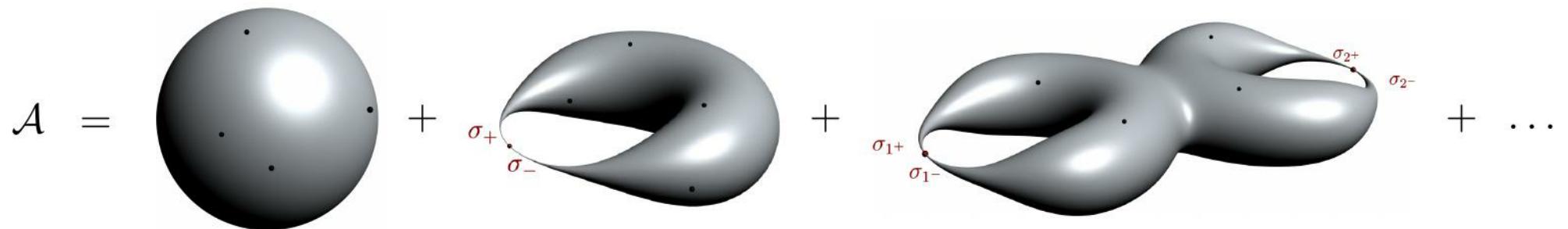


The Amplitude Games

MIP School 2021

AMPLITUDES FROM THE NODAL SPHERE



II. Worldsheet model: Ambitwistor string

So far:

1) Scattering equations

$$E_i = \text{Res}_{\sigma_i} P^2$$

$P^2 = 0$ on CTP¹

$$P_\mu(\sigma) = \sum_{i=1}^n \frac{k_{i\mu}}{\sigma - \sigma_i} d\sigma$$

2) Action :

$$S = S_A + S_{\text{matter}}$$



- should be:
- theory of maps $X \rightarrow M_0$
 - encode $P^2 = 0$
 - chiral

First attempt

$$S_A \sim \frac{1}{2\pi} \int_{\Sigma} P \cdot \bar{\partial} X - \frac{\tilde{e}}{2} P^2$$

Lagr. mult
for P^2

no α'

\hookrightarrow field theory

$$S \stackrel{?}{=} \frac{1}{2\pi} \int_{\Sigma} P \cdot \bar{\partial} X - \frac{\tilde{e}}{2} P^2$$

Conformal weight

$$P_\mu: h_P = (1, 0) \quad \text{or equiv. } P_\mu \in \mathcal{L}^\circ(\Sigma, K_\Sigma)$$

$$X^\mu: h_X = (0, 0) \quad X^\mu \in \mathcal{L}^\circ(\Sigma)$$

$$\tilde{e}: h_{\tilde{e}} = (-1, 1)$$

*2 c.f. Beltrami diff.
in shing*

Aside: primary fields in 2d CFT of conf. wt. (h, \tilde{h})

$$\phi'(z'; \bar{z}') = \left(\frac{\partial z'}{\partial z} \right)^{-h} \left(\frac{\partial \bar{z}'}{\partial \bar{z}} \right)^{-\tilde{h}} \phi(z, \bar{z})$$

$$\text{so conformal weight} \longleftrightarrow \text{form degree} \quad \text{i.e.} \quad \begin{aligned} h > 0: \quad & \phi \in \mathcal{L}^\circ(\Sigma, K_\Sigma^h) \\ h < 0: \quad & \phi \in \mathcal{L}^\circ(\Sigma, \bar{K}^{-h}) \end{aligned}$$

To promote this from CFT to worldsheet model:

gauge (chiral) stress-energy tensor $\bar{T} = -\bar{P} \cdot \partial X$

check: $\text{hw}:$

$$T(z) \phi(w) = \frac{h \phi(w)}{(z-w)^2} + \frac{\partial_w \phi(w)}{z-w}$$

\Rightarrow

$$S_A = \int_{\Sigma} \bar{P} \cdot \bar{\partial} X - \tilde{\frac{e}{2}} \bar{P}^2 - e \bar{P} \cdot \partial X$$

bosonic ambipolar
string action

Symmetries

$$S = \frac{1}{2\pi} \int P \cdot \bar{\partial} X - \frac{\tilde{e}}{2} P^2 - e P \cdot \partial X$$

