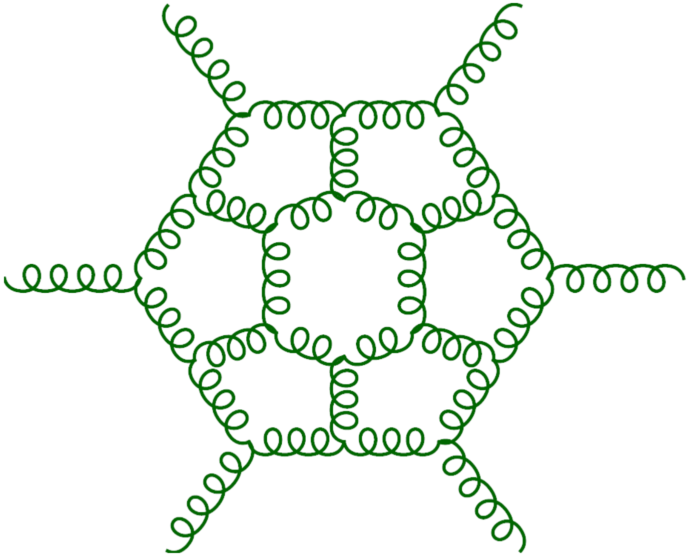
[Amplitude] Bootstrap
Lecture 4



Lance Dixon (SLAC)

"The Amplitude Games"
Mainz Institute for Theoretical Physics
19-20 July, 2021

SLAC NATIONAL ACCELERATOR LABORATORY

Hexagon functions and 6 gluon amplitudes

- Many similarities to *Hggg* form factor; for example, alphabet has 9 letters, 6 of which are the same:

$$\{u, v, w, 1-u, 1-v, 1-w\}$$

except now w is independent of u, v:

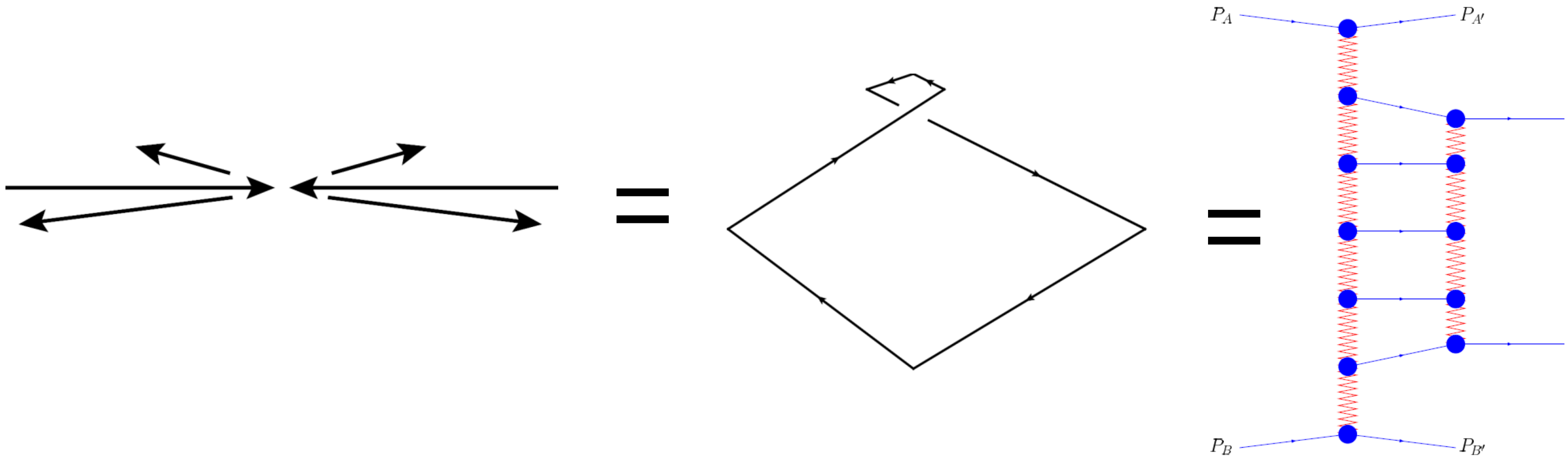
$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12}s_{45}}{s_{123}s_{345}}$$

$$v = \frac{s_{23}s_{56}}{s_{234}s_{123}}$$

$$w = \frac{s_{34}s_{61}}{s_{345}s_{234}}$$

- Will get to the other 3 letters shortly.
- With 1 more kinematical variable, several more limits are accessible

Multi-regge limit

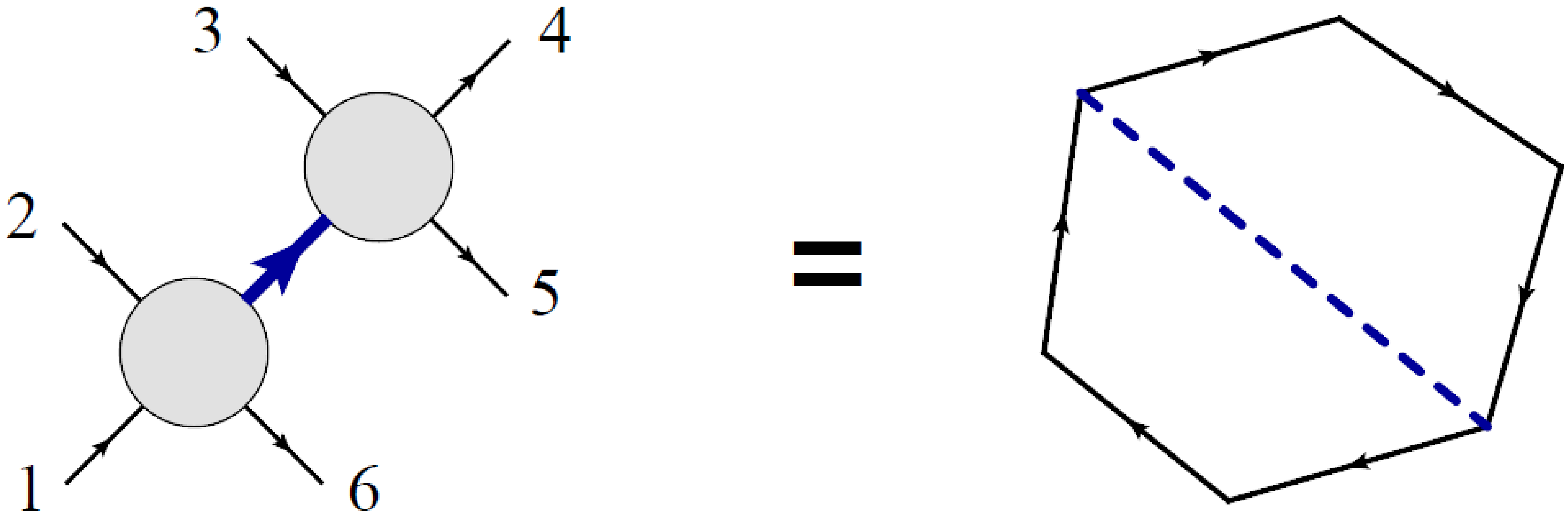


- Amplitude factorizes in Fourier-Mellin space

Bartels, Lipatov, Sabio Vera, 0802.2065, Fadin, Lipatov, 1111.0782;
LD, Duhr, Pennington, 1207.0186; Pennington, 1209.5357;
Basso, Caron-Huot, Sever, 1407.3766 (analytic continuation from OPE limit);
Broedel, Sprenger, 1512.04963; Lipatov, Prygarin, Schnitzer, 1205.0186;
LD, von Hippel, 1408.1505; Del Duca, Druc, Drummond, Duhr, Dulat, Marzucca, Papathanasiou, Verbeek, 1606.08807;…

Factorization on multi-particle pole

Bern, Chalmers, hep-ph/9503236; LD, von Hippel, 1408.1505;
Basso, Sever, Vieira (Sever talk at Amplitudes 2015)



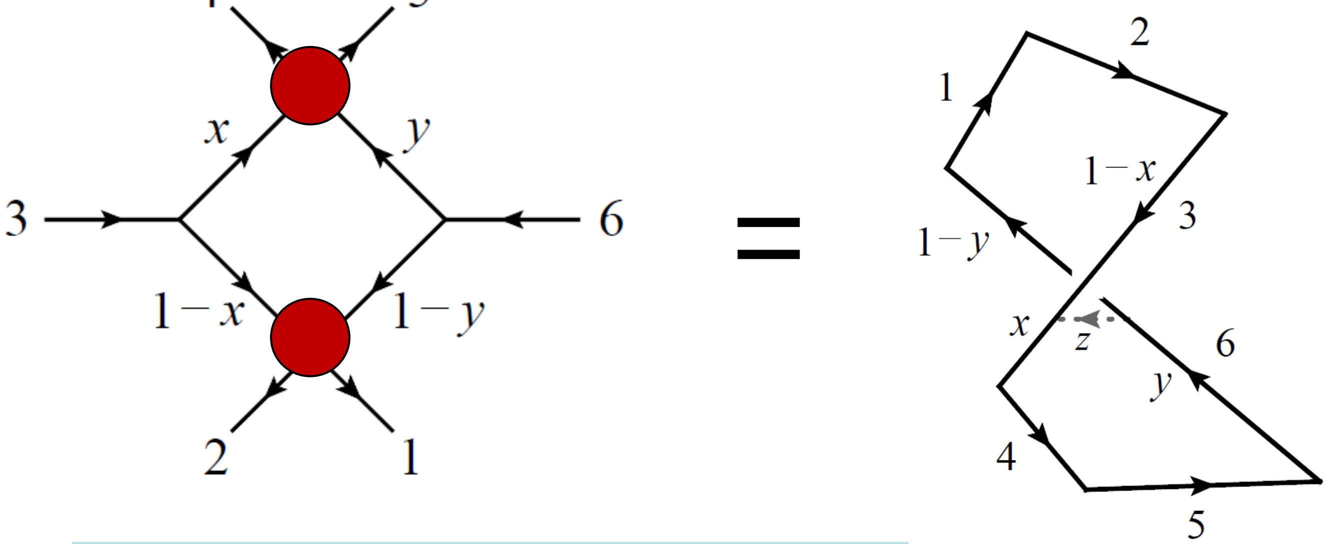
- Virtual Sudakov region, $A \sim \exp[-\ln^2 \delta],$

