

The Amplitude Games

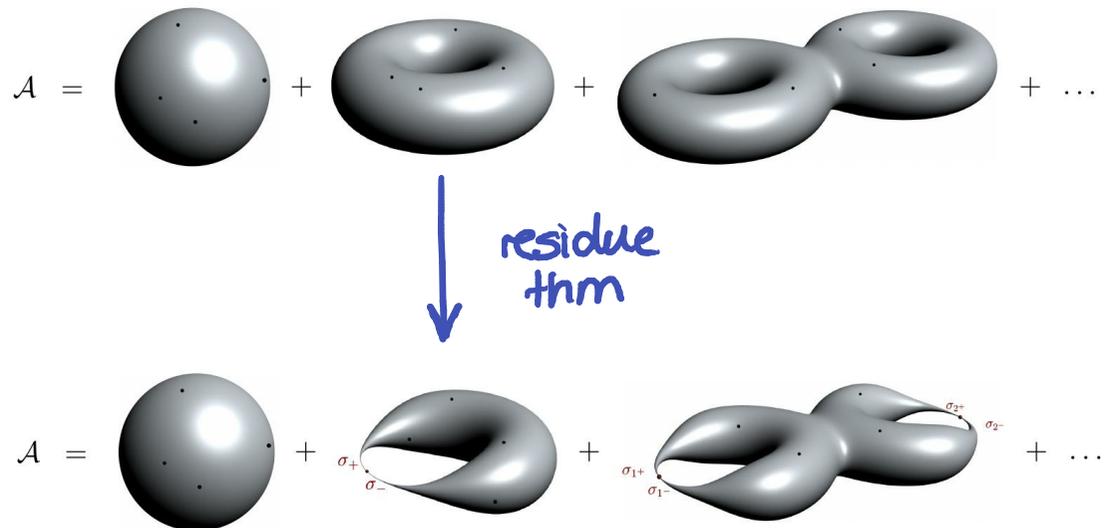
MITP School 2021

AMPLITUDES FROM THE NODAL SPHERE

$$A = \text{Sphere} + \text{Pinched Sphere} + \text{Pinched Torus} + \dots$$

V. Loop integrands from the nodal sphere

Last lecture:



Goals this lecture:

- 1) other theories
- 2) understand integrand rep.
- 3) Misc. topics

1) Beyond 10d supergravity

↳ Recall: heterotic spectrum \supset higher order grav.
 ↳ how to get YM? NS grav? ...

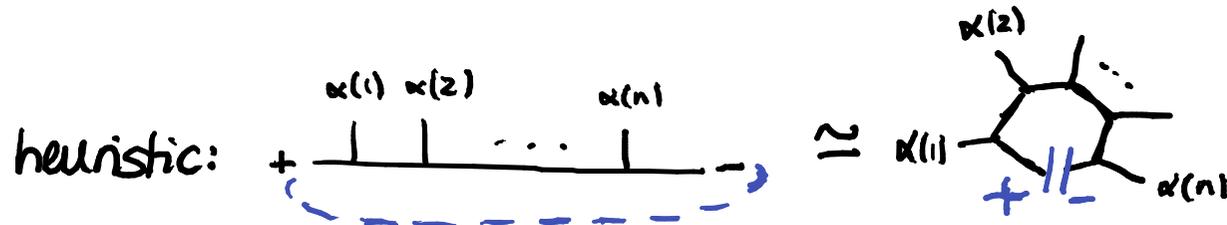
Idea: SINGLE COPY to get (s) YM

$$\mathcal{F}_n^{(1)} = \frac{1}{\ell^2} \int_{M_{n+2}} d\mu_n^{(1)} \mathbb{I}_{\frac{1}{2}}^{(1)} \tilde{\mathbb{I}}_{\frac{1}{2}}^{(1)}$$

with
$$\mathbb{I}_{\text{usy},d}^{(1)} = \underbrace{\sum_r \text{Pf}' M_{NS}^r}_{\mathbb{I}_{NS,d}^{(1)}} - \frac{c_d \text{Pf} M_2}{\sqrt{v_{+-}}}$$

$$\begin{array}{ll} c_0 = 8 & c_6 = 2 \\ c_8 = 8 & c_4 = 2 \end{array}$$

Colour factor:
$$C_n = \sum_{\alpha \in S_n} \frac{C_{(\alpha)}^{(1)}}{(+\alpha-)}$$
 with $C_{(\alpha)}^{(1)} = C_{(+\alpha-)}^{(0)} \delta^{a_+ a_-}$



