now want to perform *two* residue theorems, in z_1 and z_2. First, make sure that you understand why we still had to specify all three variables $z_{1,2,3}$. Then, perform the residue argument for the diagram above (with trivial numerator and $K^2 \neq 0$), and shift the loop momenta in the different terms such that you can extract an overall factor of $1/(\ell_1^2 \ell_2^2)$. What poles do you find in the integrand?

(b) *Two-loop scattering equations I:* Consider now the following generalization of the tree-level scattering equations to the bi-nodal sphere,

$$E_{I^\pm} = \sum_{J \neq I^\pm} \frac{L_{I^\pm J}}{\sigma_{I^\pm J}} \pm \sum_{i=1}^{n} \frac{2\ell_I \cdot k_i}{\sigma_{I^\pm i}}\,, \qquad\qquad E_i = \sum_{J^\pm} \frac{\pm 2\ell_J \cdot k_i}{\sigma_{iJ^\pm}} + \sum_{j \neq i} \frac{2k_i \cdot k_j}{\sigma_{ij}}$$

Here, the sums J run over the nodal points $1^\pm$, $2^\pm$; and $L_{IJ} = L_{(IJ)}$ are parameters that we want to determine. Since these equations are defined on the (bi-nodal) sphere, we still want them to transform covariantly under Möbius transformations as in exercise 1 (b). What condition does this impose on the parameters L?

(c) *Two-loop scattering equations II:* We will now fix the remaining freedom we have in the scattering equations by requiring that they encode the correct pole structure for a linearized integrand. For this, we first need to extend the relation we found in exercise 1 (e) to these more general scattering equations. Show that the scattering equations now relate a boundary divisor $\mathfrak{M}_{0,\tilde{n}_L+1} \times \mathfrak{M}_{0,\tilde{n}_R+1}$ to the following kinematic configuration[4]

$$\tilde{s}_L := \sum_{I^\pm, J^\pm \in L} L_{I^\pm J^\pm} + \sum_{I^\pm \in L} \big(\pm 2\ell_I \cdot K_L \big) + K_L^2 \,. \tag{6}$$

Here, we used the notation $\tilde{n} = n + 4$ to include the nodal points. Next consider the following boundary components L, where i and j are arbitrary external particles:

(i) $L = \{1^+, 1^-, i, j\}$ and $L = \{2^+, 2^-, i, j\}$
(ii) $L = \{1^+, 2^+\}$

By demanding that $\tilde{s}_L$ agrees with the expected propagators K_L^2 for factorization channels involving only external momenta, and agrees with the 'linear' propagators found in (a), determine from this the form of the two-loop scattering equations. Part (i) also gives you a physical interpretation for why we found the linear integrand representation at one loop!

(d) **Bonus:** *Unphysical poles:* Now that we have understood the form of the scattering equations at two loops, can we also say something about the other new ingredient we saw in the lectures; the cross-ratio

$$c^{(2)} = \frac{\sigma_{1^+2^-}\sigma_{1^-2^+}}{\sigma_{1^+1^-}\sigma_{2^+2^-}} \,,$$

appearing as a factor in the integrand? To see what role this plays, consider now the boundary component $L = \{1^+2^-, i, j\}$ of the moduli space. What pole do you find from (6) and your explicit form of the two-loop scattering equations? As you can see, this has no equivalent in the linear representation we discussed in part (a), and is thus unphysical. Can you understand from our factorization discussion in exercise 1 (e) why this cross-ratio ensures that these poles cannot appear? (Under reasonable assumptions for the two-loop integrand $\mathcal{I}_n^{(2)}$.)

There is one aspect that you may have noticed that I didn't have time to mention in the lectures: What if we had reversed the orientation of one of the loop momenta in our diagram? We could have repeated all the same steps, but arrived at different scattering equations (and also a different cross-ratio). Indeed, both of these give valid representations, and you can choose to work in either.

[4] This is the reason we took $L_{(IJ)}$ to be symmetric. While it's a reasonable assumption when generalizing from tree-level, most importantly it is what makes this calculation here feasible. If we relax our assumption and consider generic L_{IJ}, it is much more difficult to find the singular kinematic configuration corresponding to the boundary of the moduli space. As far as I know, no-one even looked at this.

Choose **one** of the exercises 3 and 4:

Exercise 3 *Ambitwistor string CFT and the gluing operator*

In this exercise, we'll first verify some of the CFT calculations from the lectures.

