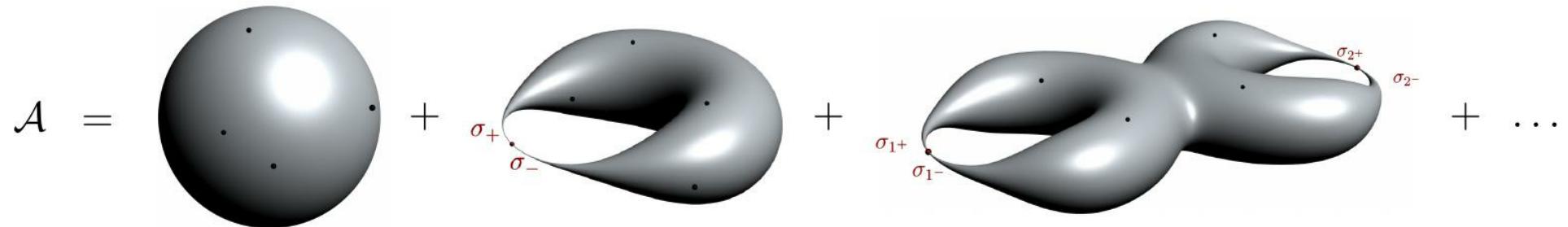


The Amplitude Games  
MITP School 2021

AMPLITUDES FROM THE  
NODAL SPHERE



## Field theory

$$\mathcal{A} = \text{(0)} + \text{(1)} + \text{(2)} + \dots$$

- graph combinatoric problem

$$\begin{array}{c} \uparrow \\ \downarrow \\ = \end{array}$$

$$\mathcal{A} = \text{Sphere} + \text{Doughnut} + \text{Trefoil} + \dots$$

?

$$\xleftarrow{\alpha' \rightarrow 0}$$



$$\mathcal{A} =$$

$$\text{Sphere} + \text{Doughnut} + \text{Trefoil} + \dots$$

[see Oli]

## String theory

- integral over  $M_{b,n}$
- geometric problem

### More detail

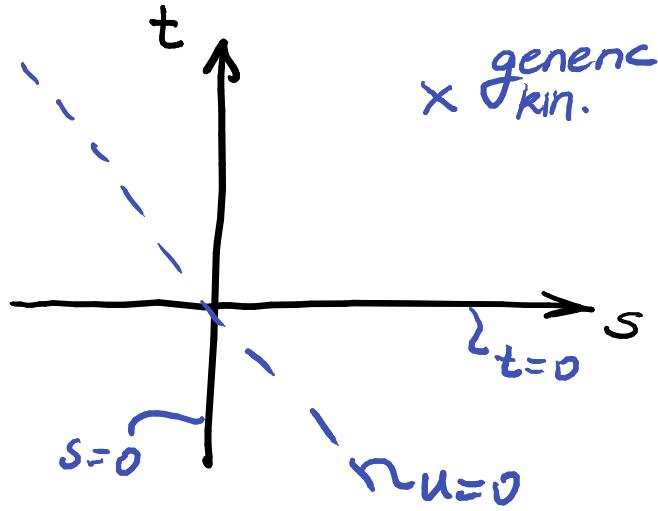
- QFT in string theory
- in string: Mandelstams  $\alpha' k_i \cdot k_j$
- as  $\alpha' \rightarrow 0$   
all of  $\alpha' k_i \cdot k_j \rightarrow 0$

$\Rightarrow$  multi-factorization limit

# Worldsheet methods for field theory

= only approach  $\hat{\mathcal{M}}_{0,n}$  "when necessary"