$$\delta \sim s_{345}$$

- Can study to very high accuracy in planar N=4 SYM

Double-parton-scattering-like limit

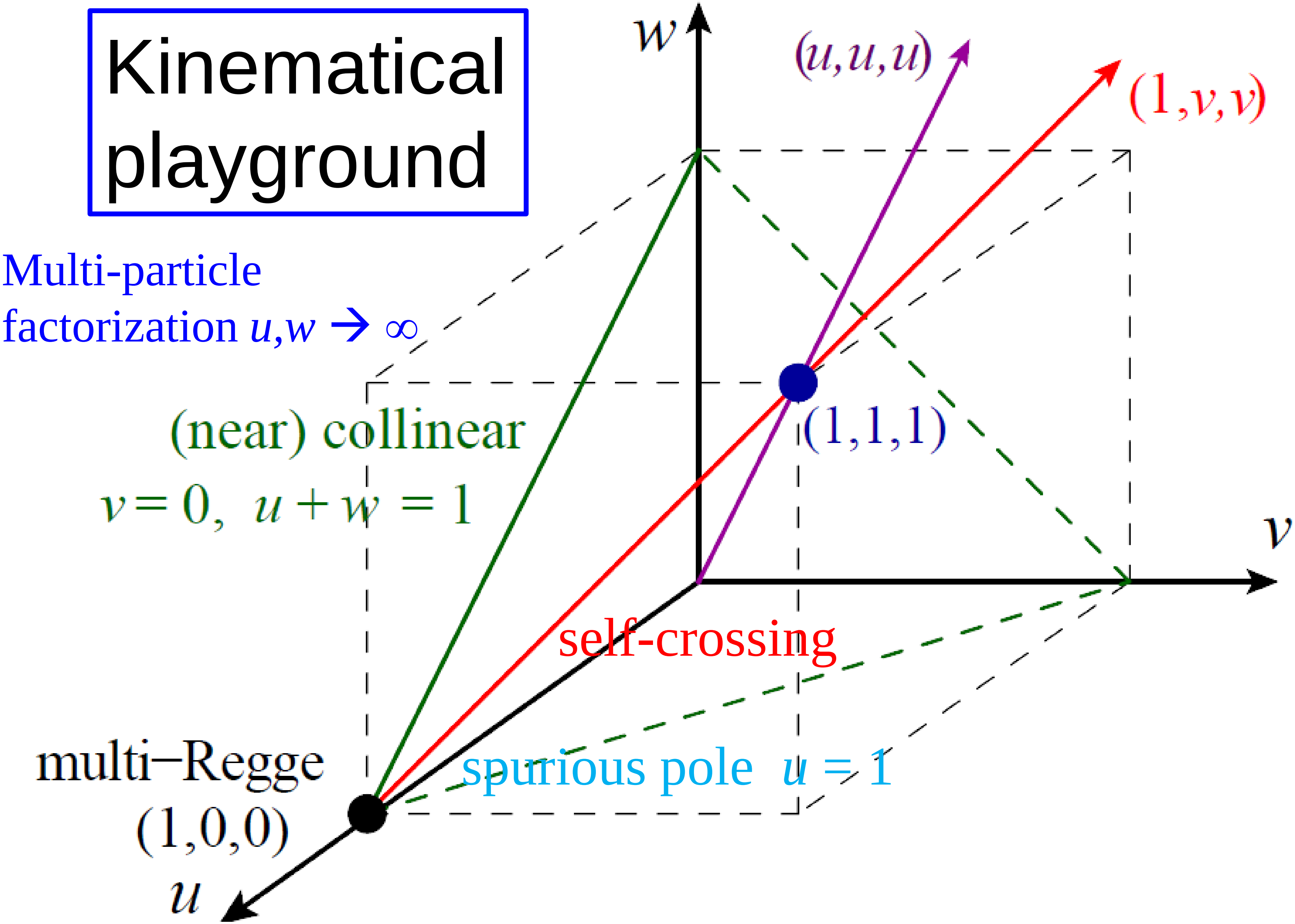
Georgiou, 0904.4675; LD, Esterlis, 1602.02107



- Self-crossing limit of Wilson loop, $\delta \sim |z|^2 \rightarrow 0$
- Overlaps MRK limit
- Another Sudakov region
- Singularities ~ Wilson line RGE

Korchemsky and Korchemskaya hep-ph/9409446

Kinematical playground



Key “initial” condition

- Two-loop 6-gluon result first computed numerically from both amplitude and Wilson loop pictures
  Bern, LD, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1465;
  Drummond, Henn, Korchemsky, Sokatchev, 0803.1466
- Wilson loop side then evaluated analytically
  → 17 pages of [Goncharov] polylogarithms
  Del Duca, Duhr, Smirnov, 0911.5332, 1003.1702
- Simplified to a few lines in term of classical polylogs $\mathrm{Li}_n(x)$, demonstrating power of symbol
  Goncharov, Spradlin, Vergu, Volovich, 1006.5703
- Exposed the 9 letter symbol alphabet

Symbol of $R_6^{(2)}(u,v,w)$

GSVV, 1006.5703

$$
\begin{aligned}
-8\,\mathcal{S}[R_6^{(2)}] \;=\;& u \otimes (1-u) \otimes \frac{u}{(1-u)^2} \otimes \frac{u}{1-u} \\
&+ 2\,(u \otimes v + v \otimes u) \otimes \frac{w}{1-v} \otimes \frac{u}{1-u} \\
&+ 2\,v \otimes \frac{w}{1-v} \otimes u \otimes \frac{u}{1-u} \\
&+ u \otimes (1-u) \otimes y_u y_v y_w \otimes y_u y_v y_w \\
&- 2\,u \otimes v \otimes y_w \otimes y_u y_v y_w
\end{aligned}
$$

$$+\ 5 \text{ permutations of } (u, v, w)$$

Nine hexagon symbol letters

$$\boxed{\mathcal{S} = \{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}}$$

- y_i not independent of u_i :

$$y_u \equiv \frac{u - z_+}{u - z_-}, \ \ldots \quad \text{where}$$

$$z_\pm = \frac{1}{2}\left[-1 + u + v + w \pm \sqrt{\Delta}\right]$$

$$\Delta = (1-u-v-w)^2 - 4uvw$$

All are projectively invariant combinations of 4 brackets $< ijkl > \equiv \varepsilon_{ABCD} Z_i^A Z_j^B Z_k^C Z_l^D$ of momentum-twistors Z_i^A:

$$u = \frac{<1234><1456>}{<1245><1346>} \qquad 1 - u = \frac{<1246><1345>}{<1245><1346>} \qquad y_u = \frac{<1234><1356><2456>}{<1235><1456><2346>}$$

Branch cuts

- Again, branch cuts only for $s_{i,i+1}, s_{i,i+1,i+2} = 0$, so $u, v, w \to 0, \infty$.

→ first entries limited to $\{u, v, w\}$

$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12} s_{45}}{s_{123} s_{345}}$$

$$v = \frac{s_{23} s_{56}}{s_{234} s_{123}}$$

$$w = \frac{s_{34} s_{61}}{s_{345} s_{234}}$$

- To expose Steinmann relations, switch to

$$\mathcal{S}' = \{a, b, c, d, e, f, y_u, y_v, y_w\}$$

$a = \frac{u}{vw} = (s_{234})^2 \times (s_{i,i+1}\ stuff)$ has only one 3-particle invariant!

$d = \frac{1-u}{u}$

BDS-like normalization $\mathrm{Disc}_{s_{234}}\left[\mathrm{Disc}_{s_{123}} \mathcal{E}_6\right] = 0$

→ $\boxed{\mathrm{Disc}_a[\mathrm{Disc}_c \mathcal{E}_6] = 0 \quad + \text{ dihedral images}}$

40 pairs

- Steinmann says $a \otimes \cancel{b \otimes} \ldots$ + dihedral
- "Extended" Steinmann observed in $\mathcal{E}_6^{(L)}$:

$$\ldots \cancel{a \otimes b} \otimes \ldots \qquad + \text{ dihedral}$$

⇔ Steinmann relations hold on arbitrary Riemann sheets

Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou, 1806.01361, 1903.10890, 1906.07116

- Also interpreted mathematically as "cluster adjacency"
  Drummond, Foster, Gürdoğan, 1710.10953, 1810.08149
- Constructing the space of hexagon functions obeying first entry and extended Steinmann relations, only 40 of the possible 81 = 9 * 9 pairs appear. No triple relations.

Hexagon functions $\mathcal{H}$ grow more slowly than $\mathcal{C}$

weight n	0	1	2	3	4	5	6	7	8	9	10	11
symbols in $\mathcal{C}$	1	3	9	21	48	108	249	567	1290	??	??	??
symbols in $\mathcal{H}$	1	3	6	13	26	51	98	184	339	612	1083	1885

- So it is possible to construct the space to higher weights; however the multiple final entries aren't understood as well (empirically).