(a) *$\beta\gamma$-systems:* Consider for now a general (bosonic) $\beta\gamma$-systems, with conformal weights $h_\beta = h$ and $h_\gamma = 1-h$. Verify that the stress-energy tensor

$$T = -h\,\beta\partial\gamma + (1-h)(\partial\beta)\gamma\,,$$

has the expected OPEs with the fields β, γ, and calculate the central charge anomaly.

Bonus: *Central charge anomaly:* Use this to verify the critical dimensions in the three models we discussed.

(b) *BRST and vertex operators:* Let us focus for now on the bi-adjoint scalar model. Recall that in the BRST quantization, we introduced the BRST operator

$$Q = \oint cT + \tilde{c}H\,.$$

Verify that after integrating out the Nakanishi-Lautrup fields, the effective BRST operator takes the form

$$Q_{\text{eff}} = \oint cT + \frac{\tilde{c}}{2}P^2\,,$$

where $T = T_{PX} + T_{bc} + T_{\tilde{b}\tilde{c}}$ is now the full chiral stress-energy tensor of the theory. Check moreover the assertions from the lectures about the spectrum, i.e that for vertex operators of the form

$$V(\sigma) = c\tilde{c}\,w(\sigma)\,e^{ik\cdot X(\sigma)}\,,$$

w has to have conformal weight $h_w = (2,0)$, and $k^2 = 0$.

After this warm-up, we can now explore the nodal operator construction. Remember from the lectures that the gluing operator encodes the target space propagator of the theory, and therefore has to be non-local if we want to require BRST invariance. We constructed the gluing operator Δ as

$$\Delta(\sigma_+, \sigma_-) = \int \frac{d^d\ell}{\ell^2}\,W(\sigma_+, \sigma_-)\,\sum_{\text{states}} \mathcal{O}(\sigma_+)\mathcal{O}(\sigma_-)\,.$$

Here, $\mathcal{O}(\sigma_\pm)$ are the trivial off-shell extensions of the vertex operators with momentum $\pm\ell$,

$$\mathcal{O}(\sigma_\pm) = c\tilde{c}\,w(\sigma_\pm)\,e^{\pm i\ell\cdot X(\sigma_\pm)}\,,$$

and encode the off-shell state flowing through the node. In the bi-adjoint scalar, we could implement the sum over states very explicitly (how?), but since there are subtleties associated to the correlator calculation, we won't bother with this here.[5] The non-local factor W takes the form

$$W(\sigma_+, \sigma_-) = \exp\left(\frac{1}{2\pi}\int_\Sigma \frac{\tilde{e}}{2}\,\ell^2\omega_{+-}^2(\sigma)\right)\,, \qquad \text{with } \omega_{+-}(z) = \frac{\sigma_{-+}\,d\sigma}{(\sigma-\sigma_+)\,(\sigma-\sigma_-)}\,.$$

[5] If you are interested: already at tree-level, we have to restrict (by hand) to the single-trace sector for current algebras with level $\kappa \neq 0$. We find the same issue in these calculations, but now we have to restrict to single-trace contributions *after* the sum over states. All of these subtleties disappear for the RNS model, but I didn't want to burden you with the extra fermions.

We claim that this is precisely the factor needed to make Δ BRST-invariant. Notice however the explicit factor of $\tilde{e}$. So if we want to check BRST invariance, we have to work with Q, not Q_{eff} — otherwise we would have already integrated out $\tilde{e}$! This can in principle be done, but in practice it is much more straightforward to quantize the theory in the presence of Δ, and derive a new effective BRST operator Q^{Δ}_{eff}.

(c) *BRST in presence of the nodal operator:* Repeat the calculation for the effective BRST operator, but now including the contribution from the gluing operator Δ. You should find

$$Q^{\Delta}_{\text{eff}} = \oint cT + \frac{\tilde{c}}{2}\left(P^2 - \ell^2 \omega_{+-}^2(\sigma)\right) .$$

Convince yourself that the vertex operators remain in the BRST cohomology.

(d) *Scattering equations on the nodal sphere:* Repeat the analogue of our derivation for the scattering equations, but now in the presence of the gluing operator. You should recover the one-loop scattering equations discussed in the lecture,

$$E_A^{(1)} = \text{Res}_{\sigma_A}\left(P^2 - \ell^2 \omega_{+-}^2\right) , \qquad A = 1, \dots, n, +, - .$$

You could now wonder about extensions of this to two loops — after all, we just discussed the two-loop scattering equations in the last exercise! However, if even a brief look at our results there should convince you that we cannot simply insert two copies of our gluing operator. So how could it be modified? It's not hard to see what the answer should be to get the correct scattering equations, but why? This certainly goes well beyond BRST invariance. Moreover, where does the cross-ratio $c^{(2)}$ come from? If you know how to solve any of these questions, you're ahead of the field and should publish your answer.

