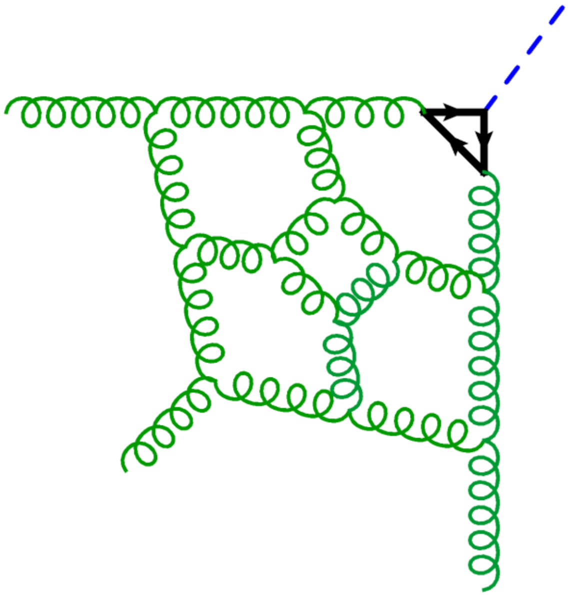
[Amplitude] Bootstrap
Lecture 3



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SLAC
NATIONAL ACCELERATOR LABORATORY

2d HPLs

Gehrmann, Remiddi, hep-ph/0008287

Space graded by weight n. Every function F obeys:

$$\frac{\partial F(u,v)}{\partial u} = \frac{F^u}{u} - \frac{F^w}{1-u-v} - \frac{F^{1-u}}{1-u} + \frac{F^{1-w}}{u+v}$$

$$\frac{\partial F(u,v)}{\partial v} = \frac{F^v}{v} - \frac{F^w}{1-u-v} - \frac{F^{1-v}}{1-v} + \frac{F^{1-w}}{u+v}$$

$$w = 1-u-v$$

where $F^u, F^v, F^w, F^{1-u}, F^{1-v}, F^{1-w}$ are weight n-1 2d HPLs.

To bootstrap *Hggg* amplitude beyond 2 loops, find as small a subspace of 2d HPLs as possible, construct it to high weight.

Symbol is too verbose → Nested representation better

loop order L	1	2	3	4	5	6	7	8
terms in $S[\mathcal{E}^{(L)}]$	6	12	636	11,208	263,880	4,916,466	97,594,968	???

- Define every function by its $\{n-1,1\}$ coproducts, i.e. its first derivatives.
- Also need to specify constants of integration at one point, e.g. $(u,v,w)=(1,0,0)$

Recall 1 and 2 loop results

$$\mathcal{S}\left[\mathcal{E}^{(1)}\right] = (-1)\ b \otimes d + \text{dihedral}$$

$$\mathcal{S}\left[\mathcal{E}^{(2)}\right] = 4\ b \otimes d \otimes d \otimes d + 2\ b \otimes b \otimes b \otimes d + \text{dihedral}$$

Final entry conditions

- Notice that, at least through 2 loops,

$$\mathcal{E}^a = \mathcal{E}^b = \mathcal{E}^c = 0 \qquad (1)$$

- Observed earlier for 2 loop remainder function R
  Brandhuber, Travaglini, Yang, 1201.4170
- Now, $\mathcal{E}^a = 0 \quad \Leftrightarrow \quad R^a = 0$

since Leibniz rule for derivatives applies to first coproducts,

so $[FG]^s = F^s G + F G^s$ and therefore (at symbol level)

$$[\mathcal{E}^{(2)}]^a = [R^{(2)}]^a + \mathcal{E}^{(1)}[\mathcal{E}^{(1)}]^a, \text{ etc.}$$

- (1) also very similar to 6-gluon relation derived from dual superconformal invariance, the $\bar{Q}$ equation
  Caron-Huot, He, 1112.1060
- Let's assume (1) holds to all loop orders!

