Prepared for submission to JHEP

Homework Solution for “Bootstrap” at MITP Amplitude Games

Abstract: How to obtain the integrability relations for the 2d HPL symbol alphabet $\mathcal{S} = \{u, v, w, 1-u, 1-v, 1-w\}$.

Contents

1 Integrability relations

The 2dHPL space has symbol alphabet

$$\mathcal{S} = \{u, v, w, 1-u, 1-v, 1-w\}, \tag{1.1}$$

and $w = 1 - u - v$. The u and v derivatives of any function F in the space are

$$\partial_u F = \frac{F^u}{u} - \frac{F^w}{1-u-v} - \frac{F^{1-u}}{1-u} + \frac{F^{1-w}}{u+v}\,, \tag{1.2}$$

$$\partial_v F = \frac{F^v}{v} - \frac{F^w}{1-u-v} - \frac{F^{1-v}}{1-v} + \frac{F^{1-w}}{u+v}\,. \tag{1.3}$$

We need to compute $\partial_v[\partial_u F]$. Take the v derivative of eq. (1.2). There are a couple of explicit v's in $1/(1-u-v)$ and $1/(u+v)$, but mainly one needs to use eq. (1.3) to evaluate the v derivatives of F^u, F^{1-u}, F^w, F^{1-w}. Given the definition of the double coproducts, to differentiate F^X, one tacks the letter X on the back of the superscripts in eq. (1.3), for $X \in \{u, 1-u, w, 1-w\}$.

The result is:

$$\begin{aligned}
\partial_v\partial_u F = {} & \frac{1}{u}\left[\frac{F^{v,u}}{v} - \frac{F^{1-v,u}}{1-v} - \frac{F^{w,u}}{1-u-v} + \frac{F^{1-w,u}}{u+v}\right] \\
& - \frac{1}{1-u-v}\left[\frac{F^{v,w}}{v} - \frac{F^{1-v,w}}{1-v} - \frac{F^{w,w}}{1-u-v} + \frac{F^{1-w,w}}{u+v}\right] \\
& - \frac{1}{1-u}\left[\frac{F^{v,1-u}}{v} - \frac{F^{1-v,1-u}}{1-v} - \frac{F^{w,1-u}}{1-u-v} + \frac{F^{1-w,1-u}}{u+v}\right] \\
& + \frac{1}{u+v}\left[\frac{F^{v,1-w}}{v} - \frac{F^{1-v,1-w}}{1-v} - \frac{F^{w,1-w}}{1-u-v} + \frac{F^{1-w,1-w}}{u+v}\right] \\
& - \frac{F^w}{(1-u-v)^2} - \frac{F^{1-w}}{(u+v)^2}\,.
\end{aligned} \tag{1.4}$$

Now subtract the same expression with $u \leftrightarrow v$. Notice that the single coproduct terms proportional to F^w and F^{1-w} cancel.

The next task is to identify the independent rational structures in this difference, and set the combinations of $F^{X,Y}$ multiplying them to zero. Partial fractioning on u and/or v first can help with this. (You may need to relabel the u's and v's in the superscripts to get Maple or Mathematica to do this for you.)

I initially found that the following set of 10 quantities have to vanish:

$$
\begin{aligned}
\{ \ & F^{u,w} + F^{v,w} - F^{w,u} - F^{w,v} - F^{w,1-u} - F^{w,1-v} + F^{1-u,w} + F^{1-v,w}, \\
& -F^{v,w} + F^{w,v} + F^{w,1-u} - F^{1-u,w}, \\
& F^{u,1-w} + F^{v,1-w} + F^{1-u,1-w} + F^{1-v,1-w} - F^{1-w,u} - F^{1-w,v} - F^{1-w,1-u} - F^{1-w,1-v}, \\
& -F^{1-u,1-w} + F^{1-v,1-w} + F^{1-w,1-u} - F^{1-w,1-v}, \\
& -F^{u,1-w} - F^{v,1-w} + F^{1-w,u} + F^{1-w,v}, \\
& F^{v,1-u} + F^{w,1-u} - F^{1-u,v} - F^{1-u,w} - F^{1-u,1-v} - F^{1-u,1-w} + F^{1-v,1-u} + F^{1-w,1-u}, \\
& -F^{1-u,1-v} + F^{1-u,1-w} + F^{1-v,1-u} - F^{1-w,1-u}, \\
& -F^{v,1-u} - F^{w,1-u} + F^{1-u,v} + F^{1-u,w}, \\
& F^{u,v} + F^{u,w} + F^{u,1-v} + F^{u,1-w} - F^{v,u} - F^{w,u} - F^{1-v,u} - F^{1-w,u}, \\
& -F^{u,v} - F^{u,1-w} + F^{v,u} + F^{1-w,u} \ \} \, .
\end{aligned}
\tag{1.5}
$$

But it's better to take linear combinations to get shorter equations that are related by dihedral symmetry.

The second and tenth equations are already of the desired type, i.e. dihedral images of

$$
F^{u,v} - F^{v,u} + F^{1-w,v} - F^{v,1-w} = 0. \tag{1.6}
$$

There is a linear combination of the second and eighth equations that is the $v \leftrightarrow w$ flip of the second equation. The sum of the first two equations is another dihedral image, as is the sum of the fifth and the tenth, and the sum of the ninth and the tenth. That completes the 6 dihedral images of eq. (1.6). They can be used to remove from the other equations everything but $F^{1-u_i,1-u_j}$. These come in quartets in eq. (1.5), but for example the sum of the third and fourth equations leads to

$$
F^{1-v,1-w} - F^{1-w,1-v} = 0, \tag{1.7}
$$

and the difference gives another dihedral image. The third and final one comes out of the sixth and seventh equations. There is one overall linear relation among the 10 equations.