Exercise 4 *Ambitwistor strings in 4d*

In this exercise, we'll take a closer look at another incarnation of the ambitwistor string – the 4d twistor model.[6] We'll start by learning some basics about twistor space,[7] which will allow us to finally understand the origin of the name 'ambitwistor'. We'll then transform our ambitwistor string action to twistor variables, and conclude by taking a closer look at what twistor theory tells us about the vertex operators.

(a) *Twistor space and the incidence relations:* For our purposes, twistor space (for 4d Minkowski space $\mathbb{M}_{\mathbb{C}}$) will be an open subset $\mathbb{PT} \subset \mathbb{CP}^3$, for which we introduce homogeneous coordinates Z^A. It will be convenient to split Z into two Weyl spinors of opposite chirality,

$$Z^A = (\mu^{\dot\alpha},\, \lambda_\alpha)\,.$$

To make this interesting and useful, we have to relate twistor space and space-time. To do so, we impose the following '*twistor correspondence*' relation between twistors Z and points $x^{\alpha\dot\alpha} = x^\mu \sigma_\mu^{\alpha\dot\alpha} \in \mathbb{M}_{\mathbb{C}}$;

$$\mu^{\dot\alpha} = x^{\alpha\dot\alpha}\lambda_\alpha\,.$$

This non-local relationship is often referred to as the *incidence relations*. The goal of this exercise is to understand these incidence relations in some detail: What do they tell us about what a point $x \in \mathbb{M}_{\mathbb{C}}$ looks like in twistor space? What does a point in twistor space (i.e. a fixed twistor Z) correspond to in space-time?

(b) *Ambitwistor space in 4d:* We can interpret twistors Z^A as spinors for the conformal group $\mathrm{SO}(6,\mathbb{C}) \simeq \mathrm{SL}(4,\mathbb{C})$. From this perspective, it is clear that there is a natural *dual twistor space* $\mathbb{PT}^* \subset \mathbb{CP}^3$ as spinors of the opposite chirality $W_A := (\tilde\lambda_{\dot\alpha}, \tilde\mu^\alpha)$. There is a natural inner product between twistors and dual twistors, given by

$$Z\cdot W = [\mu\tilde\lambda] + \langle\tilde\mu\lambda\rangle\,,$$

where we used the standard notation for Weyl spinor index contractions. Let us also introduce dual incidence relations

$$\tilde\mu^\alpha = -x^{\alpha\dot\alpha}\tilde\lambda_{\dot\alpha}\,.$$

We can now understand where the name 'ambitwistor' comes from: Show that in 4d, projective ambitwistor space can be understood as the quadric $Z\cdot W$ inside $\mathbb{PT}\times\mathbb{PT}^*$, i.e.

$$\mathbb{PA} = \left\{(Z,W) \in \mathbb{PT}\times\mathbb{PT}^* \middle|\, Z\cdot W = 0\right\}\,.$$

Comparing this with the vector representation discussed in the lectures, what is the relation between $P_{\alpha\dot\alpha}$ and the twistors/dual twistors?

[6] At tree-level, this worldsheet model can be understood as a twisted version of the Berkovits-Witten twistor string, which lies at the root of many beautiful results for $\mathcal{N}=4$ super Yang-Mills amplitudes. Historically, the twistor string predates all of the models we discussed, and was certainly an important motivation for even considering worldsheet models. If you are interested in learning more, reviews and the original papers can be found in the reference list.

[7] We will barely scratch the surface here. If you want to learn more, I can recommend Tim's lecture notes, see the list of references.

(c) *Ambitwistor string action:* Using the relation for P you found in (b) as well as the incidence relations, show that the kinematic term in the (bosonic) ambitwistor string action becomes[8]

$$S = \int W \cdot \bar{\partial} Z - e\, W \cdot \partial Z\,.$$

To complete this to the full ambitwistor string, we still need to gauge the constraint $Z \cdot W$ which we assumed in deriving the relation between P and the twistor variables,

$$S_{\mathbb{A}} = \int W \cdot \bar{\partial} Z - e\, W \cdot \partial Z + a\, Z \cdot W\,.$$

Assuming that twistors and dual twistors appear on an equal footing, what conformal weights should we assign to the fields? [9]

Congratulation, at this point you have derived the 'bare' action for the 4d ambitwistor string. For the critical model describing $\mathcal{N} = 4$ super Yang-Mills theory, we would need to (i) promote the twistors to supertwistors and (ii) add a current algebra. However, instead of going through this construction in detail, let us take a look at the basic structures we expect to find in vertex operators.