Compare numbers of independent coproducts

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$L=1$	1	3	1														
$L=2$	1	3	6	3	1												
$L=3$	1	3	9	12	6	3	1										
$L=4$	1	3	9	21	24	12	6	3	1								
$L=5$	1	3	9	21	46	45	24	12	6	3	1						
$L=6$	1	3	9	21	48	99	85	45	24	12	6	3	1				
$L=7$	1	3	9	21	48	108	236	155	85	45	24	12	6	3	1		
$L=8$	1	3	9	21	48	108	242	466	279	155	85	45	24	12	6	3	1

form factor space $\mathcal{C}$ saturates quickly on right

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$L=1$	1	3	4												
$L=2$	1	3	6	10	6										
$L=3$	1	3	6	13	23	15	6								
$L=4$	1	3	6	13	27	50	50	24	6						
$L=5$	1	3	6	13	27	54	97	117	70	24	6				
$L=6$	1	3	6	13	27	54	102	188	255	179	78	24	6		
$L=7$	1	3	6	13	27	54	102	190	337	490	409	209	79	24	6
$L\leq 7$	1	3	6	13	27	54	102	190	337	490	416	219	82	24	6

hexagon function space $\mathcal{H}$ saturates quickly on left

6 gluons: MHV and NMHV

- MHV amplitude: A supersymmetry Ward identity implies dihedral invariance of $\frac{A_6^{\mathrm{MHV}}(g^2)}{A_6^{\mathrm{MHV,tree}}}$ and hence $\mathcal{E}_6(u,v,w)$ is totally symmetric
- NMHV amplitude instead should be decomposed over dual superconformal *R*-invariants [5 brackets]

$$(f) \equiv [abcde] = \frac{\delta^4\left(\chi_a\langle bcde\rangle + \mathrm{cyclic}\right)}{\langle abcd\rangle\langle bcde\rangle\langle cdea\rangle\langle deab\rangle\langle eabc\rangle}$$

NMHV decomposition

BDS-like normalized super-amplitude

$$\hat{\mathcal{P}}_{\mathrm{NMHV}} \equiv \frac{\mathcal{A}_{\mathrm{NMHV}}}{\mathcal{A}_{\mathrm{MHV}}^{\mathrm{BDS-like}}}$$

Drummond, Henn, Korchemsky, Sokatchev, 0807.1095;
LD, von Hippel, McLeod, 1509.08127

$$\begin{aligned}\hat{\mathcal{P}}_{\mathrm{NMHV}} = \ & \frac{1}{2}\Big[[(1)+(4)]E(u,v,w)+[(2)+(5)]E(v,w,u)+[(3)+(6)]E(w,u,v) \\ & +[(1)-(4)]\tilde{E}(u,v,w)-[(2)-(5)]\tilde{E}(v,w,u)+[(3)-(6)]\tilde{E}(w,u,v)\Big]\end{aligned}$$

Rational prefactors:
Grassmann-containing
dual superconformal
invariants, (a) = [bcdef]

$E, \tilde{E}$ = hexagon functions, like $\mathcal{E}$

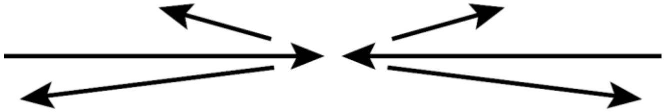
Constraints

- Again, very similar to form factor.
- Also have a multi-Regge constraint

Basso, Caron-Huot, Sever, 1407.3766 (analytic continuation from OPE)

- Technically easier to impose than OPE because it operates at “leading transcendentality”, while the OPE constraints involve transcendentality drop, leading to more complicated functional behavior.

2→4 multi-Regge limit



- Euclidean MRK limit (a soft limit) vanishes
- To get nonzero result for physical region, first let

$u \to u\, e^{-2\pi i}$, then $u \to 1, \quad v, w \to 0$

$$\frac{v}{1-u} \to \frac{1}{|1-z|^2} \qquad \frac{w}{1-u} \to \frac{|z|^2}{|1-z|^2}$$

$$\boxed{R_6^{(L)} \to (2\pi i) \sum_{r=0}^{L-1} \ln^r(1-u) \left[g_r^{(L)}(z,\bar{z}) + 2\pi i\, h_r^{(L)}(z,\bar{z}) \right]}$$

$g_r^{(L)}$ and $h_r^{(L)}$
all well understood by now;
all SVHPLs (Brown, 2004);
also NMHV behavior

Fadin, Lipatov, 1111.0782;
LD, Duhr, Pennington, 1207.0186;
Pennington, 1209.5357; Basso, Caron-Huot, Sever, 1407.3766; Broedel, Sprenger, 1512.04963

Lipatov, Prygarin, Schnitzer, 1205.0186;
LD, von Hippel, 1408.1505

MRK Master formulae

$$w = -z, \quad w^* = -\bar{z}$$

- MHV:

$$e^{R+i\pi\delta}|_{\mathrm{MRK}} = \cos\pi\omega_{ab} + i\,\frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n\left(\frac{w}{w^*}\right)^{\frac{n}{2}}\int_{-\infty}^{+\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}\,|w|^{2i\nu}\,\boxed{\Phi_{\mathrm{Reg}}(\nu,n)}$$
$$\times\left(-\frac{1}{1-u}\,\frac{|1+w|^2}{|w|}\right)^{\boxed{\omega(\nu,n)}}$$

NLL: Fadin, Lipatov, 1111.0782;
Caron-Huot, 1309.6521

- NMHV:

$$\exp(R^{\mathrm{NMHV}}+i\pi\delta)|_{\mathrm{MRK}} = \boxed{\mathcal{P}}\exp(R^{\mathrm{MHV}}+i\pi\delta)$$
$$= \cos\pi\omega_{ab} - i\frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n\left(\frac{w}{w^*}\right)^{\frac{n}{2}}\int_{-\infty}^{+\infty}\boxed{\frac{d\nu}{(i\nu+\frac{n}{2})^2}}|w|^{2i\nu}$$
$$\times\,\Phi_{\mathrm{Reg}}^{\boxed{\mathrm{NMHV}}}(\nu,n)\left(-\frac{1}{1-u}\,\frac{|1+w|^2}{|w|}\right)^{\omega(\nu,n)}$$

LL: Lipatov, Prygarin, Schnitzer, 1205.0186

NMHV MRK limit

Like g, h for R_6:
Extract p, q from V, $\tilde{V}$
→ linear combinations of SVHPLs [Brown, 2004]

$$R_6^{(L)} \to (2\pi i) \sum_{r=0}^{L-1} \ln^r(1-u) \left[g_r^{(L)}(w,w^*) + 2\pi i\, h_r^{(L)}(w,w^*) \right]$$

$$\begin{aligned} \mathcal{P}_{\mathrm{MRK}}^{(L)} =& (2\pi i) \sum_{r=0}^{L-1} \ln^r(1-u) \Bigg[\frac{1}{1+w^*} \Big(p_r^{(L)}(w,w^*) + 2\pi i\, q_r^{(L)}(w,w^*) \Big) \\ & + \frac{w^*}{1+w^*} \Big(p_r^{(L)}(w,w^*) + 2\pi i\, q_r^{(L)}(w,w^*) \Big) \Big|_{(w,w^*)\to(\frac{1}{w},\frac{1}{w^*})} \Bigg] + \mathcal{O}(1-u) \end{aligned}$$

- Then match p, q to master formula for factorization in Fourier-Mellin space

MRK limits agree with all-orders predictions

Basso, Caron-Huot, Sever 1407.3766

- BFKL eigenvalue:

$$E^{(1)}(\nu,n),\ E^{(2)}(\nu,n),\ E^{(3)}(\nu,n)\ ,\ldots$$

LL, NLL, NNLL, NNNLL

- Impact factors:

$$\Phi_{\rm Reg}^{\rm (N)MHV,(1)}(\nu,n),\ \Phi_{\rm Reg}^{\rm (N)MHV,(2)}(\nu,n),\ \Phi_{\rm Reg}^{\rm (N)MHV,(3)}(\nu,n),\ \Phi_{\rm Reg}^{\rm (N)MHV,(4)}(\nu,n),\ldots$$

- All zeta-valued linear combinations of:

derivatives of $\ln\Gamma(1 \pm i\nu + \frac{n}{2})$ $\quad \dfrac{i\nu}{\nu^2+\frac{n^2}{4}},\quad \dfrac{n}{\nu^2+\frac{n^2}{4}}$