1) Chiral worldsheet diffeo's

$$\delta X^\mu = v \partial X^\mu$$

$$\delta P_\mu = \partial(v P_\mu)$$

$$\delta e = \bar{\partial} v + v \partial e - e \partial v$$

$$\delta \tilde{e} = v \partial \tilde{e} - \tilde{e} \partial v$$

$$(\delta \phi = v \partial \phi + h_\phi \partial v \phi)$$

2) "Ambitwistor" sym., gen. by P^2

$$\delta X^\mu = \alpha \eta^{\mu\nu} P_\nu$$

$$\delta P_\mu = 0$$

$$\delta e = 0$$

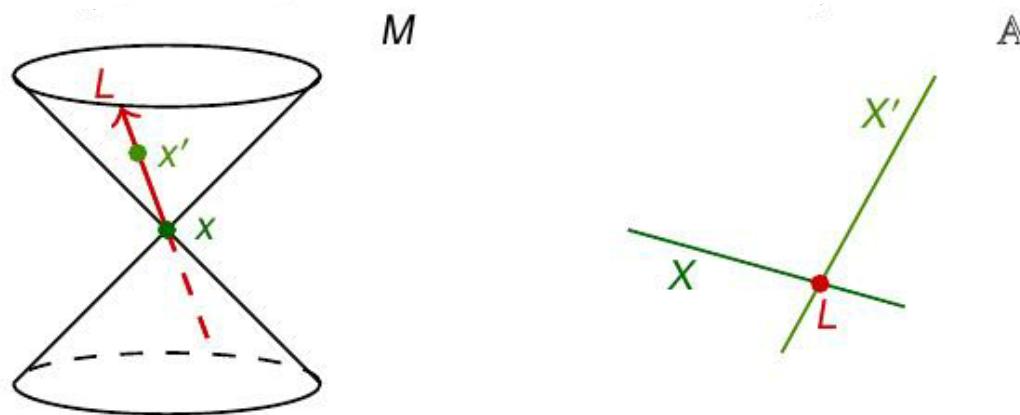
$$\delta \tilde{e} = \bar{\partial} \alpha - \alpha \partial e + e \partial \alpha$$

→ generate transl. along null rays

→ target space = space of (complex) null geodesics
=: ambitwistor space

target space = projective ambitwistor space := $T\mathbb{A} = \mathbb{A}/\{P \cdot \partial_P\}$

$$\mathbb{A} := \{ (P, x) \in T^* M \mid P^2 = 0 \} / \{ P \cdot \partial_x \}$$



- $\dim \mathbb{A} = 2d - 2$, $\dim T\mathbb{A} = 2d - 3$
(symp. manif.) (contact manif.)
- intuition: natural space to describe massless field theories

Intermission:

Basics about $B\bar{\chi}$ systems

\sum = chiral CFTs we'll
 encounter everywhere
 \sim CFT of ghosts [Dil]

Consider the action

$$S = \frac{1}{2\pi} \int b \bar{\partial} c$$

with conformal weights $h_b = (\lambda, 0)$ and $b, c \left\{ \begin{array}{l} \text{fermions: } \epsilon = -1 \\ \text{bosons: } \epsilon = 1 \end{array} \right.$ and $h_c = (1-\lambda, 0)$

Then: • OPE: $c(z) b(w) = \frac{1}{z-w}$ $b(z) c(w) = -\frac{\epsilon}{z-w}$

- stress-energy: $T_{bc} = -\lambda b \bar{\partial} c + (1-\lambda) (\bar{\partial} b) c$

- central charge: $c = 2\epsilon (6\lambda^2 - 6\lambda + 1)$

- zero-modes: $n_c - n_b = \frac{1}{2} (2\lambda - 1) x$ $x = 2(1-g)$

C.f. Riemann-Roch

BRST Quantization

$$S = \frac{1}{2\pi} \int -\frac{\tilde{e}}{2} P^2 - e P \cdot \partial X$$

Introduce std ghosts (\bar{b}, \bar{c}) and (b, c)

with

$$h_b = h_{\bar{b}} = (2, 0)$$

$$h_c = h_{\bar{c}} = (-1, 0)$$

BRST:

$$Q_0 b = T$$

$$Q_0 \bar{b} = H$$

$$Q_0 P_\mu = \partial(c P_\mu)$$

$$Q_0 X^\mu = \bar{c} P^\mu + c \partial X^\mu$$

$$Q_0 \bar{e} = \bar{\partial} \bar{c} + e \partial c - c \partial e$$

T, H: Nakaniishi -
Lautrupp.