Multiple final entry conditions

- Inspecting the double final entries (double coproducts) of $\mathcal{E}^{(2)}$ leads to the relations:

$$\mathcal{E}^{b,f} = \mathcal{E}^{b,d}, \ \mathcal{E}^{a,e} = \mathcal{E}^{a,f}, \ \ \mathcal{E}^{c,d} = \mathcal{E}^{c,e} \qquad (2)$$

- Let’s assume (2) holds to all loop orders!
- These are on top of the “generic” 21-pair relations.
- Then there are only 6 independent double coproducts, in 2 orbits under the dihedral group:

$$\mathcal{E}^{d,d}, \ \mathcal{E}^{e,e}, \ \ \mathcal{E}^{f,f}$$

$$\mathcal{E}^{b,d}, \ \mathcal{E}^{c,e}, \ \ \mathcal{E}^{a,f}$$

Starting the bootstrap

- Although it's not exactly "fair", from higher loop investigations we also found 4 generic "triple" relations, and one other set of constraints at weight 4, obeyed by the space $\mathcal{C}$ needed by planar N=4 SYM:

weight n	0	1	2	3	4	5	6	7	8	9
symbols in $\mathcal{M}$	1	3	9	27	81	243	729	2187	6561	19683
symbols in $\mathcal{C}$	1	3	9	21	48	108	249	567	1290	????

- Whereas $\mathcal{M}$ is the "right space" for QCD (at least through 2 loops). It has only the conditions $F^{d,e} = 0$ (+ dihedral)
- How many initial parameters do we need to get to 3 loops?
- Naively 249, since $\mathcal{E}^{(3)}$ has weight 6.

Starting at lower weight

- But we don't need to construct the full weight 6 space for 3 loops, if we adopt the double final entry conditions (2).
- We only need the 48-dimensional weight 4 space, once each to parametrize $\mathcal{E}^{(3)d,d}$, $\mathcal{E}^{(3)b,d}$,

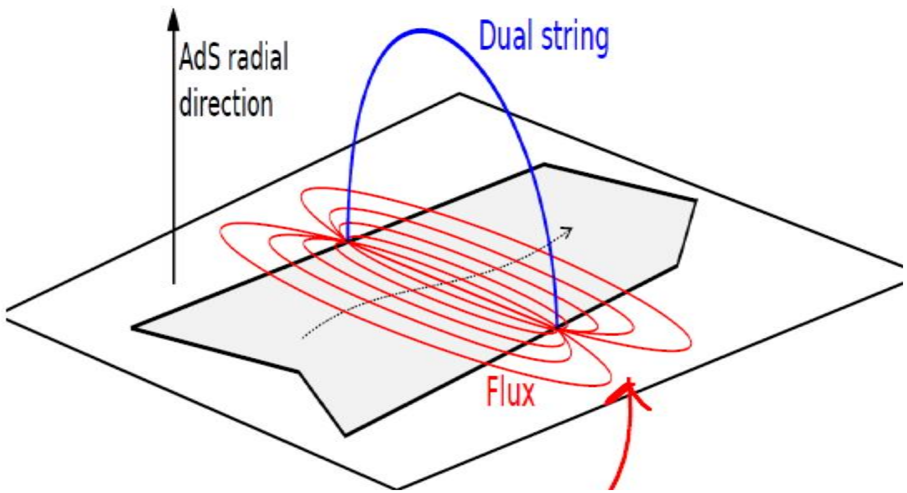
  so 48 * 2 = 96 initial parameters.
- The other double coproducts follow from dihedral transformations and/or linear relations.
- Once we impose pair relations in the 4-5 slot, dihedral, and branch cut conditions, only 9 parameters survive.
- Requiring $\mathcal{E}(u_i) \rightarrow \exp[\frac{\Gamma_{\text{cusp}}}{4}\mathcal{E}^{(1)}]$ in the collinear limit then fixes all but 1 of these. Still need a little more data!