Master Table

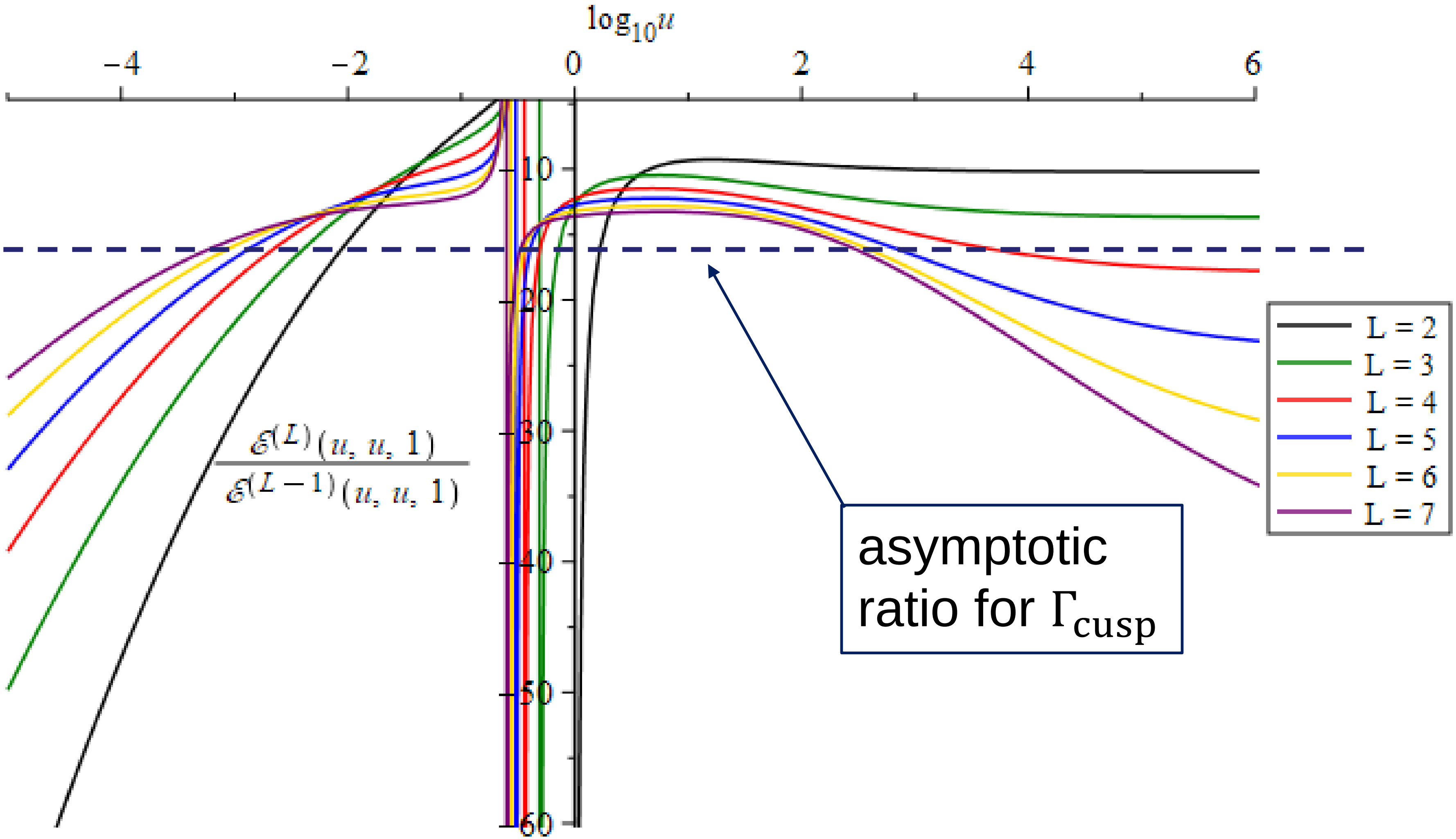
(MHV,NMHV): parameters left in $(\mathcal{E}^{(L)},\ E^{(L)} \& \tilde{E}^{(L)})$

Constraint	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$	$L=6$
1. All functions	(6,6)	(25,27)	(92,105)	(313,372)	(991,1214)	(2951,3692?)
2. Symmetry	(2,4)	(7,16)	(22,56)	(66,190)	(197,602)	(567,1795?)
3. Final entry	(1,1)	(4,3)	(11,6)	(30,16)	(85,39)	(236,102)
4. Collinear limit	(0,0)	(0,0)	$(0^*, 0^*)$	$(0^*, 2^*)$	$(1^{*3}, 5^{*3})$	$(6^{*2}, 17^{*2})$
5. LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*, 0^*)$	$(1^{*2}, 2^{*2})$
6. NLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*, 0^*)$	$(1^*, 0^*)$
7. NNLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
8. N^3LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
9. all MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
10. T^1 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
11. $T^2F^2\ln^4 T$ OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
12. all T^2F^2 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

(0,0) → amplitude uniquely determined

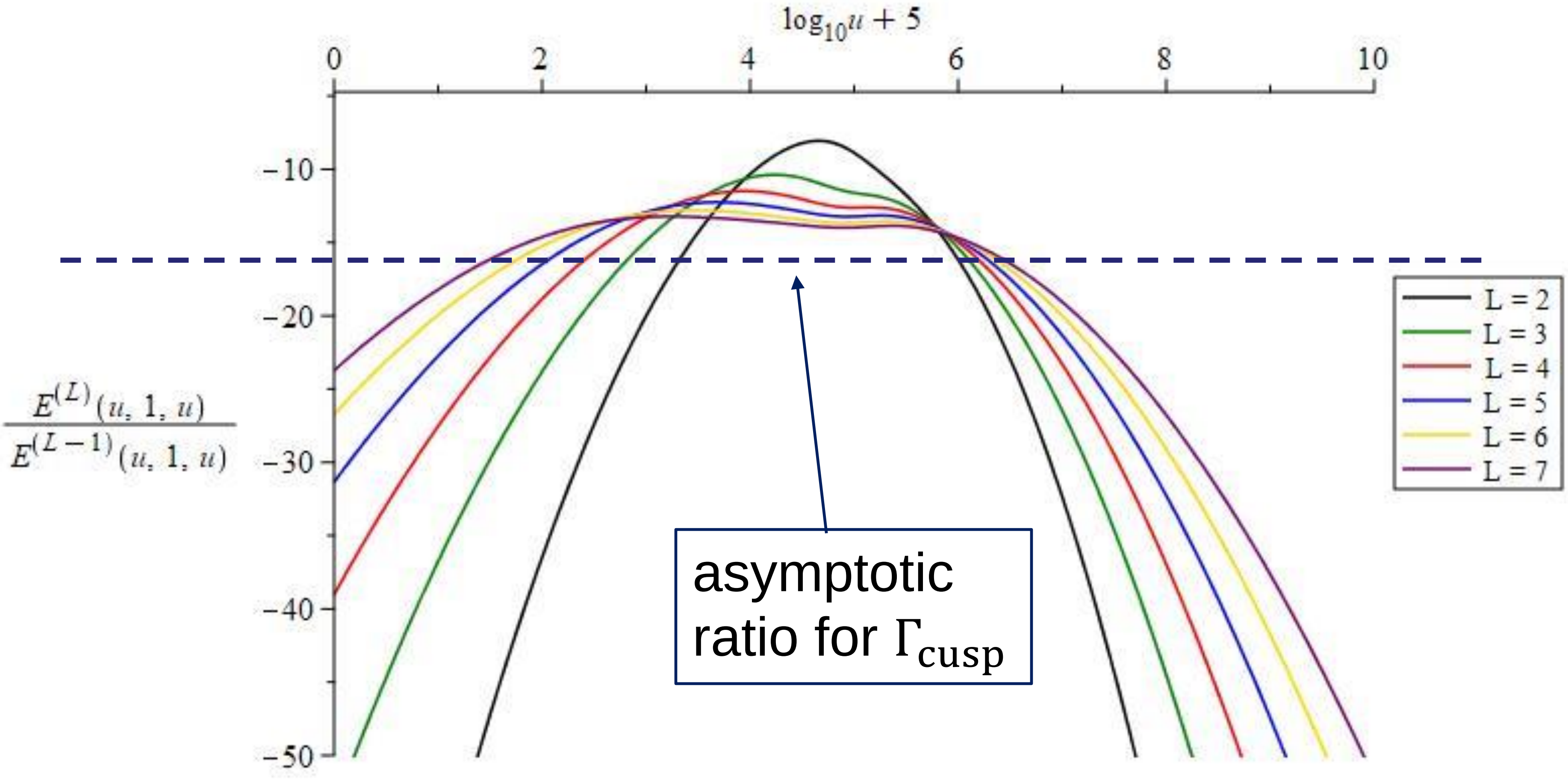
$L = 7$ too

MHV, Euclidean region



NMHV, Parity Even

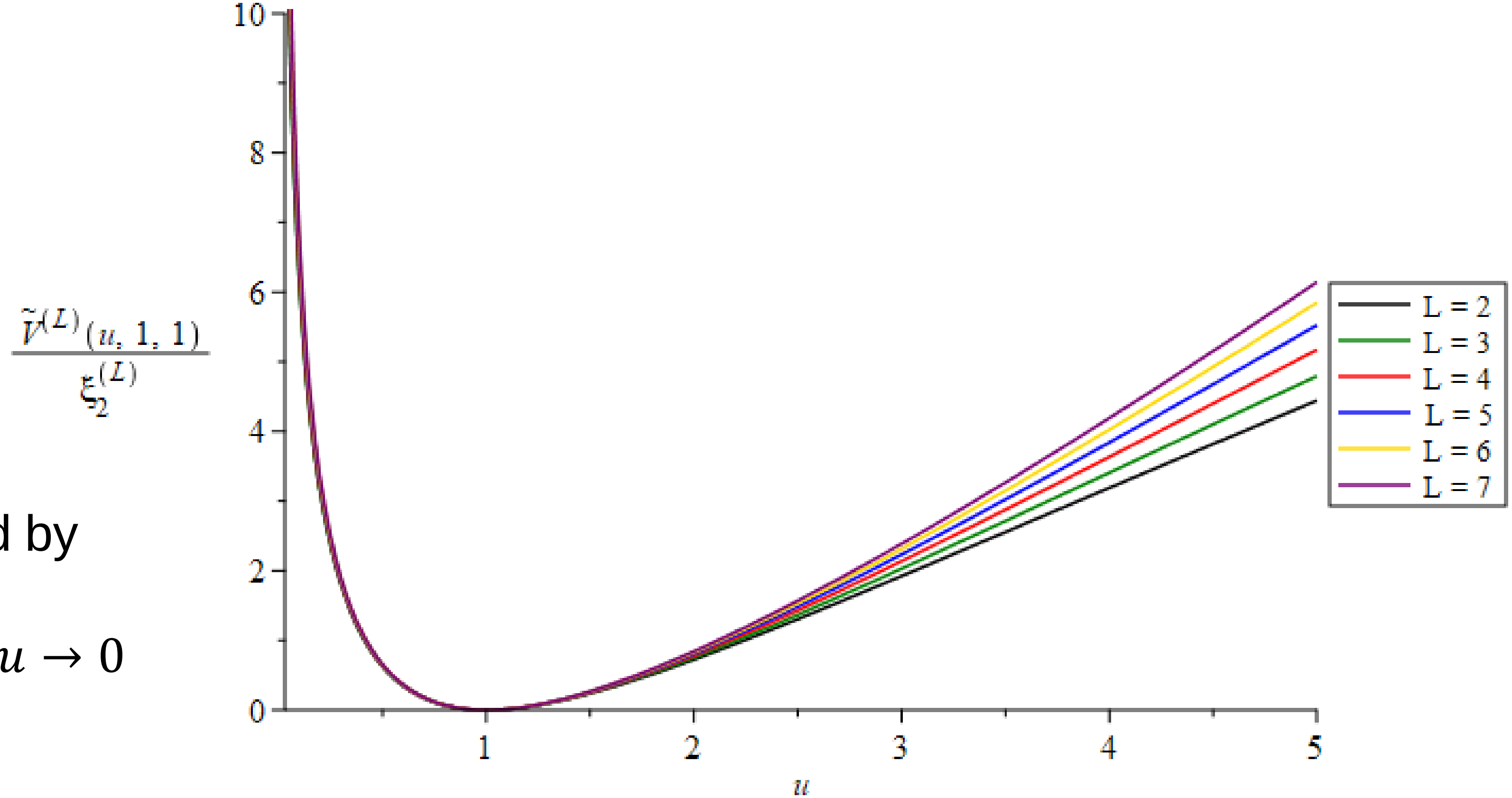
LD, Dulat, to appear



NMHV, Parity Odd

LD, Dulat, to appear

$$\tilde{V} = \frac{\tilde{E}}{\mathcal{E}} = parity\ odd\ part\ of\ "ratio\ function"$$



