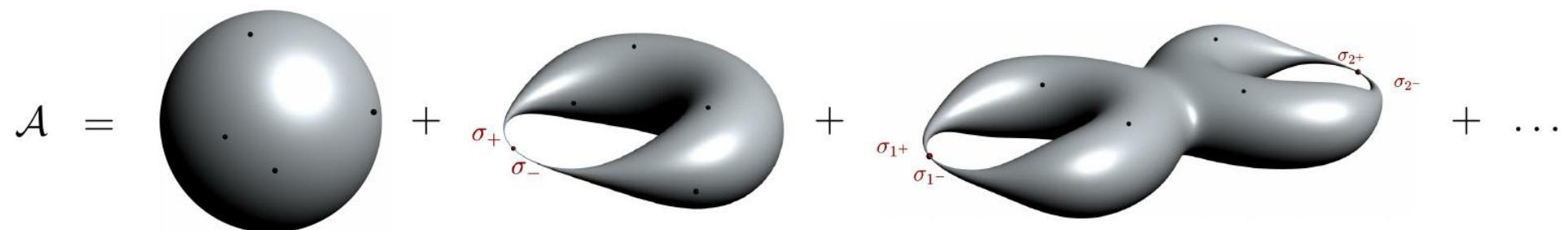


The Amplitude Games
MITP School 2021

AMPLITUDES FROM THE
NODAL SPHERE



Last lecture: ambitwistor string

- $S = S_A + S_m$

$$S_A = \frac{1}{2\pi} \int \vec{P} \cdot \partial \vec{x} - \frac{\bar{e}}{2} \vec{P}^2 - e \vec{P} \cdot \partial \vec{x}$$

$$S_m^{\text{ens}} = S_f + S_{\bar{f}}$$

- $\mathcal{A} = \bullet + \bullet + \bullet + \dots$

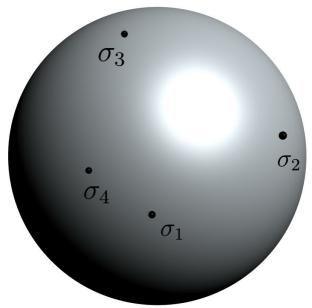
$A_n^{\text{CHY}} = \left\langle \Pi c \bar{c} V_i \Pi \bar{V}_j \right\rangle_A$

THIS
LECTURE

CHY formula

= ambitwistor string
correlator

Cartoon of tree-level correlator

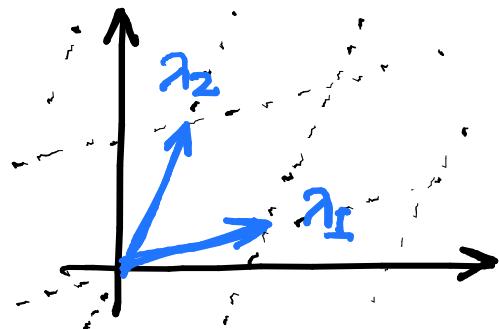


field	relevant part of action	in amplitude
e	$S = \frac{1}{2\pi} \int_{\Sigma} e T$	int. over $\prod_{0,n}$
\tilde{e}	$S = \frac{1}{2\pi} \int_{\Sigma} \frac{\tilde{e}}{2} P^2$	localization on SE $\prod_{i=1}^n \delta(R_{\text{res},i} P^2)$
X_{RNS}	$S_{\text{eff}} = \frac{1}{2\pi} \int_{\Sigma} P \cdot \bar{\partial} X +$ $+ 2\pi i \sum k_i \cdot X \bar{\delta}(r - r_i) dr$	momentum conservation $\delta^d(\sum k_i) \quad (d=10 \text{ for RNS})$
(P, X)	— " —	localization on EoM $\bar{\partial} P_\mu = 2\pi i \sum_i k_{i\mu} \bar{\delta}(r - r_i) dr$
ψ		Pfaffian $\text{Pf}' M$