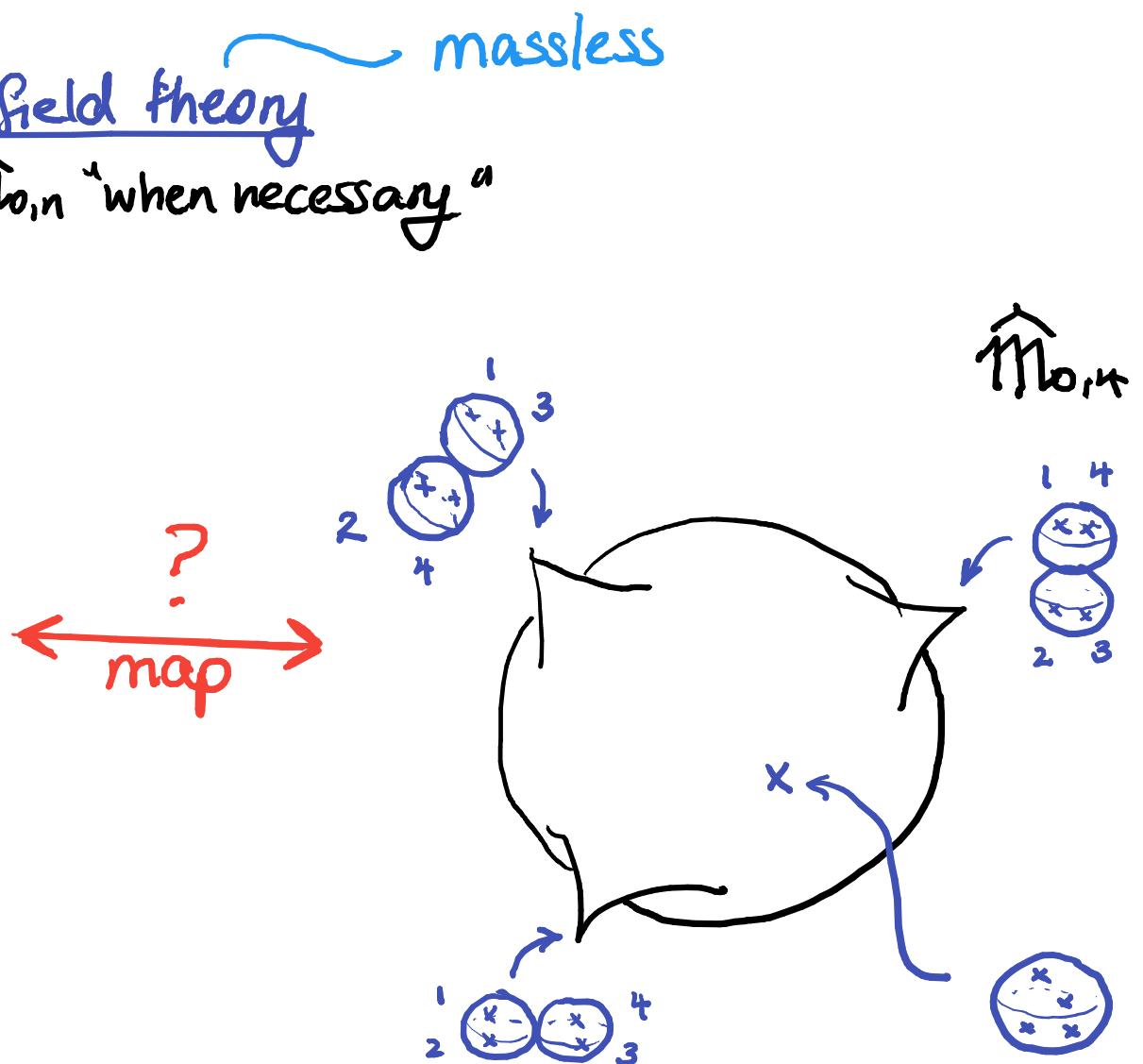
kin space ( $n=4$  pts)



$$s = (k_1 + k_2)^2$$

$$t = (k_1 + k_4)^2$$

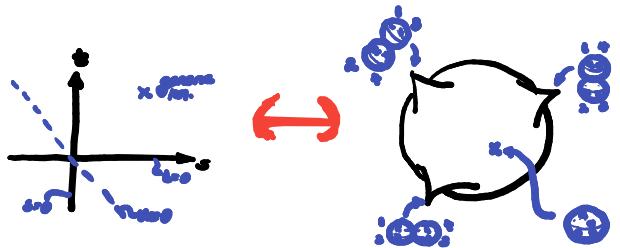
(any dim d)



Claim: map given by SCATTERING EQUATIONS (SE)

$$E_i = \sum_{\substack{j \neq i \\ j=1}}^n \frac{k_i \cdot k_j}{r_i - r_j}$$

$i=1 \dots n$



via SE:  $E_i = \sum_{j \neq i} \frac{k_i \cdot k_j}{\nabla_i - \nabla_j}$

- For  $n=4$ :

Look at  $s \rightarrow 0$



$\leftarrow$ :

$$E_1 = \frac{k_1 \cdot k_2}{\varepsilon} + \left( \frac{k_1 \cdot k_3}{\nabla_{13}} + \frac{k_1 \cdot k_4}{\nabla_{14}} \right)$$

$$E_2 = -\frac{k_1 \cdot k_2}{\nabla_{12}} + \left( \frac{k_2 \cdot k_3}{\nabla_{23}} + \frac{k_2 \cdot k_4}{\nabla_{24}} \right)$$

$$\Rightarrow 0 = \varepsilon(E_1 - E_2) = 2k_1 \cdot k_2 + \varepsilon f$$

parametrized by  
 $\nabla_i = \nabla_2 + \varepsilon$   
for  $\varepsilon \ll 1$