normalized by coefficient of $\ln^2 u$ as $u \to 0$

Beyond 6 gluons

- Cluster algebras provide strong clues to right polylogarithmic function space
  Golden, Goncharov, Paulos, Spradlin, Volovich, Vergu, 1305.1617, 1401.6446, 1411.3289; Drummond, Foster, Gurdogan, 1710.10953
- Symbol of 3-loop MHV 7-point amplitude bootstrapped first. 42 letter alphabet.
  More rigid: No need for OPE constraints
  Drummond, Papathanasiou, Spradlin 1412.3763
- With Steinmann relations, could go to 4-loop MHV and 3-loop non-MHV LD, Drummond, McLeod, Harrington, Papathanasiou, Spradlin, 1612.08976
- Extended Steinmann → 4-loop NMHV
  Drummond, Gurdogan, Papathanasiou, 1812.04640
- Now lifted from symbols → actual functions!
  LD, Liu, 2007.12966

7 gluons: 6 variables, 42 letters

$$a_{11} = \frac{\langle 1234\rangle\langle 1567\rangle\langle 2367\rangle}{\langle 1237\rangle\langle 1267\rangle\langle 3456\rangle}, \qquad a_{41} = \frac{\langle 2457\rangle\langle 3456\rangle}{\langle 2345\rangle\langle 4567\rangle},$$

$$a_{21} = \frac{\langle 1234\rangle\langle 2567\rangle}{\langle 1267\rangle\langle 2345\rangle}, \qquad a_{51} = \frac{\langle 1(23)(45)(67)\rangle}{\langle 1234\rangle\langle 1567\rangle}$$

$$a_{31} = \frac{\langle 1567\rangle\langle 2347\rangle}{\langle 1237\rangle\langle 4567\rangle}, \qquad a_{61} = \frac{\langle 1(34)(56)(72)\rangle}{\langle 1234\rangle\langle 1567\rangle}$$

$$\langle a(bc)(de)(fg)\rangle \equiv \langle abde\rangle\langle acfg\rangle - \langle abfg\rangle\langle acde\rangle$$

- plus cyclic, $i \rightarrow i+1$ (mod 7), $a_{ji} \rightarrow a_{j,i+1}$ (6 x 7 = 42)

Number of (first 2 entry) Steinmann heptagon symbols

Weight $k =$	1	2	3	4	5	6	7	$7''$
parity $+$, flip $+$	4	16	48	154	467	1413	4163	3026
parity $+$, flip $-$	3	12	43	140	443	1359	4063	2946
parity $-$, flip $+$	0	0	3	14	60	210	672	668
parity $-$, flip $-$	0	0	3	14	60	210	672	669
Total	7	28	97	322	1030	3192	9570	7309

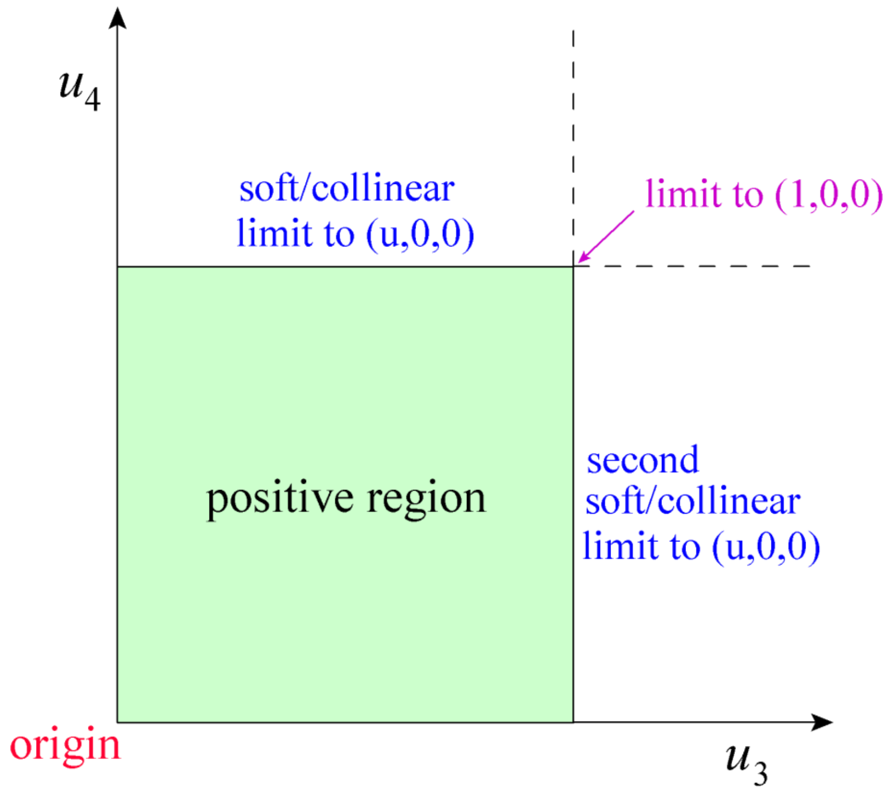
Table 1. Number of Steinmann heptagon symbols at weights 1 through 7, and those satisfying the MHV next-to-final entry condition at weight 7.

Enough to get symbols of 4 loop MHV & 3 loop NMHV amplitude.
Even less boundary data needed: just well-defined collinear limits.

Very useful 2d slice of heptagon kinematics:

"CO" surface on boundary

interpolates between Soft/Collinear limits and Origin



$u_{1,2,5,6} \ll 1 \qquad u_7 = 1$

Symbol letters on CO surface

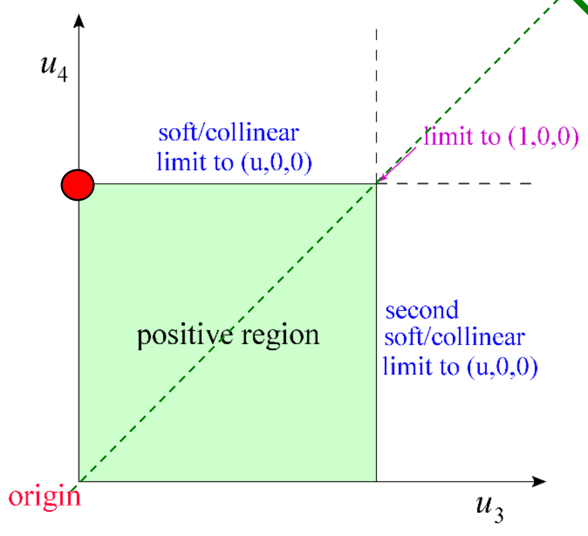
$$\{u_1, u_2, u_5, u_6, u_3, 1-u_3, u_4, 1-u_4, 1-u_7\}$$

- Remarkably simple! In fact, coefficients of $1-u_7$ coproduct always vanish, as a branch-cut constraint.
- Function space is just single-argument harmonic polylogs:

$$\{\ln(u_i), i=1,2,5,6\} \otimes \{H_{\vec{w}}(u_3)\} \otimes \{H_{\vec{w}}(u_4)\}, \quad w_k \in \{0,1\}$$

- **"CO" surface ~ two copies of 6-point $(u,0,0)$ line**

Dihedral D_7 symmetry and CO surface



$u_{1,2,5,6} << 1$ $u_7 = 1$

Flip

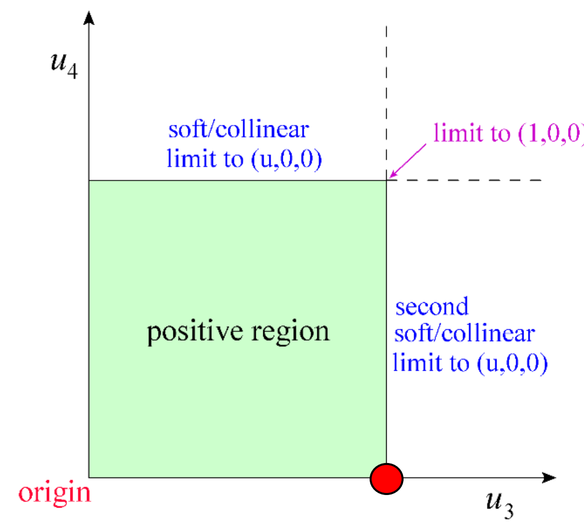
$u_3 \leftrightarrow u_4$

$u_2 \leftrightarrow u_5$

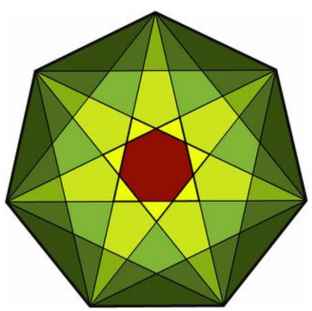
$u_1 \leftrightarrow u_6$

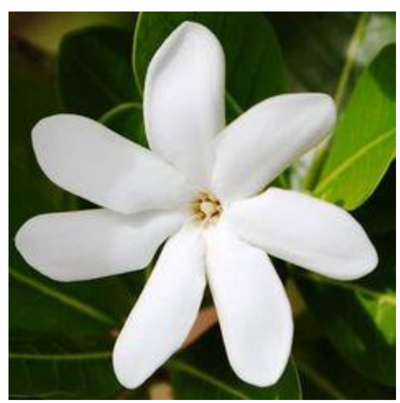
Cycle by 3

$u_4 \rightarrow u_7 \rightarrow u_3 \rightarrow u_6 \rightarrow u_2 \rightarrow u_5 \rightarrow u_1 \rightarrow u_4$