One-loop single (& 0th) copy on the WS

$$I_{\text{BAS}}^{(1)} = C_n^{(1)} \tilde{C}_n^{(1)}$$

$$I_{\text{SYM}}^{(1)} = C_n^{(1)} I_{\text{susy}}^{(1)}$$

$$I_{\text{YM}}^{(1)} = C_n^{(1)} I_{\text{NS}}^{(1)}$$

$$I_{\text{sigra}}^{(1)} = I_{\text{susy}}^{(1)} \tilde{I}_{\text{susy}}^{(1)}$$

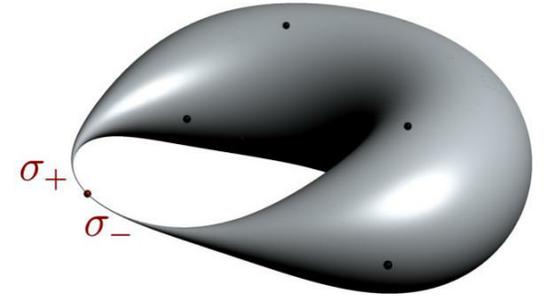
$$I_{\text{NS-NS}}^{(1)} = I_{\text{NS}}^{(1)} \tilde{I}_{\text{NS}}^{(1)}$$

⇒ "free lunch": We can construct many more theories
(& lower dim)
on the nodal sphere!

Aside:

Is there a way to calculate amplitudes from nodal sphere

WITHOUT tons calculation?



[see homework]

YES!

Gluon operator $\Delta \sim$ target space propagator

- Construction:
- BRST-invariant
 - must be **non-local!**

required by BRST inv.

$$\Delta(\sigma_+, \sigma_-) = \int \frac{d^d \ell}{\ell^2} \sum_{\text{states}} \tilde{V}(\sigma_+) \tilde{V}(\sigma_-) W(\sigma_+, \sigma_-)$$

(trivial) off-shell extension of VOs:

$$\tilde{V}(\sigma_+) \Big|_{\ell^2=0} = V(\sigma_+)$$

$$= \exp \left(\frac{\ell^2}{2} \int_{\Sigma} \tilde{e} \omega_{+-}^2 \right)$$

2) Understanding the integrand representation

Nodal sphere representation:

$$f_n^{(1)} = \frac{1}{l^2} \int_{\mathbb{M}_{0, n+2}} d\mu_n^{(1)} I_n^{(1)}$$

localized on scattering equations

$$E_A = \text{Res}_{\sigma_A^{\frac{1}{2}}} \left(P^2 - l^2 \omega_{+-}^2 \right)$$

Explicitly:

$$E_{\pm} = \pm \sum_{i=1}^n \frac{l \cdot k_i}{\sigma_{\pm i}}$$
$$E_i = l \cdot k_i \omega_{+-}(\sigma_i) + \sum_{j \neq i} \frac{k_i \cdot k_j}{\sigma_{ij}}$$

Puzzle:

$$f_n^{(1)} \sim \frac{N}{l^2 \Pi f_I(l)} \quad \text{with } f_I(l) \text{ linear in } l!$$

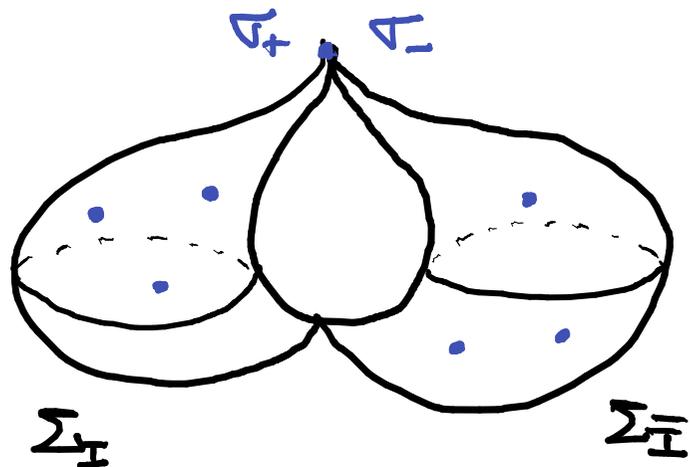
(vs. Feynman props $D_I = (l + \underline{K}_I)^2$)

We can make this precise:

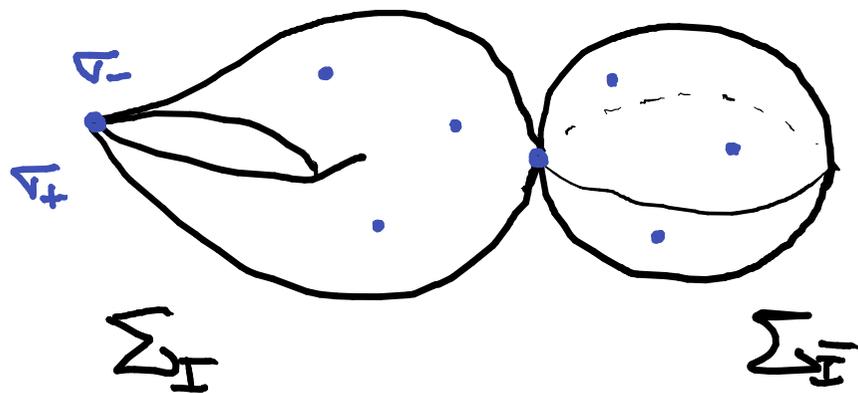
(similar to hw ex. 1)

boundary $\widehat{2M}_{0,n+2}$

pole $\rightarrow 0$



$$SE \ E_n^{(1)} > 2l \cdot K_I + K_I^2$$



$$\xrightarrow{SE \ E_n^{(1)}} K_I^2$$

so $f_n^{(1)} \sim \sum \frac{N(l)}{l^2 \pi (2l \cdot K_I + K_I^2)}$ "linear" integr. repres.

Relation to std "Feynman prop." integrand

$$\mathcal{J}_{FP}^{(1)} = \frac{N(l, l^2)}{\prod_{I=1}^m D_I}$$

with $D_I = (l + K_I)^2$
 $K_I = \sum_{i \in I} k_i$

(and poles $\frac{1}{K_I^2}$ absorbed in N)

Then lin. rep. from residue thm!

Deform $l \rightarrow l + \eta$, with $l \cdot \eta = k_i \cdot \eta = 0$, $\eta^2 = z$

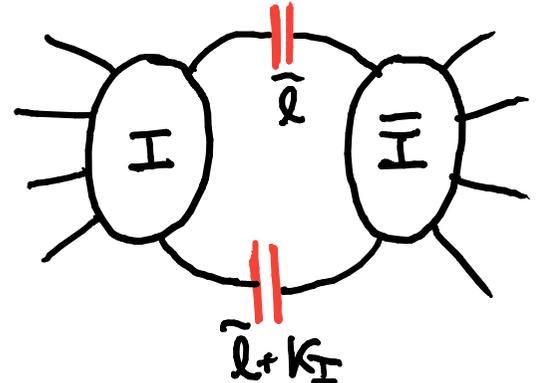
$$\mathcal{J}_{FP}^{(1)} = \oint_{z=0} \frac{dz}{z} \frac{N(l, l^2 + z)}{\prod (D_I + z)} \stackrel{\text{Res thm}}{=} \sum_{I=1}^m \frac{N(l, l^2 - D_I)}{D_I \prod_{J \neq I} (D_J - D_I)}$$

quadr. in l } } all lin. in l

shifts $l \rightarrow l - K_I$

$$\approx \frac{1}{l^2} \sum_{I=1}^m \frac{N(l - K_I, -2l \cdot K_I + K_I^2)}{\prod_{J \neq I} (2l \cdot (K_J - K_I) + (K_J - K_I)^2)}$$

Closely related: Q-cut decomposition

$$\mathcal{F}_n^{(1)} = \sum_{\mathbb{I}} \text{Diagram} \frac{1}{l^2 (2l \cdot K_{\mathbb{I}} + K_{\mathbb{I}}^2)}$$


The diagram shows a bubble with two internal lines. The top line is labeled \tilde{l} and the bottom line is labeled $\tilde{l} + K_{\mathbb{I}}$. Both lines are drawn with double red vertical bars. The bubble is divided into two regions, each labeled with \mathbb{I} . There are several external lines radiating from the left and right sides of the bubble.