III. One-loop integrands

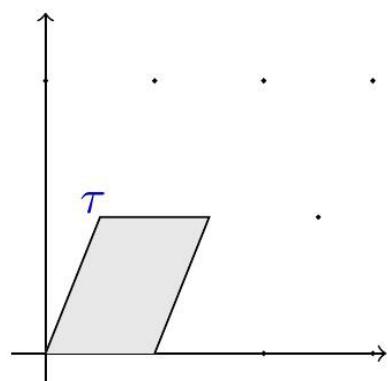
i) Moduli space at g=1

[c.f. Claude,
Oli]



torus = lattice \mathbb{C}/Λ

conformal
transformation



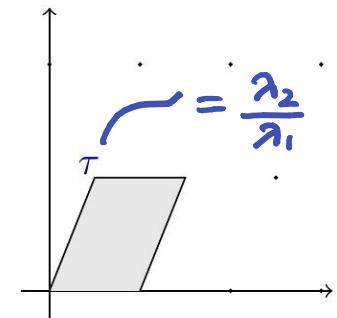
$z \sim z + 1 \sim z + \tau$

- $\Lambda_\tau = \mathbb{Z} \oplus \tau \mathbb{Z}$

- So naively

$$\begin{aligned}\mathcal{T} &= \{\tau \mid \operatorname{Im} \tau > 0\} \\ &= \text{Teichmüller space}\end{aligned}$$

But not all $\tau \in \mathcal{T}$ gen. different lattices!



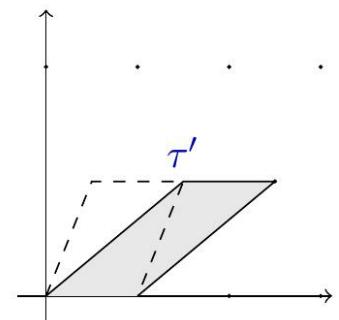
Dehn twists

"cut along cycle, twist, glue back"

$T:$

$$\begin{aligned}\lambda_1 &\rightarrow \lambda_1 \\ \lambda_2 &\rightarrow \lambda_2 + \lambda_1\end{aligned}$$

$$\tau \rightarrow \tau + 1$$

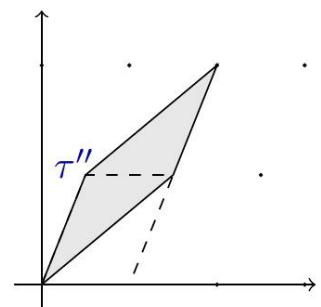


$B:$

$$\begin{aligned}\lambda_1 &\rightarrow \lambda_1 + \lambda_2 \\ \lambda_2 &\rightarrow \lambda_2\end{aligned}$$

$$\tau \rightarrow \frac{\tau}{\tau+1}$$

$$\left(\text{simpler: } S: \tau \mapsto -\frac{1}{\tau} \right)$$



$$\longrightarrow \tau \sim \frac{a\tau+b}{c\tau+d} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\mathbb{Z} \text{ modular group}$$