$u_{1,2,5,6} << 1$ $u_7 = 1$





touching at points is enough to carry constants of integration from one “petal” to another

CO surface as triple scaling limit

- BSV parametrization: $T_i, S_i, F_i, i = 1,2$

Basso, Sever, Vieira 1306.2058

$$(Z_1 \ \ldots \ Z_7) = \begin{pmatrix} \frac{S_1}{\sqrt{F_1}} & 1 & -1 & -S_2\sqrt{F_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{T_2\sqrt{F_2}} & \frac{S_2+T_2F_2}{T_2S_2\sqrt{F_2}} & 1 & \frac{1}{S_1\sqrt{F_1}} \\ \frac{\sqrt{F_1}}{T_1} & 0 & 0 & -\frac{1}{T_2\sqrt{F_2}} & -\frac{1}{T_2\sqrt{F_2}} & 0 & \frac{\sqrt{F_1}}{T_1} \\ T_1\sqrt{F_1} & 0 & 1 & \frac{1+T_2S_2F_2+T_2^2}{T_2\sqrt{F_2}} & \frac{1}{T_2\sqrt{F_2}} & 0 & 0 \end{pmatrix}$$

- OPE limit: $T_i \to \epsilon\, T_i, \quad \epsilon \to 0$

BSV, 1407.1736

- Double scaling limit: $T_i \to \epsilon\, T_i, \quad F_i \to \frac{F_i}{\epsilon}, \quad \epsilon \to 0$
- CO limit: $T_i \to \epsilon\, T_i, \quad S_i \to \frac{S_i}{\epsilon}, \quad F_i \to \frac{F_i}{\epsilon^2}, \quad \epsilon \to 0$
- Similar limits for $n > 7$

Basso, LD, Liu, Papathanasiou, in progress

Branch cut conditions

- **The way to fix all ζ values, build consistent functions**
  LD, Drummond, von Hippel, Pennington, 1308.2276
- Functions $\boldsymbol{\zeta_m \ln s_k}$ for $s_k \neq u_k$ have unphysical branch cuts. Not independent functions, but get "attached" to other functions to make their branch cuts all physical.
- **Where to fix?** Near $\boldsymbol{s_k} = 0$, initially.
- **Hexagon natural base point:** $(u, v, w) = (1,1,1)$
- Finite, dihedrally invariant, all hexagon functions evaluate to multiple zeta values (MZV's) there.
- **"Bulky" choice of branch cut conditions**: $\text{no} \;\; \ln(1-u_i) \;\; \text{at} \;\; (1,1,1)$

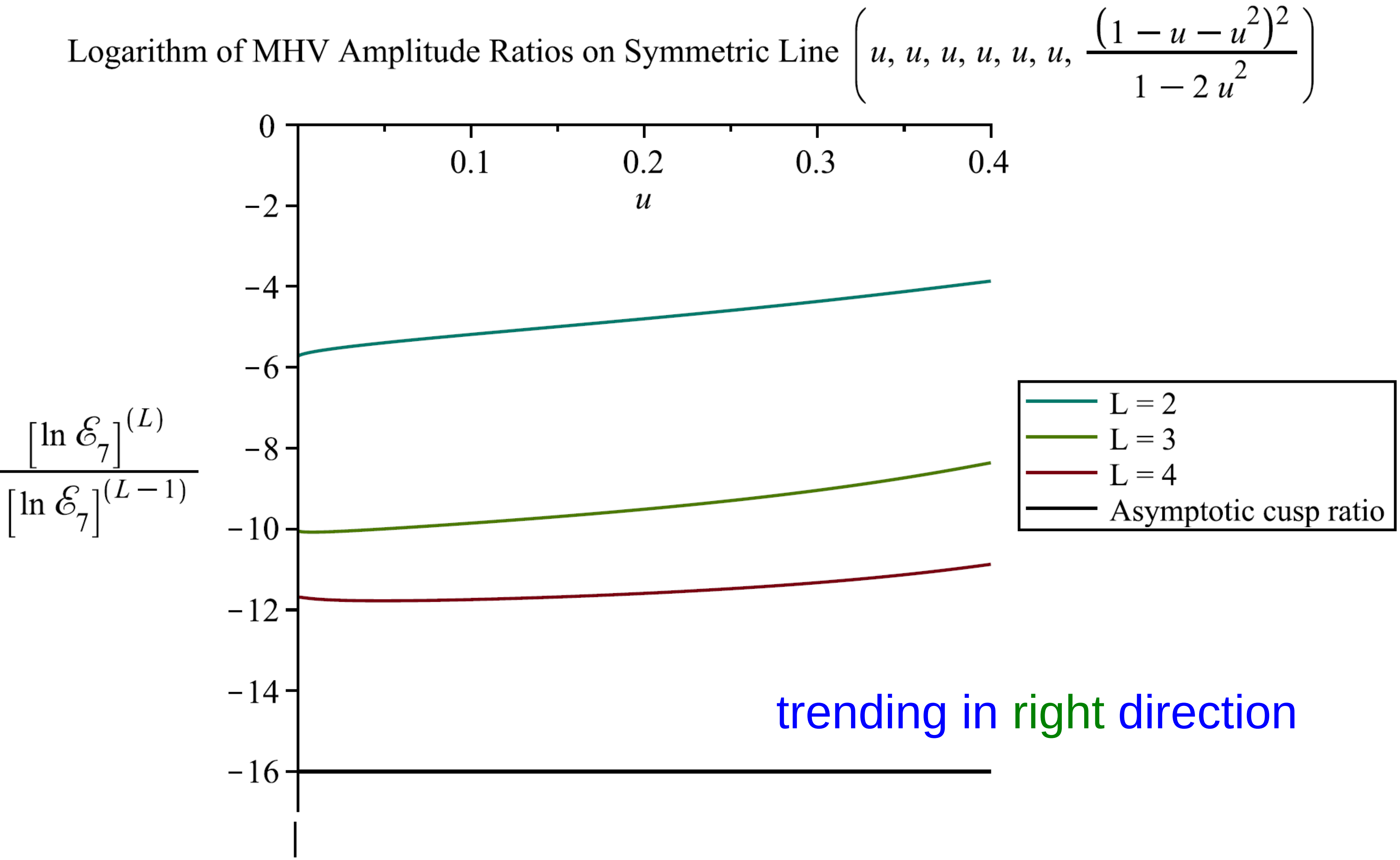
  $$\text{no} \;\; \ln(y_u) \text{ or } \ln(y_v) \;\; \text{at} \;\; (u, u, 1)|_{u\to 0}$$

- Transport results back up $(u, u, 1)$ line to $(1,1,1)$; remarkably only a restricted set of MZV's appear there: $1,\ \zeta_2,\ [no\ \ \zeta_3],\ \zeta_4,\ 5\zeta_5 - 2\zeta_2\zeta_3, \ldots$

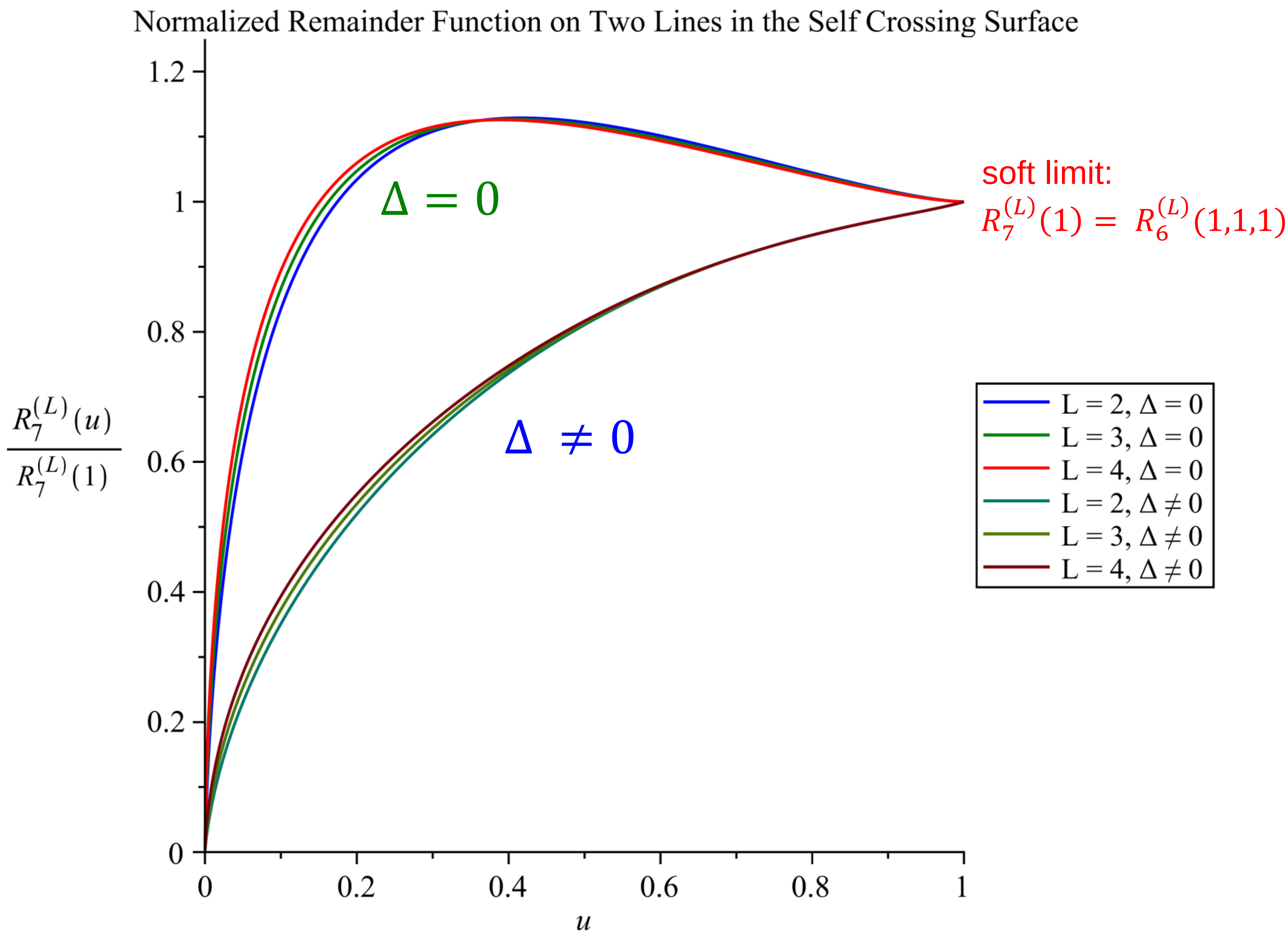
Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou, 1906.07116