Add gauge fixing term:

$$S_{GF} = \frac{1}{2\pi} Q_0 (b F(e) + \bar{b} \tilde{F}(\bar{e}))$$

in absence
of VD's:

$$\tilde{F}(\bar{e}) = \bar{e}$$

Now look at $S_A + S_{GF}$

- integrate out e, \bar{e} and N.-L. fields H, T to find

$$H = \frac{1}{2} P^2 \quad T = -P \cdot \partial X + \bar{T}_{bc} + \bar{T}_{\bar{b}\bar{c}}$$

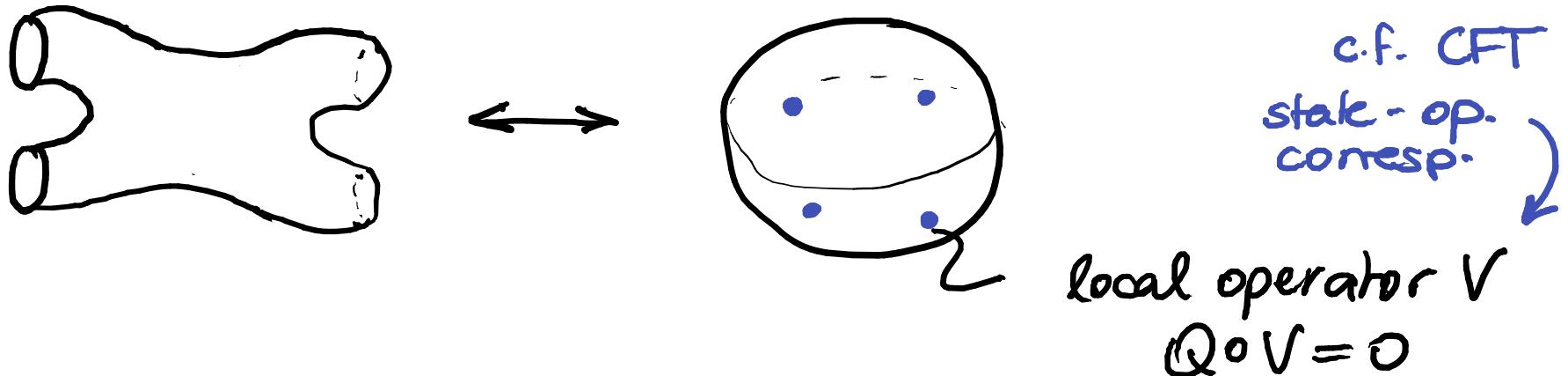
Then action

$$S_{\text{gauge}} = \frac{1}{2\pi} \int P \cdot \bar{\partial} X + b \bar{\partial} c + \bar{b} \bar{\partial} \bar{c}$$

- free, linear
- std OPE's, e.g. $X^\mu(z) P_\nu(w) = \frac{\delta^\mu_\nu}{z-w}$
- central charge anomaly:
 $Q^2=0 \iff d=26$

Structure of Vertex Operators

physical states in BRST cohomology



for BRST closure:

$$c\bar{c}V(z) = c\bar{c} w e^{ik \cdot X(z)}$$

ghost no
 $= 2$

depends
on WS matter
 S_m

momentum
eigenstate

Note:

- $T(z) e^{ik \cdot X(\omega)} = \frac{\partial (e^{ik \cdot X(\omega)})}{z - \omega}$ so $h_{\text{exp}} = (0, 0)$
- \Rightarrow Conclusion 1: $h_\omega = (2, 0)$ for $Q \circ V = 0$

- $P^e(z) e^{ik \cdot X(\omega)} = \frac{k^2}{(z - \omega)^2} e^{ik \cdot X(\omega)} + \dots$
- \Rightarrow Conclusion 2: $k^2 = 0$ for $Q \circ V = 0$

}

so BRST cohomology contains
only massless states!