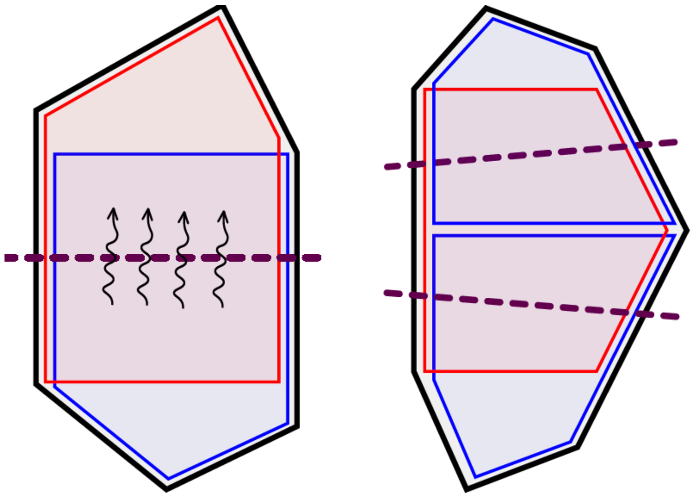
Near-collinear limit = OPE limit

- Amplitude boundary data: (near)collinear limits, related to an OPE for Wilson loops
- Until recently, no OPE for form factors

Flux tubes at finite coupling

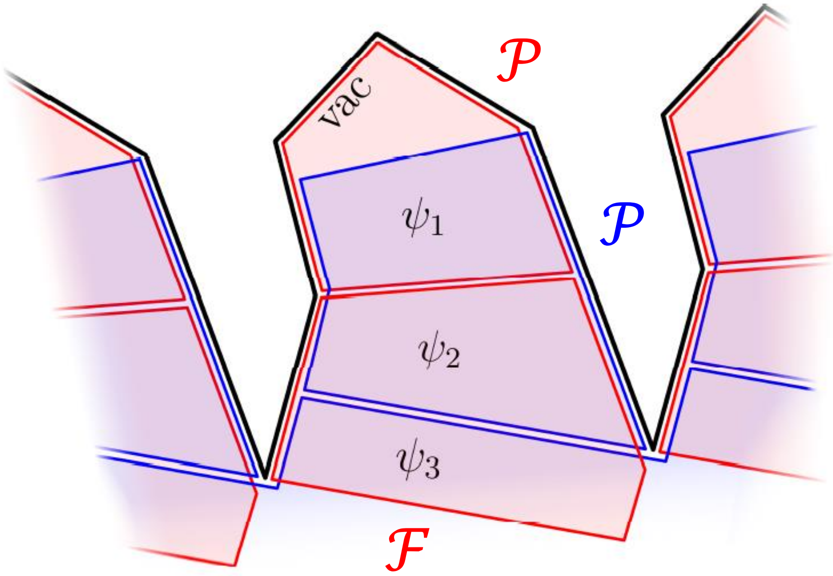
Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788;
Basso, Sever, Vieira, 1303.1396, 1306.2058, 1402.3307, 1407.1736, 1508.03045
BSV+Caetano+Cordova, 1412.1132, 1508.02987





- Tile n-gon with pentagon transitions $\mathcal{P}$.
- Quantum integrability → compute pentagons exactly in 't Hooft coupling
- 4d S-matrix as expansion (OPE) in number of flux-tube excitations = series expansion around near collinear limit

The new FFOPE



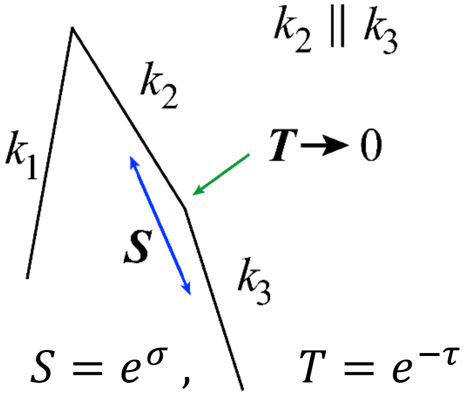
- Form factors are Wilson loops in a periodic space, due to injection of operator momentum

Brandhuber, Spence, Travaglini, Yang, 1011.1899

- Besides pentagon transitions $\mathcal{P}$, this program needs an additional ingredient, the form factor transition $\mathcal{F}$