7-point branch cut conditions

- **No simple bulk analog of (1,1,1) for heptagons, so work on boundary (CO surface)**
- Vanishing of $\ln(1-u_7)$ on entire CO surface
- No $\ln(1-u_3)$ at $u_3 = 1$
- No $\ln(1-u_4)$ at $u_4 = 1$
- Also one branch cut condition on nearby “soft” surface (or can apply additional CO conditions)
- Parity-odd functions vanish at $u_3 = u_4 = 1$
  because this point touches 6-point $\Delta = 0$ surface
- **These conditions, along with integrability and extended Steinmann relations, suffice to fix all ζ values and fully define heptagon functions**
- Big computational assist from previous symbol level information, complete through weight 6 LD, Drummond, McLeod, Harrington, Papathanasiou, Spradlin, 1612.08976; Drummond, Gurdogan, Papathanasiou, 1812.04640



Logarithm of MHV Amplitude Ratios on Symmetric Line $\left(u, u, u, u, u, u, \frac{(1-u-u^2)^2}{1-2u^2}\right)$
0
0.1
0.2
0.3
0.4
u
−2
−4
−6
−8
−10
−12
−14
−16
$\frac{[\ln \mathcal{E}_7]^{(L)}}{[\ln \mathcal{E}_7]^{(L-1)}}$
L = 2
L = 3
L = 4
Asymptotic cusp ratio
trending in right direction



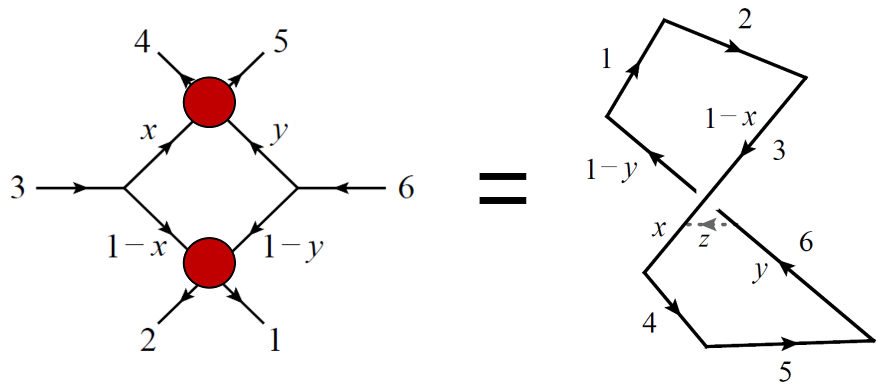
Normalized Remainder Function on Two Lines in the Self Crossing Surface
$\frac{R_7^{(L)}(u)}{R_7^{(L)}(1)}$
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
u
$\Delta = 0$
$\Delta \neq 0$
soft limit:
$R_7^{(L)}(1) = R_6^{(L)}(1,1,1)$
L = 2, $\Delta = 0$
L = 3, $\Delta = 0$
L = 4, $\Delta = 0$
L = 2, $\Delta \neq 0$
L = 3, $\Delta \neq 0$
L = 4, $\Delta \neq 0$

Summary & Outlook

- Form factors as well as scattering amplitudes in planar N=4 SYM can now be bootstrapped to high loop order
- Rich information about different kinematic limits, for example the high-energy/Regge, factorization, and self-crossing limits,…
- Great synergy with pentagon/FFOPE: perturbative information also aided in the construction of the form factor transition $\mathcal{F}_3$ beyond leading order
- Can we go to finite coupling for generic kinematics? What are the right finite-coupling functions? Clues from OPE/integrability?
- Does principle of maximal transcendentality apply to $\mathcal{F}_3$ beyond 2 loops?
- Other lessons for QCD?

Extra Slides

Self-crossing resummation



1903.10890

$$u \to u e^{-2\pi i}$$
$$(u,v,w) \to (1-|\delta|, v, v)$$

- Exact result for all leading-power terms as $\delta \to 0$:

$$\frac{1}{2\pi i}\frac{d\mathcal{E}_{3\to3}}{d\ln|\delta|} = \frac{g^2}{\rho}\exp\left[\tfrac{1}{2}\zeta_2\Gamma_{\rm cusp} + 2\Gamma_3\right] \times 2\int_0^\infty d\nu\, J_1(2\nu)\exp\left[-\frac{1}{4}\Gamma_{\rm cusp}[\lambda(\nu)]^2 - \Gamma_{\rm virt}\lambda(\nu)\right]$$

$$\lambda(\nu) = 2(\ln\nu + \gamma_E) - \ln|\delta|$$

$\Gamma_{\rm cusp}$, $\Gamma_{\rm virt}$, Γ_3 known exactly in ‘t Hooft coupling

Beisert, Eden, Staudacher hep-th/0610251; Basso, 1010.5237

- checked explicitly through 7 loops

$\bar{\mathrm{Q}}$ equation for NMHV

Caron-Huot, He, 1112.1060; S. Caron-Huot (2015);
LD, von Hippel, McLeod, 1509.08127

$$\bar{Q}\hat{\mathcal{R}}_{6,1} = \frac{\gamma_K}{8}\int d^{2|3}\mathcal{Z}_7[\mathcal{R}_{7,2} - \hat{\mathcal{R}}_{6,1}\mathcal{R}_{7,1}^{\text{tree}}] + \text{ cyclic}$$

$$\bar{Q}_a^A = \sum_{i=1}^{n}\chi_i^A\frac{\partial}{\partial Z_i^a} \qquad \hat{\mathcal{R}}_{6,1} \equiv \frac{\mathcal{A}_{\text{NMHV}}}{\mathcal{A}_{\text{BDS}-\text{like}}}$$

prevents second (simpler) term
from generating new “final entries”

→ Only 18 out of 5 x 9 = 45 possible R-invariants x final entries:

$$(1)\, d\ln(uw/v), \quad (1)\, d\ln\Big(\frac{(1-w)u}{w(1-u)y_v}\Big),$$

$$[(2)+(5)+(3)+(6)]\, d\ln\Big(\frac{v}{1-v}\Big) + (1)\, d\ln\Big(\frac{w}{y_u(1-w)}\Big) + (4)\, d\ln\Big(\frac{u}{y_w(1-u)}\Big)$$

+ cyclic

All hexagon letters are rational in terms of y_i

$$u = \frac{y_u(1-y_v)(1-y_w)}{(1-y_u y_v)(1-y_u y_w)}, \quad v = \frac{y_v(1-y_w)(1-y_u)}{(1-y_v y_w)(1-y_v y_u)}, \quad w = \frac{y_w(1-y_u)(1-y_v)}{(1-y_w y_u)(1-y_w y_v)}$$

$$1-u = \frac{(1-y_u)(1-y_u y_v y_w)}{(1-y_u y_v)(1-y_u y_w)}, \quad \text{etc.}, \quad \sqrt{\Delta} = \frac{(1-y_u)(1-y_v)(1-y_w)(1-y_u y_v y_w)}{(1-y_u y_v)(1-y_v y_w)(1-y_w y_u)}$$

$$\mathcal{S} = \{y_i,\ 1-y_i,\ 1-y_i y_j,\ 1-y_u y_v y_w\}$$

"extra" 10th letter

At $(u,v,w) = (1,1,1)$, amplitude $\rightarrow$ MZVs

Allowed MZV's obey a Galois "co-action" principle, restricting the combinations that can appear
Brown, Panzer, Schnetz

MHV

$$\mathcal{E}^{(1)}(1,1,1) = 0\,,$$
$$\mathcal{E}^{(2)}(1,1,1) = -10\,\zeta_4\,,$$
$$\mathcal{E}^{(3)}(1,1,1) = \frac{413}{3}\,\zeta_6\,,$$
$$\mathcal{E}^{(4)}(1,1,1) = -\frac{5477}{3}\,\zeta_8 + 24\left[\zeta_{5,3} + 5\,\zeta_3\,\zeta_5 - \zeta_2\,(\zeta_3)^2\right],$$
$$\mathcal{E}^{(5)}(1,1,1) = \frac{379957}{15}\,\zeta_{10} - 12\left[4\,\zeta_2\,\zeta_{5,3} + 25\,(\zeta_5)^2\right] - 96\left[2\,\zeta_{7,3} + 28\,\zeta_3\,\zeta_7 + 11\,(\zeta_5)^2 - 4\,\zeta_2\,\zeta_3\,\zeta_5 - 6\,\zeta_4\,(\zeta_3)^2\right]$$