Derivation: 1. Step linear rep of integrand (c.f. last slide)

2. Step residue thm in $l \rightarrow \alpha l$,

$$\frac{\mathcal{F}(\alpha l)}{\alpha - 1}$$

→ used to prove nodal sphere amplitudes
(similar to exercise 1)

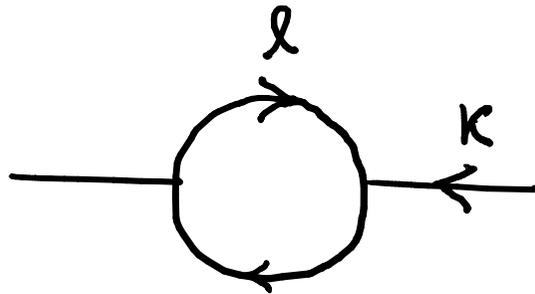
Two examples (basic)

1) = Partial fractions for $N=1$

$$\frac{1}{\pi \mathcal{D}_I} = \sum_{I=1}^m \frac{1}{\mathcal{D}_I \pi (\mathcal{D}_J - \mathcal{D}_I)} \quad (*)$$

(std partial fraction identity)

2) Off-shell bubble



$$(k^2 \neq 0)$$

$$\frac{1}{l^2 (l+k)^2} \stackrel{(*)}{=} \frac{1}{l^2 (2l \cdot k + k^2)} + \frac{1}{(l+k)^2 (-2l \cdot k - k^2)}$$

$$= \frac{1}{l^2} \left(\frac{1}{2l \cdot k + k^2} + \frac{1}{-2l \cdot k - k^2} \right)$$

Comments

1) Double Copy of one loop

↳ Remember: in LINEAR integrand rep.

Same construction as at tree-level:

$$C_n^{(1)} = \sum_{\alpha \in S_n} \frac{C_{(\alpha)}^{(1)}}{(+\alpha-)}$$

$$\sum_r \text{PF}' M_{NS}^r = \sum_{\alpha \in S_n} \frac{N_{(\alpha)}^{(1)}}{(+\alpha-)} \quad \text{mod } E_n^{(1)}$$

with $N_{(\alpha)}^{(1)} = \sum_r N_{(+\alpha-)}^{(0)}$

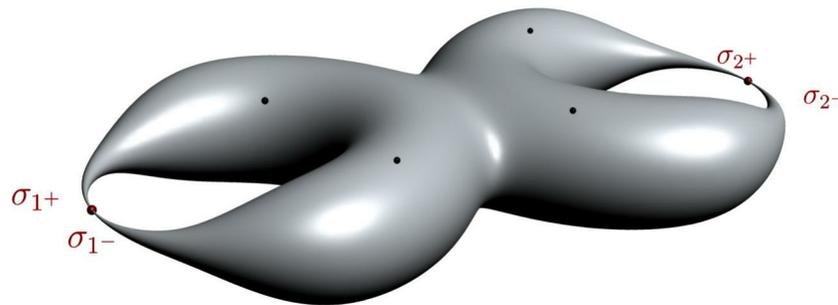
↳ BCF numerators!

(swy version exists: Oli)

→ Double copy in lin. rep more general than in FP rep.

2) Beyond one loop

Naïve proposal:
(straightforw. extension of $g=1$)



$$g_n^{(g)} \stackrel{?}{=} \frac{1}{\pi l_{\text{I}}^2} \int_{\mathcal{M}_{g,n+2g}} d\mu_n^{(g)} I_n^{(g)}$$

$$I_n^{(g)} = I_{\text{susy}}^{(g)} \tilde{I}_{\text{susy}}^{(g)}$$

- from higher g
- in gen. difficult

Proposal for \mathcal{SE} :

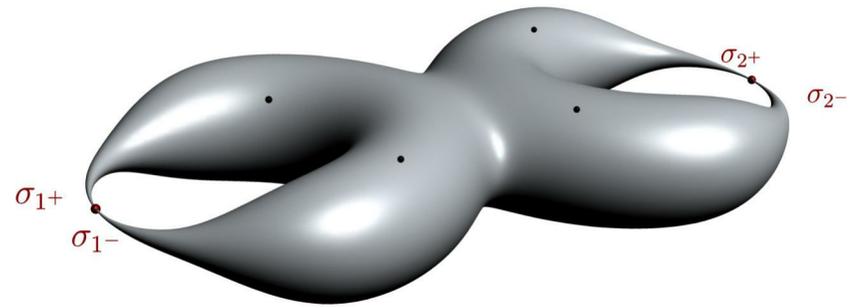
$$E_A^{(g)} = \text{Res}_{\sigma_A} \left(P^2 - \sum_{\text{I}} l_{\text{I}}^2 \omega_{\text{I}^+ \text{I}^-}^2 \right)$$

with $A = 1, \dots, n, \underbrace{+, \dots, g^+}_{\text{nodal}}, \underbrace{-, \dots, g^-}_{\text{poles}}$

This is **WRONG!**

↳ wrong poles for lin. rep.