Sever, Tumanov, Wilhelm, 2009.11297, 2105.13367

OPE is expansion around collinear limit



$$u = \frac{s_{12}}{s_{123}} = \frac{1}{1 + S^2 + T^2}$$

$$v = \frac{s_{23}}{s_{123}} = \frac{T^2}{1 + T^2} \to 0$$

$$w = 1 - u - v \to 1 - u$$

FFOPE computes “framed” (IR finite) Wilson loop,

related to R by $\mathcal{W}_3 = \exp[\frac{\Gamma_{\rm cusp}}{4} \mathcal{W}_3^{(1)} + R]$ where

$$\mathcal{W}_3^{(1)} = 4\sigma^2 - 2\mathrm{Li}_2(-e^{-2\tau}) + 2\mathrm{Li}_2(-e^{-2\tau} - e^{2\sigma}) + 2\mathrm{Li}_2(-e^{-2\tau} - e^{-2\sigma}(1 + e^{-2\tau})^2) + \frac{\pi^2}{3}$$

Flux tube representation $\mathcal{W}_3 = \sum_{\psi} e^{-E_{\psi}\tau + ip_{\psi}\sigma} \mathcal{P}(0|\psi)\mathcal{F}(\psi)$

leads to $T^2 = v^1$ terms as series expansion in S around $S = 0$

Near-collinear limit from bulk

- Integrate up functions in limits with simpler function space, using coproduct representations → first derivatives
- $v \to 0$ limit:

$$\frac{\partial F(u,v)}{\partial v} = \frac{F^v}{v} - \frac{F^w}{1-u-v} - \frac{F^{1-v}}{1-v} + \frac{F^{1-w}}{u+v} \approx \frac{F^v}{v} - \frac{F^w}{1-u} - F^{1-v} + \frac{F^{1-w}}{u}$$

→ v dependence is trivial, $\propto v^m \ln^k v$

$$\frac{\partial F(u,v)}{\partial u} = \frac{F^u}{u} - \frac{F^w}{1-u-v} - \frac{F^{1-u}}{1-u} + \frac{F^{1-w}}{u+v} \approx \underbrace{\frac{F^u+F^{1-w}}{u} - \frac{F^w+F^{1-u}}{1-u}}_{H_{\vec{w}}(u),\quad w_i \in \{0,1\}}$$

u dependence is 1d HPLs;
integrate up just using HPL definitions

General form in near collinear limit

$$R^{(L)}(T,S) = \sum_{j=1}^{\infty} \sum_{k=0}^{L-1} T^{2j} \ln^k T \, R^{(L)}_{j,k}(S)$$

- Leading power (T^2), with $x = -S^2$, can be decomposed as:

$$R^{(L)}_{1,k} = \frac{(1-x)^2}{x} A^{(L)}_k + (1-x) B^{(L)}_k + C^{(L)}_k$$

- Three loop, leading log "target":

$$\begin{aligned} A_2^{(3)} &= -12\, H_1 \ln^2(-x) - 32\left[2\, H_{1,1} + H_1\right] \ln(-x) + 8\, H_{0,0,1} - 128\, H_{1,1,1} - 8\,\zeta_2 \\ &\quad - 64\, H_{1,1} - 32\, H_1\,, \\ B_2^{(3)} &= 32\, \ln(-x)\,, \\ C_2^{(3)} &= 12\, \ln^2(-x) + 8\,\zeta_2 + 24\,, \end{aligned}$$

- Matching near-collinear limit of ansatz fixes the last parameter.

Parameters left, bootstrapping in $\mathcal{C}$

L	2	3	4	5	6	7	8
symbols in $\mathcal{C}$	48	249	1290	????	????	????	????
dihedral symmetry	11	51	247	????	????	????	????
$(L-1)$ final entries	5	9	20	44	86	???	???
$L^{\rm th}$ discontinuity	2	5	17	38	75	???	??
collinear limit	0	1	2	8	19	70	6
OPE $T^2 \ln^{L-1} T$	0	0	0	4	12	56	0
OPE $T^2 \ln^{L-2} T$	0	0	0	0	0	36	0
OPE $T^2 \ln^{L-3} T$	0	0	0	0	0	0	0
OPE $T^2 \ln^{L-4} T$	0	0	0	0	0	0	0
OPE $T^2 \ln^{L-5} T$	0	0	0	0	0	0	0