NMHV

$$E^{(1)}(1,1,1) = -2\,\zeta_2\,,$$
$$E^{(2)}(1,1,1) = 26\,\zeta_4\,,$$
$$E^{(3)}(1,1,1) = -\frac{940}{3}\,\zeta_6\,,$$
$$E^{(4)}(1,1,1) = -\frac{36271}{9}\,\zeta_8 - 24\left[\zeta_{5,3} + 5\,\zeta_3\,\zeta_5 - \zeta_2\,(\zeta_3)^2\right],$$
$$E^{(5)}(1,1,1) = -\frac{1666501}{30}\,\zeta_{10} + 12\left[4\,\zeta_2\,\zeta_{5,3} + 25\,(\zeta_5)^2\right] + 132\left[2\,\zeta_{7,3} + 28\,\zeta_3\,\zeta_7 + 11\,(\zeta_5)^2 - 4\,\zeta_2\,\zeta_3\,\zeta_5 - 6\,\zeta_4\,(\zeta_3)^2\right]$$

Cosmic Galois Theory

Studies the symmetries of 'periods' (integrals of rational functions over domains given by rational inequalities)

- The space of functions appearing in the six-point amplitude is (conjecturally) stable under the coaction
- This property can be formulated as a 'coaction principle'

$$\Delta \mathcal{H}^{\text{hex}} \subset \mathcal{H}^{\text{hex}} \otimes \mathcal{H}^{\pi}$$

  which incorporates the branch cut condition, but also constrains the constants that can show up

- This can be alternately formulated in terms of the action of the 'cosmic Galois group' C which is dual to this coaction

$$C \times \mathcal{H}^{\text{hex}} \to \mathcal{H}^{\text{hex}}$$

Cosmic Galois Theory

- The Lie algebra of C includes a set of elements ∂_{2m+1} that act on the zeta values as

$$\partial_{2m+1}\zeta_{2n+1} = \delta_{m,n}$$

  and that satisfy the Leibniz rule. So, for example,

$$\partial_3(\zeta_7\zeta_3^2) = 2\zeta_7\zeta_3$$

- There is no ∂_2, because including even zeta values on both sides of the coaction leads to contradictions
- These operators also act nontrivially on multiple zeta values

Brown, arXiv:1102.1310 [math.NT]

MZV restrictions

Weight	All MZVs	$\mathcal{H}^{\rm hex}(1,1,1)$	$\mathcal{H}^{\rm hex}$ indep.
0	1	1	1
1	—	—	—
2	ζ_2	ζ_2	—
3	ζ_3	—	—
4	ζ_4	ζ_4	ζ_4
5	$\zeta_5,\ \zeta_2\zeta_3$	$5\zeta_5 - 2\zeta_2\zeta_3$	—
6	$\zeta_6,\ (\zeta_3)^2$	ζ_6	ζ_6
7	$\zeta_7,\ \zeta_2\zeta_5,\ \zeta_3\zeta_4$	$7\zeta_7 - \zeta_2\zeta_5 - 3\zeta_3\zeta_4$	—
8	$\zeta_8,\ \zeta_{5,3},\ \zeta_3\zeta_5,\ \zeta_2(\zeta_3)^2$	$\zeta_8,\ \zeta_{5,3} + 5\zeta_3\zeta_5 - \zeta_2(\zeta_3)^2$	ζ_8

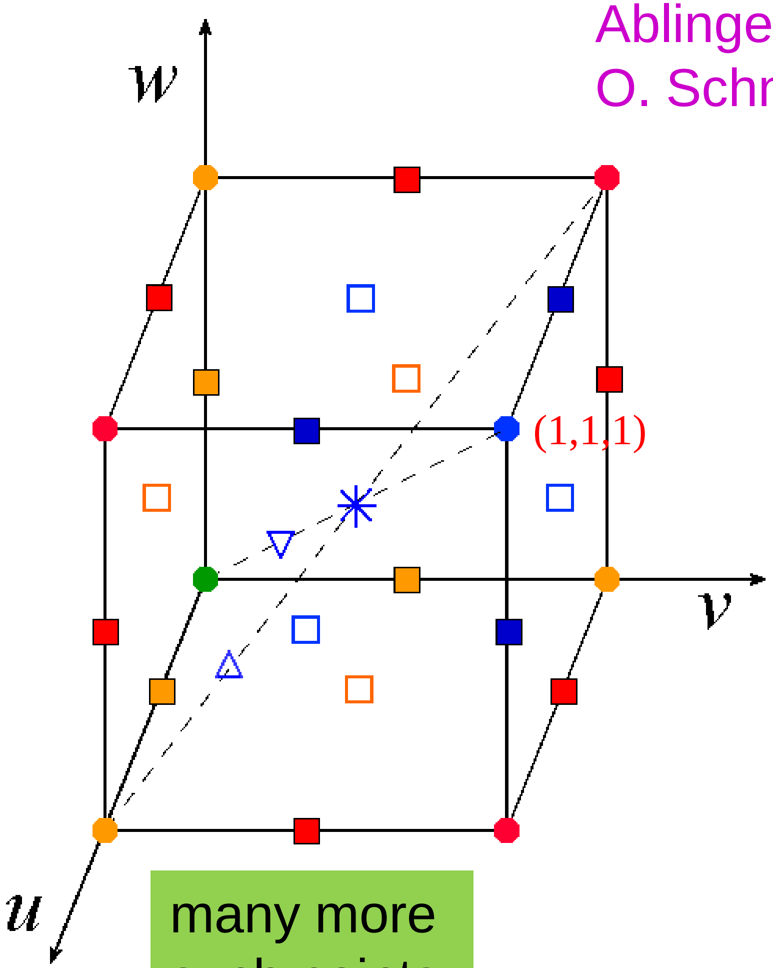
∂_3

∂_5

∂_3

Menagerie of "cyclotomic" polylogs at unity

Ablinger, Blumlein, Schneider, 1105.6063, 1310.5645;
O. Schnetz, **HyperlogProcedures**



●	MZVs
■, □	Alternating sums
✳	4th roots of unity
▽, △	6th roots of unity

finite
1 variable singular
2 variables singular
3 variables singular

Galois co-action principle applies to entire function space at every point at which we have checked it!!

many more such points

Example 1: Harmonic Polylogarithms of one variable (HPLs {0,1})

Remiddi, Vermaseren, hep-ph/9905237

- Subsector of hexagon functions.
- Gen'lize classical polylogs: $\mathrm{Li}_n(u) = \int_0^u \frac{dt}{t} \mathrm{Li}_{n-1}(t), \quad \mathrm{Li}_1(t) = -\ln(1-t)$
- Define HPLs by iterated integration:

$$H_{0,\vec{w}}(u) = \int_0^u \frac{dt}{t} H_{\vec{w}}(t), \quad H_{1,\vec{w}}(u) = \int_0^u \frac{dt}{1-t} H_{\vec{w}}(t)$$

- Or by derivatives

$$dH_{0,\vec{w}}(u) = H_{\vec{w}}(u)\ d\ln u \quad dH_{1,\vec{w}}(u) = -H_{\vec{w}}(u) d\ln(1-u)$$

- Symbol letters: $\mathcal{S} = \{u, 1-u\}$
- Weight n = length of binary string $\vec{w}$
- Number of functions at weight $\boldsymbol{n = 2L}$: $\boldsymbol{2^{2L}}$

Values of HPLs {0,1} at $u = 1$

- Classical polylogs

evaluate to Riemann zeta values

$$\mathrm{Li}_n(u) = \int_0^u \frac{dt}{t}\mathrm{Li}_{n-1}(t) = \sum_{k=1}^{\infty} \frac{u^k}{k^n}$$

$$\mathrm{Li}_n(1) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \zeta(n) \equiv \zeta_n$$

- HPL's evaluate to nested sums called multiple zeta values (MZVs):

$$\zeta_{n_1,n_2,\ldots,n_m} = \sum_{k_1>k_2>\cdots>k_m>0}^{\infty} \frac{1}{k_1^{n_1} k_2^{n_2} \cdots k_m^{n_m}}$$

Weight $n = n_1 + n_1 + \ldots + n_m$

- MZV's obey many identities, e.g. stuffle

$$\zeta_{n_1}\zeta_{n_2} = \zeta_{n_1,n_2} + \zeta_{n_2,n_1} + \zeta_{n_1+n_2}$$

- All reducible to Riemann zeta values until weight 8.
  Irreducible MZVs: $\zeta_{5,3},\ \zeta_{7,3},\ \zeta_{5,3,3},\ \zeta_{9,3},\ \zeta_{6,4,1,1},\ \ldots$

Example 2: Single-valued harmonic polylogarithms of one complex variable

Brown, C. R. Acad. Sci. Paris, Ser. I 338 (2004) 527

- Also a subsector of hexagon functions, in the "multi-Regge limit"
- 4 symbol letters: $\mathcal{S} = \{z, 1-z, \bar{z}, 1-\bar{z}\}$
- But also require function to be real analytic in
  $$(z, \bar{z}) \in \mathbb{C} - \{0, 1\}$$
- Constrains the first entry of the symbol to be
  $z\bar{z} \leftrightarrow \ln|z|^2$ or $(1-z)(1-\bar{z}) \leftrightarrow \ln|1-z|^2$
- Brown: One SVHPL for each HPL
- Powerful constraint: $4^{2L} \rightarrow 2^{2L}$ functions