- If $S_m = 0$, BRST cohom. contains 10's:

$$w = w^L w^R \quad w^{LR} = e \cdot P$$

w^L ↗
 $(1, 0)$

↗ higher order
gravity

For interesting theories, add "WS matter" S_m :

- current algebra S_j

- with current $j \in \mathcal{L}^0(\Sigma, k_{\Sigma} \otimes g)$, satisfying

$$j^a(z) j^{(b)}(\omega) = \underbrace{\frac{\kappa \delta^{ab}}{(z-\omega)^2}}_{\text{level}} + \underbrace{\frac{i f^{abc} j_c(\omega)}{(z-\omega)}}_{\text{structure const.}}$$

- Fermions

$$S_4 = \frac{1}{2\pi} \int \bar{\psi}_\mu \partial^\mu \psi - \sum_e \bar{\psi} \cdot \not{e} \psi - x \not{P} \cdot \psi$$

gauged WS
diffeos

new fermionic constraint

with $h \psi = (\gamma_2, 0)$

$$h x = (-\gamma_2, 1)$$

- New fermionic symmetry generated by P·4

$$\delta X^\mu = \epsilon \gamma^\mu$$

$$\delta e = 0$$

check

$$\delta p_\mu = 0$$

$$\delta \bar{e} = 2e\bar{e}$$

$$\delta \gamma^\mu = \epsilon \eta^{\mu\nu} P_\nu$$

$$\delta X = \bar{e}e + e\bar{e} - \frac{1}{2}\epsilon\bar{\epsilon}e$$

→ WS gauge superalgebra

$$(P\cdot 4)(z) (P\cdot 4)(w) = \frac{P^2}{z-w}$$

NOT $T(w)$!
so no super
Riemann-surf.

- BRST quantiz. now also introduces ghosts ($\beta\gamma$)

$$h_\beta = (\tfrac{3}{2}, 0) \quad h_\gamma = (-\tfrac{1}{2}, 0)$$

- Then Vertex operators: $w^{L,R} = \delta(\gamma) e \cdot \gamma$

3 models

THEORY	ACTION S_m	CENTRAL CH.
• bi-adj. scalar*	$S_j + S_{\bar{j}}$	$c = 2(d-26) + g_j + g_{\bar{j}}$
• super Yang-Mills*	$S_j + S_4$	"heterotic"
• supergravity	$S_4 + \tilde{S_4}$	"RNS"
		$c = 3(d-10)$

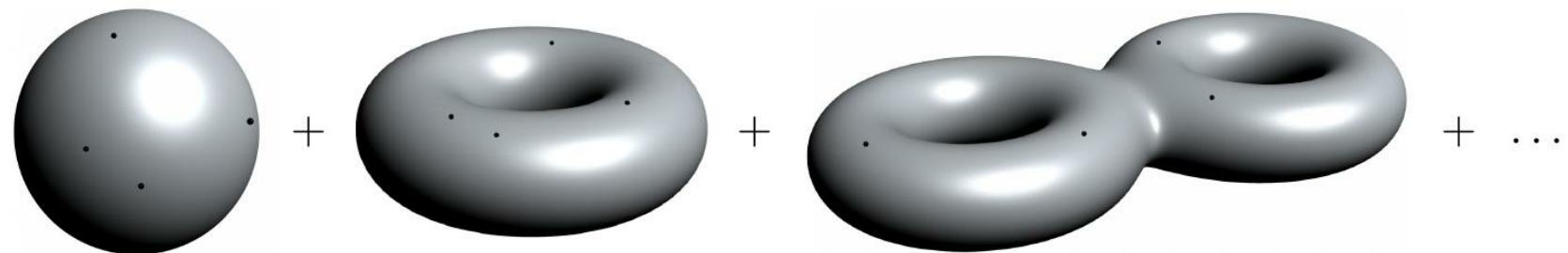
*: containing higher-order gravity in spectrum

Amplitudes

Schematically: [c.f. Oli's lectures]



↓
conformal
transformation



TODAY

NEXT
LECTURE

BRST quantization in presence of VOs

- on Σ_n : • cannot choose $\tilde{F}(\tilde{e}) = \tilde{e}$
because $\tilde{e} \in H^{0,1}(\Sigma, \overline{\Omega}(-z_1 - \dots - z_n))$
 \hookrightarrow c.f. $\delta \tilde{e} = \bar{\partial} \alpha$

- choose instead

- BRST: $Q \circ s_r = g_r$

\hookrightarrow ferm.
params

$$\tilde{F}(\tilde{e}) = \tilde{e} - \sum_{r=1}^{n-3+3g} s_r \mu_r$$

*bosonic
params*

*basis for
 $H^{0,1}(\Sigma, \overline{\Omega}(\dots))$*

- Correlator includes (finite-dim) integrals over

$$\prod_{r=1}^{n-3} ds_r dq_r$$

Repeat now BRST quantis:

$$S_A + S_{GF} \supset \frac{1}{2\pi} \int -\frac{\tilde{e}}{2} P^2 + \tilde{e} H - \underbrace{\sum_{r=1}^{n-3} (s_r \mu_r H + g_r \mu_r \tilde{b})}_{Q \circ \tilde{b} \tilde{F}(\tilde{e})}$$

↑
only look
at terms
with \tilde{e}

Integrate out $\{s_r, g_r\}$:

insertion in correlator of operators

$$\prod_{r=1}^{n-3} \bar{\delta}(\mu_r P^2) \langle \mu_r \tilde{b} \rangle$$

↓
start looking
like SE!