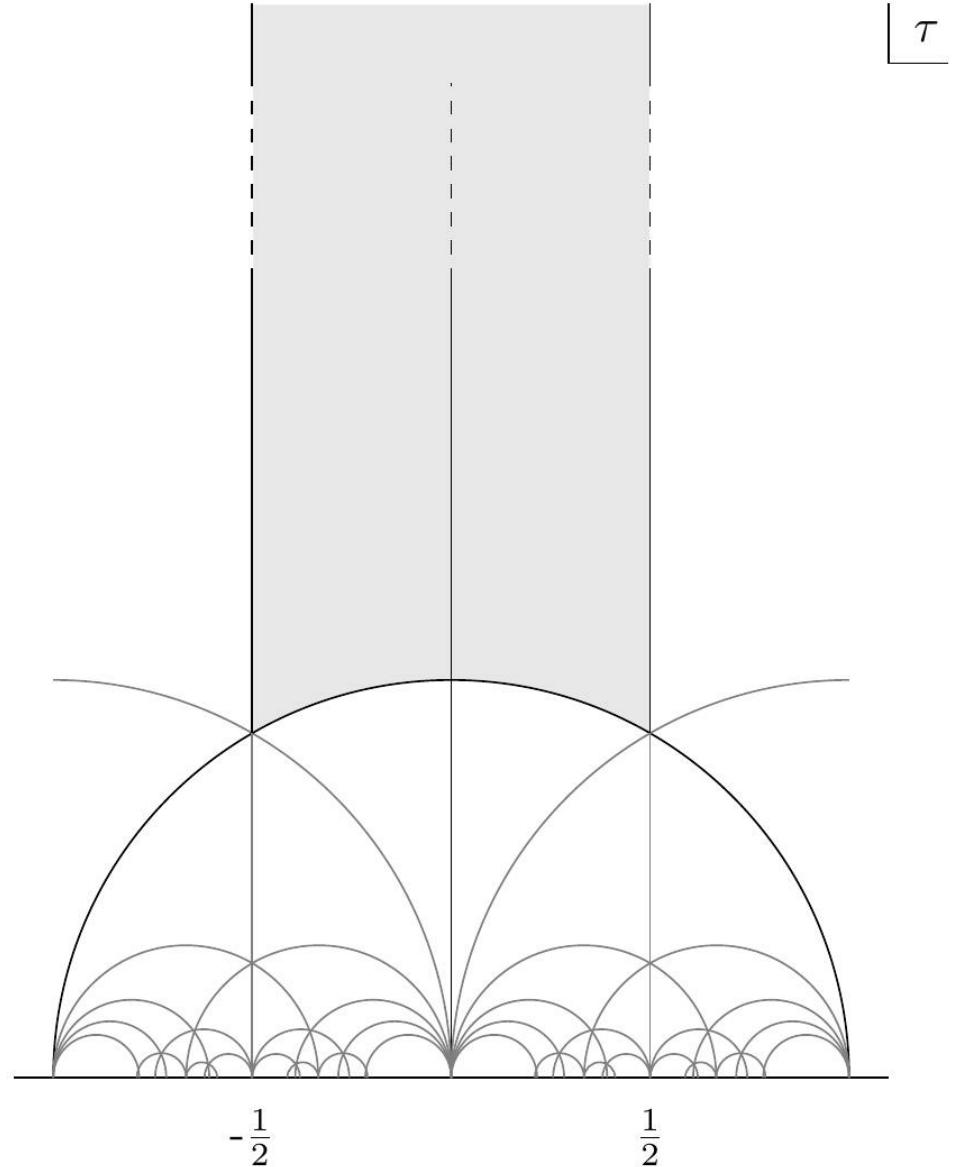
Moduli space of torus

= fundamental domain

$$\mathcal{F} = \left\{ \tau \mid \operatorname{Re} \tau \in [-\frac{1}{2}, \frac{1}{2}], |\tau| \geq 1 \right\}$$