(d) *Momentum eigenstates:* It turns out that there are two types of vertex operators in this model (corresponding to positive and negative helicity states),[10]

$$\begin{aligned}
\mathcal{V}_i &= \int \frac{ds_i}{s_i}\, \bar{\delta}^2\big(\kappa_i^\alpha - s_i \lambda^\alpha(\sigma_i)\big)\, w\, e^{is_i[\mu(\sigma_i)\tilde{\kappa}_i]} \\
\tilde{\mathcal{V}}_i &= \int \frac{ds_i}{s_i}\, \bar{\delta}^2\big(\tilde{\kappa}_i^{\dot\alpha} - s_i \tilde{\lambda}^{\dot\alpha}(\sigma_i)\big)\, w\, e^{is_i\langle\tilde{\mu}(\sigma_i)\kappa_i\rangle}\,.
\end{aligned}$$

Here, the momentum of the particle is given in spinor-helicity notation by $k_i^{\alpha\dot\alpha} = \kappa_i^\alpha \tilde{\kappa}_i^{\dot\alpha}$. Show that on the support of the delta-functions and the incidence relations, the exponential factors indeed give the standard momentum eigenstate.

These vertex operators have in fact a very intuitive interpretation from twistor space: they are the natural pull-back to the worldsheet of free fields! This relies on a particularly beautiful result, the Penrose transform, which tells us that massless free fields on spacetime correspond to *cohomology classes* on twistor space. If you are interested in learning more, Tim's lecture notes are a great place to start, and you'll find a detailed discussion on momentum eigenstates in twistor space (in the form of an exercise) at the end of section 3. Instead, we'll look here at what the form of these vertex operators implies for correlators and amplitudes in these models.

(e) **Bonus:** *Reduced scattering equations:* Consider now (schematically) a correlator of the form

$$\left\langle \prod_{i=1}^{k} \tilde{\mathcal{V}}_i \prod_{p=k+1}^{n} \mathcal{V}_p \right\rangle .$$

[8] I've dropped the overall factor of $(2\pi)^{-1}$ for convenience later in the exercise

[9] For the *twistor* string, this is precisely the requirement we would change.

[10] As in the lectures, I've hidden some factors in w: it contains the current algebra contribution $t^{\mathfrak{a}} j_{\mathfrak{a}}$ familiar from the lectures, as well as some ghost and supersymmetry contributions.

If we had done the calculation carefully, this would give an N^{k-2}MHV amplitude in maximal super Yang-Mills. Even in our simplified format however, we can draw some important conclusions. Assume that we have BRST-quantized the theory (or do it, if you feel ambitious), and are left with a free action after gauge-fixing and integrating out all of our Nakanishi-Lautrup fields, i.e. $S_{\mathbb{A}}$ with $e = a = 0$ and additional ghost contributions. Assume further that w is independent of all twistor and dual twistor variables. We can then perform the same trick as in the lectures to integrate out the ZW-system. Follow this procedure, and derive in this manner the *reduced scattering equations* for $i = 1, \ldots, k$ and $p = k+1, \ldots, n$;

$$s_i\, \tilde{\lambda}^{\dot{\alpha}}(\sigma_i) = \tilde{\kappa}_i^{\dot{\alpha}}\,, \qquad\qquad s_p\, \lambda^{\alpha}(\sigma_p) = \kappa_p^{\alpha}\,,$$

where

$$\tilde{\lambda}^{\dot{\alpha}}(\sigma) = \sum_{p=k+1}^{n} \frac{s_p\, \tilde{\kappa}_p^{\dot{\alpha}}}{\sigma - \sigma_p}\,, \qquad\qquad \lambda^{\alpha}(\sigma) = \sum_{i=1}^{k} \frac{s_i\, \kappa_i^{\alpha}}{\sigma - \sigma_i}\,.$$

Check that these imply the familiar scattering equations (1).

If we had gone through the above calculation with more care, we would have found the renown RSVW formulæ (Roiban-Spradlin-Volovich-Witten) for maximal super Yang-Mills. However, the whole calculation including the underlying twistor theory is easily enough to fill another lecture series. I hope I managed here to give you at least a small taste, and enough background to explore the literature on your own.