$=: \int \mu_r \tilde{b}$

- Now:
 - choose Beltrami diff's to extract residue at ∇_i
 - absorb into VD's:

$$\mathcal{V}(\nabla_i) := \int \bar{\delta}(\text{Res}_i P^2) V_i$$

For fermionic $P \cdot \gamma$ constraints: same principle
 = "descent" for WOs

Leads to insertion of PCOs (= Picture Changing Operators)

$$\Upsilon = S(\beta) P \cdot \gamma$$

Absorb again into WO's ("descent")

$$V_{(\tau_i)}^{(0,0)} := \lim_{z \rightarrow \tau_i} V(\tau_i) \Upsilon(z)$$

$$V_{RNS}^{(0,0)} = (\epsilon \cdot P + k \cdot \gamma \epsilon \cdot \gamma) (\tilde{\epsilon} \cdot P + k \cdot \tilde{\gamma} \tilde{\epsilon} \cdot \tilde{\gamma}) e^{ik \cdot X}$$

$$V_{\text{het}}^{(0)} = T^a j^a (\epsilon \cdot P + k \cdot \gamma \epsilon \cdot \gamma) e^{ik \cdot X}$$

Finally ready: Correlator

$$A_n = \left\langle c_1 \bar{c}_1 V_1 \quad c_2 \bar{c}_2 V_2 \quad c_3 \bar{c}_3 V_3^{(0,0)} \prod_{i=4}^n V_i^{(0,0)} \right\rangle$$

Easy parts first:

- bc-system: $\nabla_{12} \nabla_{23} \nabla_{31} = \text{Jac from } SL(2, \mathbb{C})$
- $\bar{b} \bar{c}$ -system: $\nabla_{12} \nabla_{23} \nabla_{31} = \text{Jac from } \Pi' \bar{\delta}(E_i)$
- X zero modes: momentum conserv.
 $\delta^d(\sum k_i)$ with $d_{\text{ans}} = 10$
- Rest?

PX-system

Naively this looks hard, but we can use a neat trick:
 Form of VOs:

$$V = f(P) e^{ik \cdot X}$$

$\in P, \nabla, \dots$ but not X

\Rightarrow Take exp. into effective action

$$S_{\text{eff}} \supset \frac{1}{2\pi} \int P \cdot \bar{\partial} X + 2\pi i \sum_{i=1}^n k_i \cdot X \delta(\tau - \tau_i) d\tau$$

\Rightarrow Then easy to integrate out X (no dep. in correl), P

\Rightarrow P localizes onto EoM

$$\bar{\partial} P_\mu = 2\pi i \sum_{i=1}^n k_{i\mu} \delta(\tau - \tau_i) d\tau$$

\hookrightarrow solve on : $P_\mu = \sum_{i=1}^n \frac{k_{i\mu}}{\tau - \tau_i} d\tau$ + SE!

(CP)

Finally correlator:

$$A_n = \int_{M_{0,n}} d\mu_n^{\text{CHY}} \left\langle w_1^L \dots w_n^L \right\rangle \left\langle w_1^R \dots w_n^R \right\rangle \quad \left| \quad P = \sum_{i=1}^n \frac{k_i dv}{\sigma - v_i} \right.$$

Now just calculate! We find

$$C_n = \left\langle T^{a_1 j^{a_1}}(\sigma_1) \dots T^{a_n j^{a_n}}(\sigma_n) \right\rangle$$

$$\begin{aligned} Pf'M &= \left\langle e_1 \cdot \gamma(\sigma_1) e_2 \cdot \gamma(\sigma_2) \prod_{i=3}^n (e_i \cdot P + k_i \cdot \gamma e_i \gamma)(\sigma_i) \right\rangle \\ L &= -\frac{1}{\sigma_{12}} Pf M^{[12]} \end{aligned}$$

Conclusion:

$$A_n^{\text{CHY}} = \left\langle \dots \right\rangle_{\text{ambiguistor string}}$$