$\tau: \tau \sim \tau + 1$

$s: \tau \sim -\frac{1}{\bar{\tau}}$

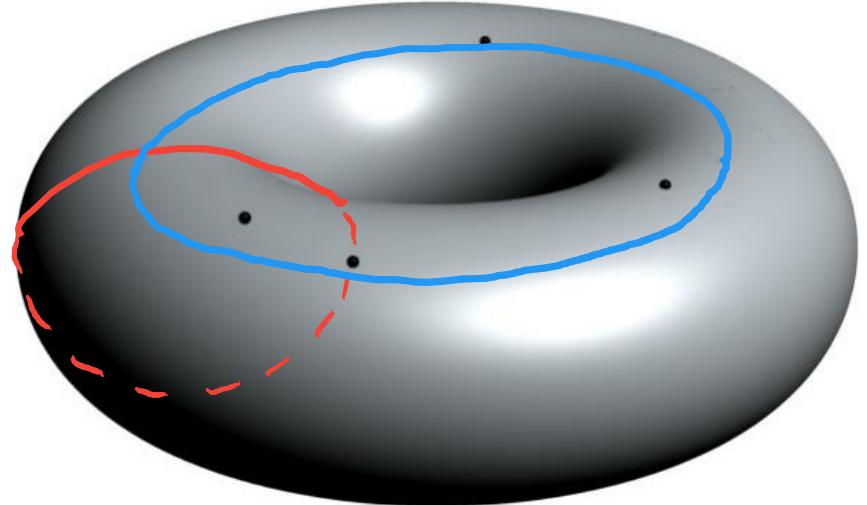


2) Spin structures

Homology basis:

(A_I, B_I) -cycles, $I=1\dots g$

such that $\star(A_I, B_J) = \delta_{IJ}$



- genus g : 2^{2g} spin structures
 - $2^{g-1}(2g+1)$ even
 - $2^{g-1}(2g-1)$ odd

notation: $\alpha = (\alpha' | \alpha'')$ with $\alpha', \alpha'' \in (\mathbb{Z}/2\mathbb{Z})^g$

- $g=1$: $2\alpha_1 = (1, 1)$ even

$$\psi(z+1) = (-1)^{2\alpha_1^1+1} \psi(z)$$

$$\psi(z+\tau) = (-1)^{2\alpha_1^2+1} \psi(z)$$

- $2\alpha_2 = (1, 0)$
- $2\alpha_3 = (0, 0)$
- $2\alpha_4 = (0, 1)$

$\alpha' = 0$: NS state
 $\alpha' = 1/2$: R state along B -acyc.

Modular transformations

permute even spin structures:

$$S: \quad \alpha_1 \rightarrow \alpha_1$$

$$\begin{aligned}\alpha_2 &\rightarrow \alpha_4 \\ \alpha_3 &\rightarrow \alpha_3 \\ \alpha_4 &\rightarrow \alpha_2\end{aligned}$$

$$T: \quad \alpha_1 \rightarrow \alpha_1$$

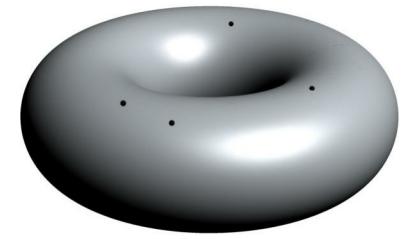
$$\begin{aligned}\alpha_2 &\rightarrow \alpha_2 \\ \alpha_3 &\rightarrow \alpha_4 \\ \alpha_4 &\rightarrow \alpha_3\end{aligned}$$

→ Sum over spin structures

necessary for
modular inv.

$$Z = \left(Z_i + \sum_{\alpha=2,3,4} (-1)^\alpha z_\alpha \right) \left(Z_i \pm \sum_{\beta=2,3,4} (-1)^\beta z_\beta \right)$$

Toolkit at $g=1$



) Objects on the torus [see also Oli]

- holomorphic differential $\frac{dz}{w_I}$ for $I=1\dots g$

- Jacobi ϑ -function

$$\vartheta_\alpha(z|\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n+\alpha')^2} e^{2\pi i (n+\alpha')(z+\alpha'')}$$

with $q = e^{\frac{2\pi i \tau}{\tau}}$

- Prime form: $E(z, w) = \frac{\vartheta_i(z-w|\tau)}{\vartheta_i'(0|\tau)} dz^{-\frac{1}{2}} dw^{-\frac{1}{2}} \sim \frac{z-w}{2\pi} dz^{-\frac{1}{2}} dw^{-\frac{1}{2}}$