Beyond one loop (resolution at $g=2$)



Several new features:

$$g_n^{(2)} = \frac{1}{l_1^2 l_2^2} \int d\mu_n^{(2)} \mathbb{M}_{0,n+4}$$

$$I_n^{(2)}$$

$$C^{(2)}$$

$$= \frac{\nabla_{1+2-} \nabla_{1-2+}}{\nabla_{1+1-} \nabla_{2+2-}}$$

(role: cancels unphys poles from $\mathcal{J}E$)

scattering equations:

$$E_A^{(2)} = \text{Res}_{\sigma_A} \mathcal{P}^{(2)}$$

with $\mathcal{P}^{(2)} = \mathcal{P}^2 - \sum_I l_I^2 \omega_{I+I-}^2 + (l_1^2 + l_2^2) \omega_{1+1-} \omega_{2+2-}$

role: provide correct poles for lin. rep.

simplifies for $n=4$:

$$I_4^{(2)} = R_4 (\mathcal{J}^{(2)} \mathcal{Y}^{(2)})^2$$

std factor $A_4^{(0)} \sim R_4 \frac{1}{s_{12} s_{34} s_{13}}$

Jacobian from going to nodal sph.

$$= (\nabla_{1+2-} \nabla_{1+2-} \nabla_{1-2+} \nabla_{2-})^{-1}$$

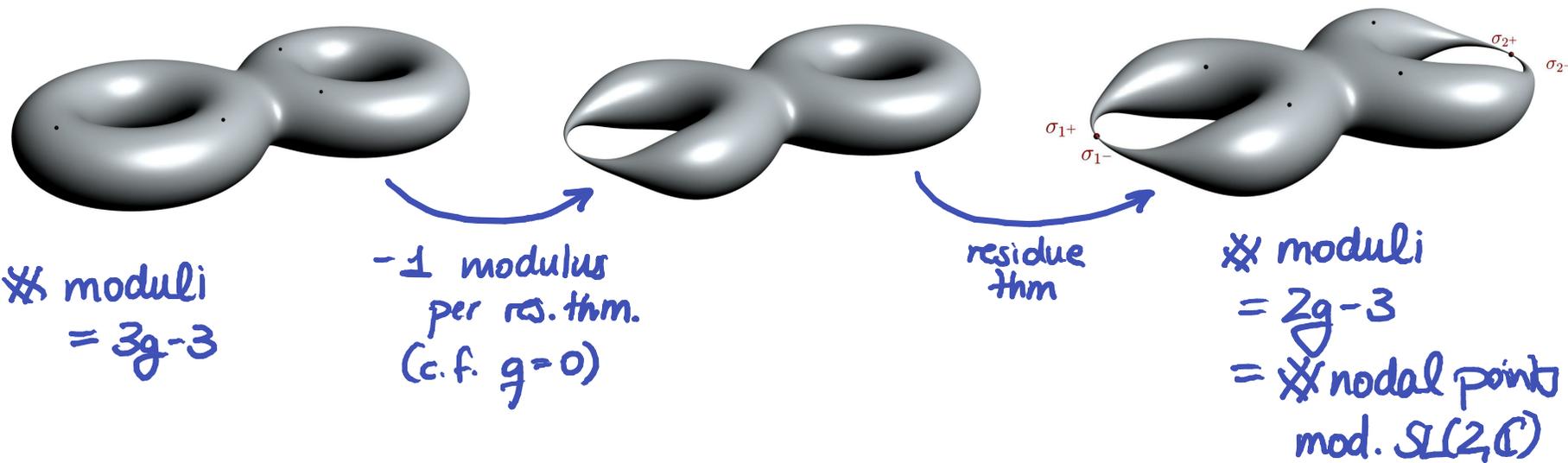
same as string!

$$\mathcal{Y}^{(2)} = \frac{1}{3} (s_{13} - s_{14}) \Delta_{12} \Delta_{34} + \text{cyc}(234)$$

with $\Delta_{ij} = \varepsilon^{\mathbb{D}} \omega_{I+I-(\sigma_i)} \omega_{J+J-(\sigma_j)}$

Comments:

i) Can be derived from residue thm



2) Physical interpret. of new features

- poles in linear rep:
$$\begin{cases} 2l_I \cdot k + k^2 & \text{for } I=1,2 \\ (l_1 + l_2 + k)^2 \end{cases}$$

↙ need additional terms in SE

- now look at all poles encoded in SE:
 > unphysical poles, canceled by $c^{(2)}$

Outlook :

Loop integrands from the Worldsheet

genus