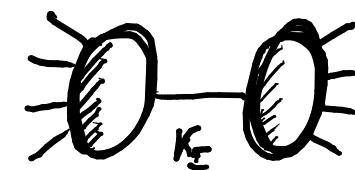
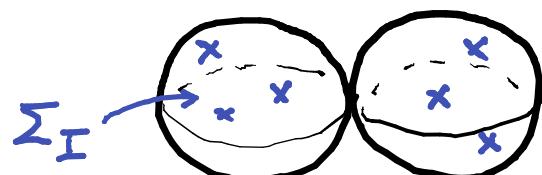
order  $O(1)$

$\rightarrow$ :

$$E_i = k_i \cdot k_1 \left( \frac{1}{\nabla_{i4}} - \frac{1}{\nabla_{i3}} \right) + O(s) \quad \text{for } i=1,2$$

$$\Rightarrow \nabla_i = \nabla_2 + O(s)$$

- In general:



as  $K_F^2 \rightarrow 0$

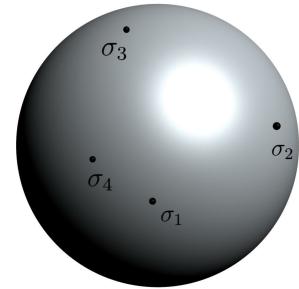
Comment:

Properties of the scattering equations

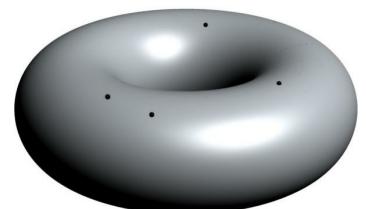
- 1) encode pole structure of massless FTs  
(as we've just seen)
- 2) Möbius invariant  
 $\Rightarrow$  (n-3) of the n scattering equ.  $E_i$  are indep  
 $\Rightarrow$  see homework
- 3) #of solutions =  $(n-3)!$   
 $\Rightarrow$  check, e.g. using soft limit & induction

# Outline

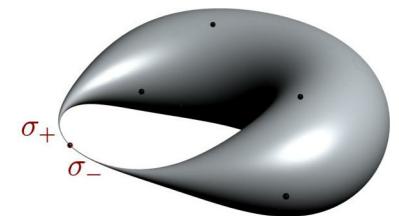
I. Tree-level amplitudes from the sphere  
= CHY formulae



II. Worldsheet model  
= Ambitwistor strings

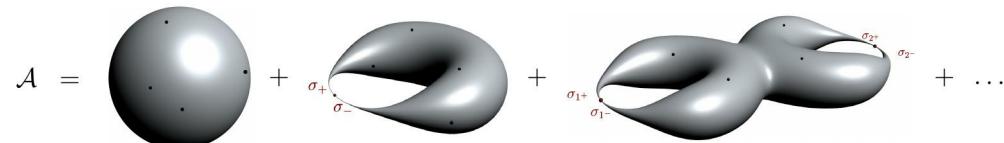


III. One-loop amplitudes from the torus



IV. From the torus to the nodal sphere

V. Some additional aspects

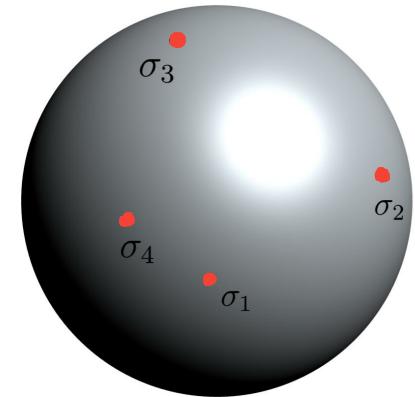


# I. Amplitudes from the sphere

= CHY formula  
✓ Cachazo, He, Yuan

only  $(n-3)$  indep  
 $\cong \sigma_2 \sigma_{23} \sigma_{31} \prod_{j=4}^n \delta(E_j)$

hol.  $\delta$ -fn.  
 $\bar{\delta}(z) := \bar{\sigma} \left( \frac{1}{2\pi i z} \right)$



$$A_n = \int_{M_{0,n}} \frac{\prod_{i=1}^n d\sigma_i}{\text{vol } \text{SL}(2, \mathbb{C})} \prod_{j=1}^n \bar{\delta}(E_j) I_n(r_i, k_i, \epsilon_i)$$

n particles,  
any dim  $d \geq 4$

$=: d\mu_n^{\text{CHY}}$

universal for  
massless FTs

"integrand":

- theory-dependent
- polyn in  $(r_i, \epsilon_i)$

Note: fully localized:  $A_n = \sum_{\text{solns} \neq} \frac{I_n(\sigma_i^*, r_i, \epsilon_i)}{J_{SE}(\sigma_i^*, r_i)}$