- Szegő Kernels: $S_\alpha(z, w) = \frac{\vartheta_\alpha(z-w|\tau)}{\vartheta_\alpha'(0|\tau) E(z, w)} \sim \frac{2\pi i}{z-w} dz^{\frac{1}{2}} dw^{\frac{1}{2}}$

meromorphic differential

$$\omega_{z_1, z_2}(z) = dz \partial_z \ln \frac{E(z, z_1)}{E(z, z_2)}$$

$$\sim \frac{(-1)^a dz}{z - z_a}$$

2) CFT on the torus

- Green's functions:

$$\langle \psi^\mu(z) \psi^\nu(w) \rangle_\alpha = S_\alpha(z, w) \eta^{\mu\nu}$$

$$\langle X^\mu(z) P_\nu(w) \rangle = S_\nu^\mu \omega_{z, z_0}(w)$$

- Partition functions for $B\gamma$ -system ($h_\beta = (\lambda, 0)$, $h_\gamma = (1-\lambda, 0)$)

$$Z_{\lambda=2} = Z_{\lambda=1} = \eta^2(\tau)$$

$$Z_{\lambda=\gamma_2} = \sum_\alpha Z_{\lambda=\beta_2} = \frac{J_\alpha(0|\tau)}{\eta(\tau)}$$

Dedekind η -function
 $\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$

\Rightarrow ambihwistor string: $Z_\alpha^A = \frac{Z_2}{(Z_1)^5} \frac{(Z_{\gamma_2, \alpha})^5}{Z_{\beta_2, \alpha}} = \frac{J_\alpha(0|\tau)}{\eta^2(\tau)}$

(chiral part. fn)

One-loop correlator in the ambihorizon string

$$A_n^{(i)} = \int_{m_{i,n}} d\tau \left\langle \mu_\tau b \right\rangle \left\langle \mu_0 \tilde{b} \right\rangle \bar{s}(\left\langle \mu_0 P^2 \right\rangle) c \tilde{c}_i V_i \prod_{i=2}^n \bar{s}(\left\langle \mu_i P^2 \right\rangle) V_i \right\rangle$$

↑
from e

basis of Beltrami diffs.
 $\tilde{F}(\tilde{e}) = \tilde{e} - s_0 \mu_0 - \sum_{i=2}^n s_i \mu_i$

- quantiz. of e, \tilde{e} ✓
- X zms: momentum conservation
(as on the sphere)

$$\delta^d(\sum k_i)$$

PX system etc.

- for non-zeros : repeat same trick as from LECTURE 2 :

$$\bar{\partial} P_\mu = 2\pi i \sum_{i=1}^n k_{i\mu} \bar{\delta}(z - z_i) dz$$

BUT: Solve now on torus:

$$P_\mu(z) = 2\pi i \oint_\gamma l_\mu dz + \sum_{i=1}^n k_{i\mu} \omega_{z_i, z_0}(z)$$

zero-mode params. } holom. differential

- Choose $(n-1)$ Beltrami-diffs (again, as on sphere)

$$\int \mu_i P^2 = \text{Res}_{z_i} P^2 \longrightarrow \text{SE: } E_i^{(1)} = \text{Res}_{z_i} P^2 = 2k_i \cdot P(z_i)$$

For last SE: Note that $P^2(z) = u dz^2$ on support of $E_i^{(1)}$

$$\int \mu_i P^2 = u \longrightarrow \text{SE: } E_i^{(0)} = u$$

for some $u = u(l, k_i, z_i)$

Putting this all together (for even α)

$$A_n^{(1)} = \delta^{10}(\sum k_i) \underbrace{\int d^{10}l \int_{m_{n,n}} dt \bar{S}(u) \prod_{i=2}^n \bar{S}(\text{Res}_i P^2)}_{=: f_n^{(1)} \text{ integrand}} I_n^{(1)}$$

with $I_n^{(1)} = \sum_{\alpha, \beta} (-1)^{\alpha + \beta} \text{Pf}(M_\alpha) \text{Pf}(M_\beta) Z_\alpha Z_\beta$

$$M_\alpha = \begin{pmatrix} A & -C^\top \\ C & B \end{pmatrix} \quad \text{and}$$