very recently made it to $L = 8$ using $\mathcal{C}$

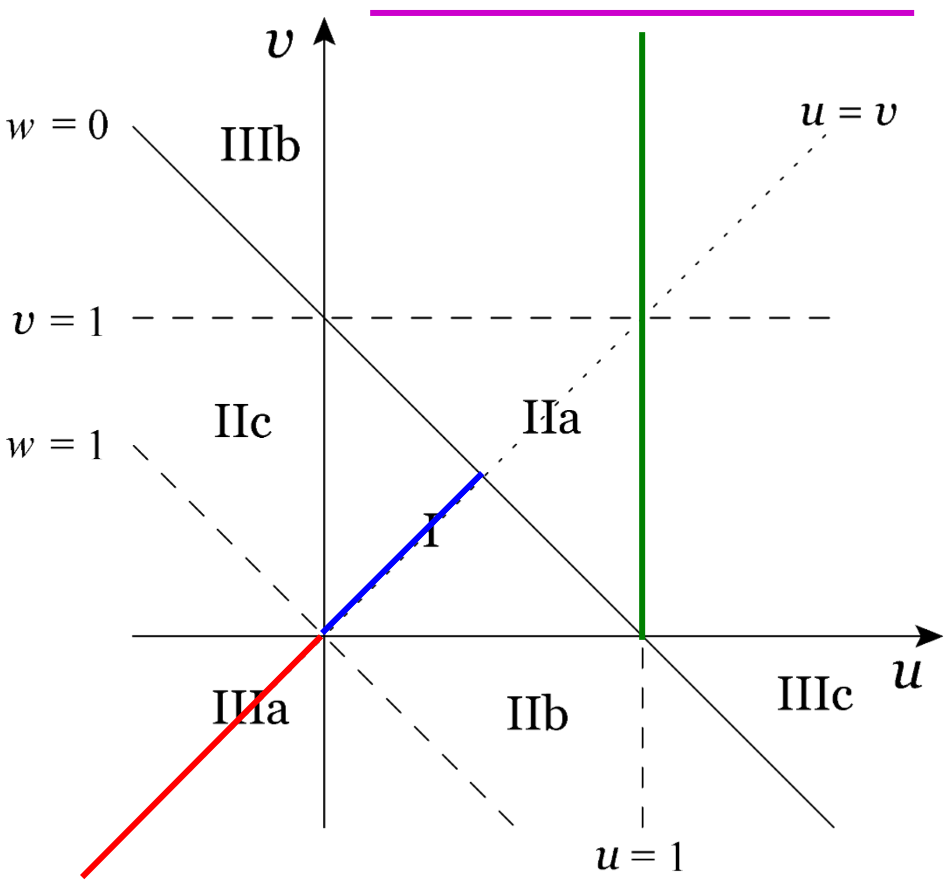
amazingly little OPE data needed for $L = 8$!

Number of (symbol-level) linearly independent $\{n, 1, \dots, 1\}$ coproducts ($2L - n$ derivatives)

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$L=1$	1	3	1														
$L=2$	1	3	6	3	1												
$L=3$	1	3	9	12	6	3	1										
$L=4$	1	3	9	21	24	12	6	3	1								
$L=5$	1	3	9	21	46	45	24	12	6	3	1						
$L=6$	1	3	9	21	48	99	85	45	24	12	6	3	1				
$L=7$	1	3	9	21	48	108	236	155	85	45	24	12	6	3	1		
$L=8$	1	3	9	21	48	108	242	466	279	155	85	45	24	12	6	3	1

- N=4 form factors belong to much smaller space $\mathcal{C} \subset \mathcal{M}$ than QCD amplitudes (48 < 81).
- Amplitude obeys multiple-final-entry relations