Integrand  $I_n = I_n^L I_n^R$

Building blocks / "half-integrands"

PT Factor  
 $\equiv (\alpha(1) \dots \alpha(n))$   
 $\equiv (\alpha)$   
 (notation)

- Colour:  $C_n = \sum_{\alpha \in S_n / Z_n} \frac{\text{tr}(T^{\alpha_{n(1)}} \dots T^{\alpha_{n(n)}})}{(v_{\alpha(1)} - v_{\alpha(2)}) \dots (v_{\alpha(n)} - v_{\alpha(1)})}$

- "Kinematic" (spin):  $\text{Pf}' M := \frac{(-1)^{i+j}}{v_{ij}} \text{Pf } M^{[i,j]}$  remove rows/cols  $i, j$

$2n \times 2n$   
 with kernel  
 $(1 \dots 1 | 0 \dots 0)$   
 $(v_1 \dots v_n | 0 \dots 0)$

$$M = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}$$

with  $\begin{cases} A_{ij} = \frac{k_i \cdot k_j}{v_{ij}} & A_{ii} = 0 \\ B_{ij} = \frac{\epsilon_i \cdot \epsilon_j}{v_{ij}} & B_{ii} = 0 \\ C_{ij} = \frac{\epsilon_i \cdot k_j}{v_{ij}} & C_{ii} = -\sum_{j \neq i} C_{ij} \end{cases}$

Integrand : DOUBLE COPY

$$I^{\text{BAS}} = C_n \tilde{C}_n$$

$$I^{\text{YM}} = C_n \text{Pf}' M$$

$$I^{\text{grav}} = \text{Pf}' M \text{Pf}' \tilde{M}$$

## A couple of comments

### 1) Worldsheet vs standard double copy

- alternative representation of building blocks:

$$C_n = \sum_{\alpha \in S_{n-2}} \frac{C(1 \alpha n)}{(1 \alpha n)}$$

with  $C(1 \alpha n) = \sum_{b_1 \dots b_{n-2}} f^{a_1 a_{\alpha(2)} b_1} \dots f^{b_{n-2} a_{\alpha(n-1)} a_n}$

$$PF'M = \sum_{\alpha \in S_{n-2}} \frac{N(1 \alpha n)}{(1 \alpha n)}$$

BCJ numerator

- this is the KK / half-ladder rep!



Why BCJ?

$$A_n^{\text{YM}} = \sum_{\alpha \in S_{n-2}} C(1 \alpha n) A_n^{\text{YM}}(1 \alpha n)$$

$$A_n^{\text{grav}} = \sum_{\alpha \in S_{n-2}} N(1 \alpha n) A_n^{\text{YM}}(1 \alpha n)$$

## 2) Other theories

$I^*$	$Pf'M$	$\det'A$	$Pf' \times Pf'M_{red}$	$C_1 \dots C_m Pf'\Pi$	$C$
$Pf'M$	E				
$\det'A$	BI	Galileon			
$Pf' \times Pf'M_{red}$	$EM _{U(1)^m}$	DBI	$EMS _{U(1)^m \otimes U(1)^m}$		
$C_1 \dots C_m Pf'\Pi$	EYM	extended DBI	$EYMS _{SU(N) \otimes U(1)^{m'}}$	$EYMS _{SU(N) \otimes SU(N')}$	
$C$	YM	NLSM	$YMS _{SU(N) \otimes U(1)^{m'}}$	gen. $YMS _{SU(N) \otimes SU(N')}$	BS

3) Proof that  $CTY = \text{amplitude} ?$

→ You! ↗

Two main tools:

- BCFW
- scatt. equ.

To go beyond tree-level:

(Remember: loops range from not-easy to hard,  
we won't be able to guess answer)

Question: Is there an underlying worldsheet model?

Attempt: Construct a model based on

1) universal measure vs. theory-dep. integrand

$$S = S_{\text{bare}} + S_{\text{matter}}$$

2) striking feature:

localization on hol. f-fns

$$\prod_{i=1}^n \delta(E_i)$$

} geometric?  
interpret?

## Another look at the Scattering Equations

$$E_i = \sum_{j \neq i} \frac{k_i \cdot k_j}{\sigma_{ij}}$$

More geometric construction:

1) Define one-form  $P_\mu(\sigma) = \sum_{i=1}^n \frac{k_{i\mu}}{\sigma - \sigma_i} d\sigma$

2) Then Scattering equations:

$$E_i = \text{Res}_{\sigma_i} P^2$$

3) Geometrically, this implies  $P^2(\sigma) = 0 \quad \forall \sigma \in \mathbb{CP}^1$

- $P^2$  quadr. diff., so at least 4 poles  
• SE set  $(n-3)$  of  $n$  residues to zero.  $\square$