$$A_{ij} = k_i \cdot k_j S_\alpha(z_i, z_j | \tau)$$

$$B_{ij} = G \cdot g S_\alpha(z_i, z_j | \tau)$$

$$C_{ij} = G_i \cdot k_j S_\alpha(z_i, z_j | \tau)$$

$$A_{ii} = 0$$

$$B_{ii} = 0$$

$$G_i = -G \cdot P$$

Note: • $f_n^{(1)}$ is fully localized

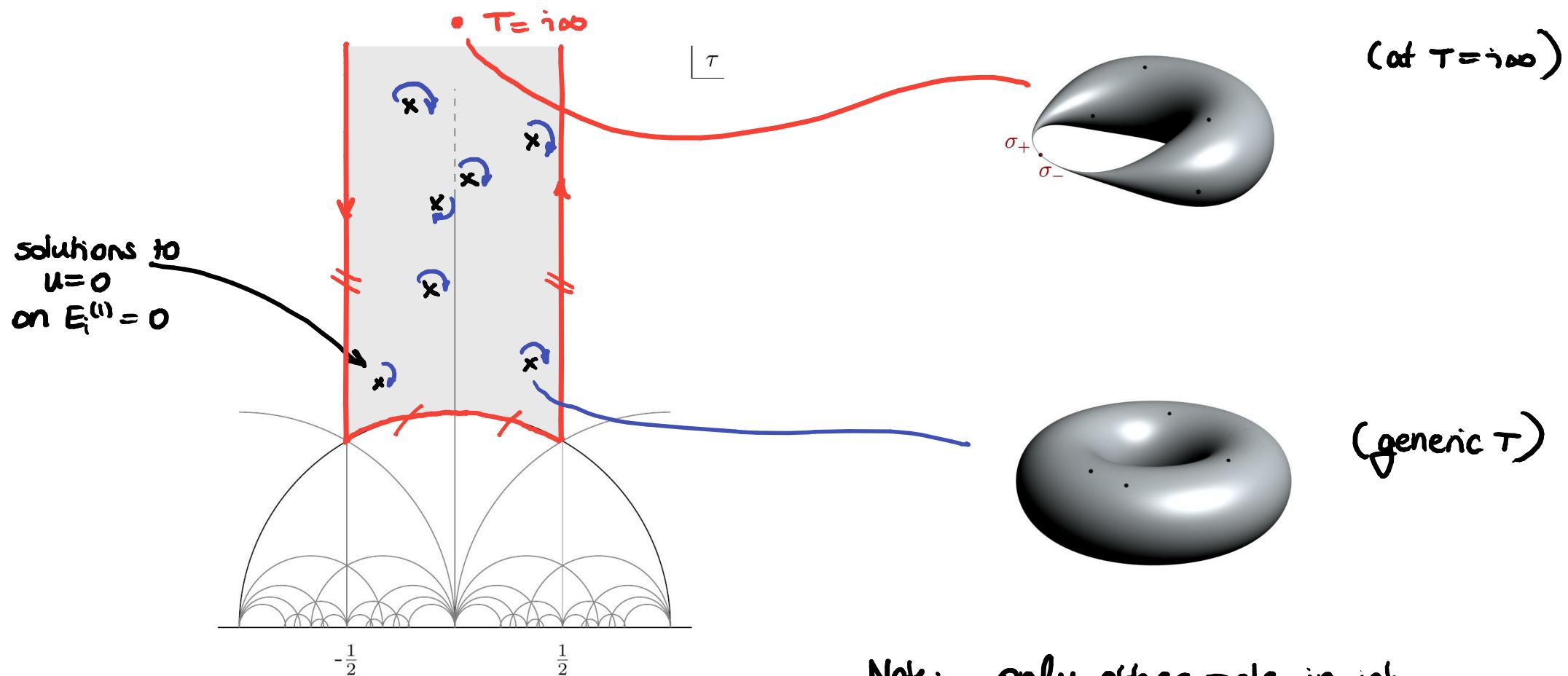
• modular invariance (check: $\tau \rightarrow \tau + 1$ $dz \rightarrow dz$

$$\tau \rightarrow -1/\tau \quad dz \rightarrow \frac{dz}{z^2}, \quad l_\mu \rightarrow \tau l_\mu$$