Some numerics



simplest line, analytically

$H_{\vec{w}}(u), \quad w_i \in \{0,1\}$

I = decay / Euclidean

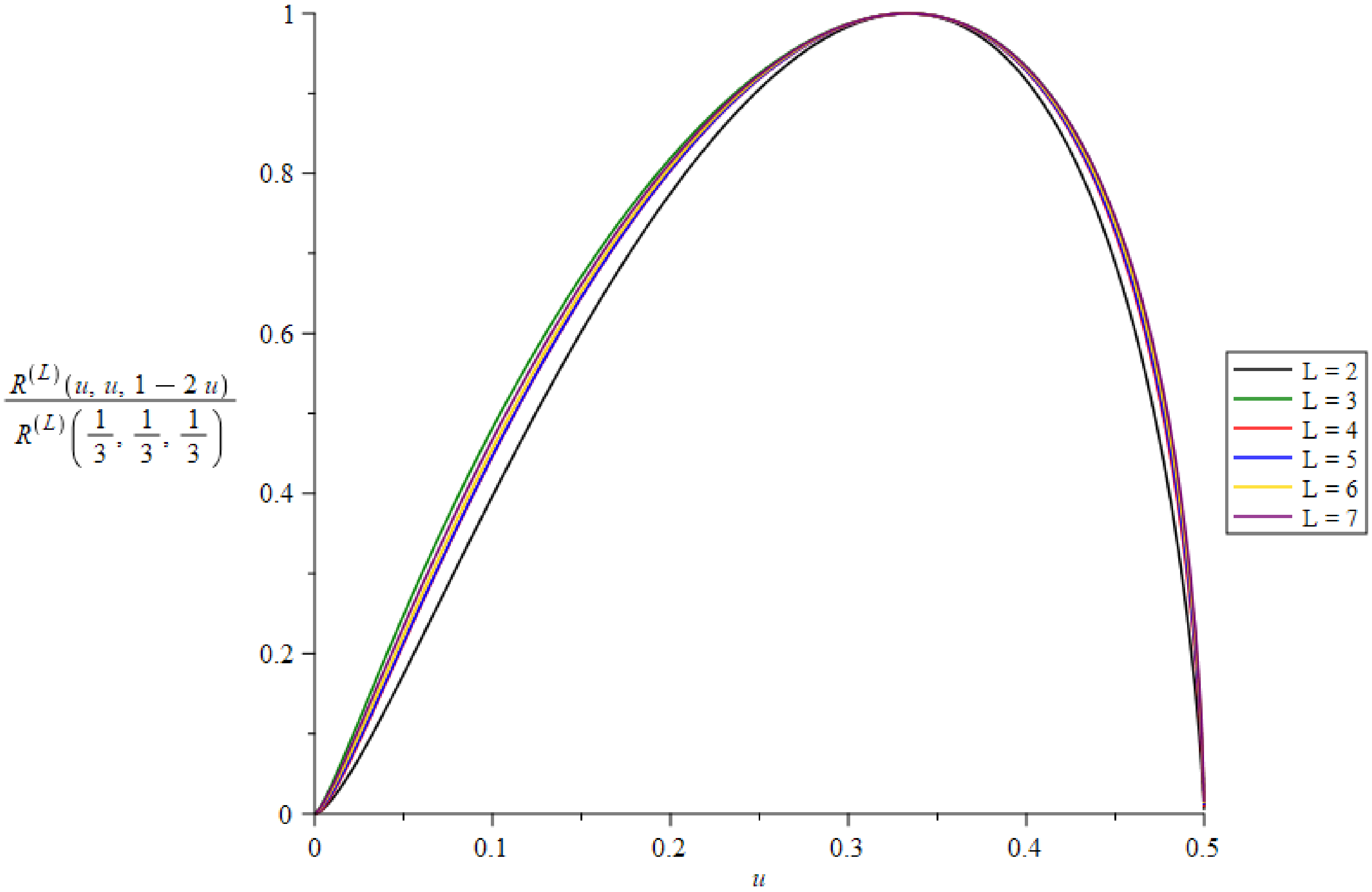
IIa,b,c = scattering / spacelike operator