IV. From torus to nodal sphere

Idea: Use
 • modular invariance
 • localization of $\mathcal{F}_n^{(1)}$

→ residue thm in fund. dom \mathcal{F}



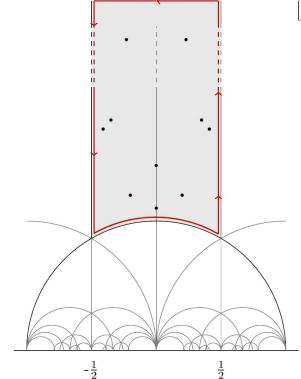
In more detail:

For convenience, define shorthand notation:

$$J_n^{(1)} = \int_{M_{n,n}} \frac{dq}{q} \bar{\delta}(u) I(q) = \int_{M_{n,n}} dt \bar{\delta}(u) \prod_{i=2}^n \bar{\delta}(E_i) I_n$$

$\sim q = e^{2\pi i t}$

(definition)



$$= - \int_{M_{n,n}} dq \bar{\delta}(q) \frac{I(q)}{u} \Big|_{q=0}$$

$$= - \int_{M_{0,n+2}} \frac{I(0)}{u} \Big|_{q=0}$$

$$= - \frac{1}{l^2} \int_{M_{0,n+2}} I(0)$$



What is u on $q = 0$?

$$\text{Recall } P^2(z) = u dz^2$$

$$\Rightarrow \text{implies } u = l^2$$

(e.g. using details from next sl.)

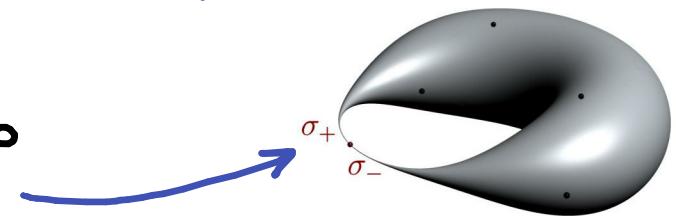
Now: Find form of $I(0)$ that manifestly lives on the (nodal) sphere

Change variables to map to \mathbb{CP}^1 :

$$\text{map: } z \in \mathbb{C}/\Lambda_T \rightarrow \tau \in \mathbb{CP}^1 \quad (\text{as } \tau \rightarrow i\infty)$$

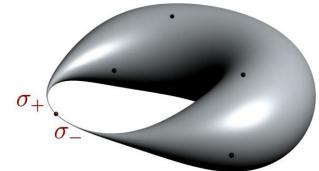
achieved by: $\tau = e^{2\pi i(z - T/2)}$

- for generic T , this maps to $\{e^{-\pi \operatorname{Im} \tau} \leq |\tau| \leq e^{\pi \operatorname{Im} \tau}\}$, with $\tau \sim q\tau$
- so as $q \rightarrow 0$, recover $\tau \in \mathbb{CP}^1$
with $\tau_+ = 0$ and $\tau_- = \infty$
identified



Look at important objects:

- holom. differential $dz \longrightarrow \frac{1}{2\pi i} \underbrace{\left(\frac{1}{\tau - \tau_+} - \frac{1}{\tau - \tau_-} \right) d\tau}_{=: \omega_{+-}(\tau)}$
- merom. diff. $\omega_{z_i, j}(z) \longrightarrow \underbrace{\left(\frac{1}{\tau - \tau_i} - \frac{1}{\tau - \tau_j} \right) d\tau}_{=: \omega_{ij}(\tau)}$



So on the nodal sphere:

1)

$$P_\mu(\sigma) = l_\mu w_{+-}(\sigma) + \sum_{i=1}^n \frac{k_{i\mu}}{\sigma - \sigma_i} d\sigma$$

- looks like tree-level, $n+2$ pts
- BUT: $\ell^2 \neq 0$

2)

Scattering Equations

$$\frac{1}{2} E_i^{(1)} = k_i \cdot P(\sigma_i) = k_i \cdot l w_{+-}(\sigma_i) + \sum_{j \neq i} \frac{k_i \cdot k_j}{\sigma_j} d\sigma_j$$

Rewrite these SE again as residue of sth:

$$E_A^{(1)} = \text{Res}_{\sigma_A} (P^2 - \ell^2 w_{+-}^2)$$

$A = 1 \dots n, +, -$

Why?

- On torus, $P^2 = 0$
- res. thm: $u=0 \leftrightarrow q=0$
so now $P^2 = \ell^2 w_{+-}^2$

A closer look at the SE on the nodal sphere

$$E_A^{(1)} = \text{Res}_{\nabla_A} (P^2 - l^2 \omega_r^2)$$

$A=1\dots n, +, -$

Explicitly: (dropped form deg.)

$$E_i^{(1)} = 2l \cdot k_i \left(\frac{1}{\nabla_{+}} - \frac{1}{\nabla_{-}} \right) + \sum_{j \neq i} \frac{k_i \cdot k_j}{\nabla_{ij}}$$
$$E_{\pm}^{(1)} = \pm \sum_{i=1}^n \frac{2l \cdot k_i}{\nabla_{\pm i}}$$

- Comments:
- Möbius Inv. (this is how we wrote down $E_{\pm}^{(1)}$)
 - \sim tree-level $(n+2)$ -pts (so can now calculate!)
but with $l^2 \neq 0$
 - ~~sol's~~ $= (n-1)! - (n-2)!$
 $(n+2-3)!$ 2 from back-to-back kinematics

At this stage

$$:= p^2 - l^2 \omega_{+-}^2$$

$$J_n^{(1)} = \frac{1}{l^2} \int_{M_{0,n+2}} \frac{\pi d\sigma_A}{\text{vol } SL(2, \mathbb{C})} \left| \pi' \bar{s}(\text{Res}_B p_1) \right| I_n^{(1)}$$

Integrand: (straightforward but lengthy calc.)

$$I_n^{(1)} = I_{\text{susy}}^{(1)} \tilde{I}_{\text{susy}}^{(1)}$$

$$I_{\text{susy}}^{(1)} = \underbrace{\sum_r Pf'(M_{NS}^r)}_{\text{NS, from } \alpha_3, \alpha_4} - \underbrace{\frac{8}{\pi_{+-}^2} Pf M_2}_{R, \text{ from } \alpha_2}$$

$$M_{NS}^r := M_{n+2}^{\text{tree}} \quad \left| \begin{array}{l} k_{n+1} = l, k_{n+2} = -l \\ e_{n+1} = e^r, e_{n+2} = (e^r)^+ \end{array} \right.$$

$$\sum_i e_i^r (e^r)_j^\dagger = \Delta_{\mu\nu} = \eta_{\mu\nu} - \frac{l q_\mu + l q_\nu}{l \cdot g}$$

- as on torus, but with
- $S_2 \rightarrow S_2(\tau_{ij}) := \frac{1}{\tau_{ij}} \left(\sqrt{\frac{\tau_{++}}{\tau_{+-}\tau_{+-}}} + \sqrt{\frac{\tau_{+-}}{\tau_{++}\tau_{+-}}} \right)$
 - $P_\mu \rightarrow P_\mu(\tau)$

Summary so far

$$A = \text{ (sphere)} + \text{ (torus)} + \dots$$

only in RNS 1A-string

↓
residue thm
on \mathcal{F}

$$A = \text{ (sphere)} + \text{ (surface with handles, labeled } \sigma_+, \sigma_-) + \text{ (surface with handles, labeled } \sigma_{1+}, \sigma_{1-}, \sigma_{2+}, \sigma_{2-}) + \dots$$

This suggests:
expansion in ~~nodes~~
instead of genus!

generalizations
beyond sugra?

next lecture