IIIa,b,c = scattering / timelike operator

other lines are $H_{\vec{w}}, \quad w_i \in \{-1,0,1\}$

computationally more difficult: $3^{14} \gg 2^{14}$

Euclidean Region



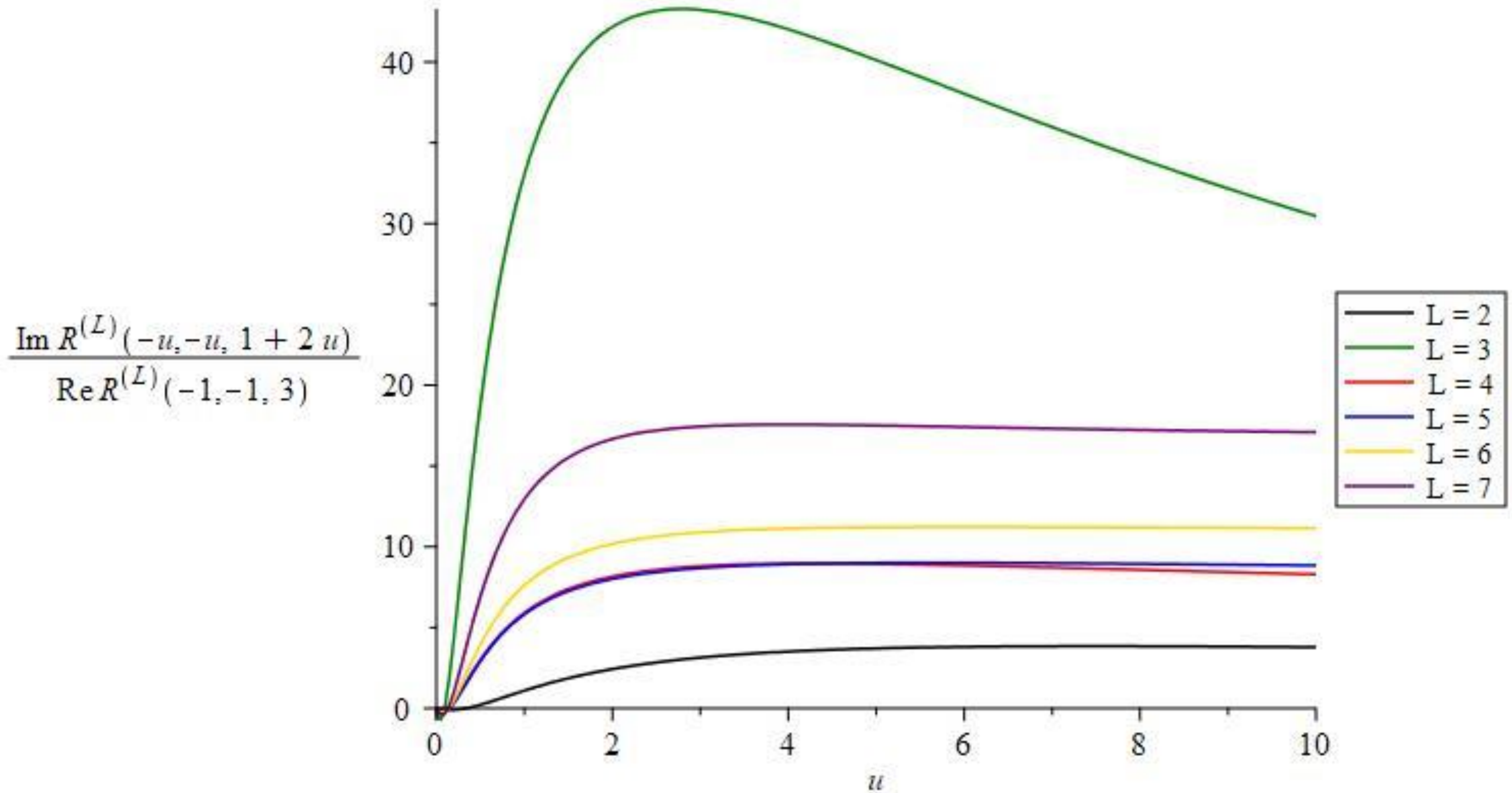
Amplitude real here; shapes very similar!

Normalization factor at peak

L	$R^{(L)}(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$R^{(L)}(\frac{1}{3},\frac{1}{3},\frac{1}{3})/R^{(L-1)}(\frac{1}{3},\frac{1}{3},\frac{1}{3})$
2	-0.148966439	–
3	-8.372052781	$+56.20093$
4	166.894338831	-19.93470
5	-2666.615691	-15.97787
6	40557.38379	-15.20931
7	-608664.5908	-15.00749

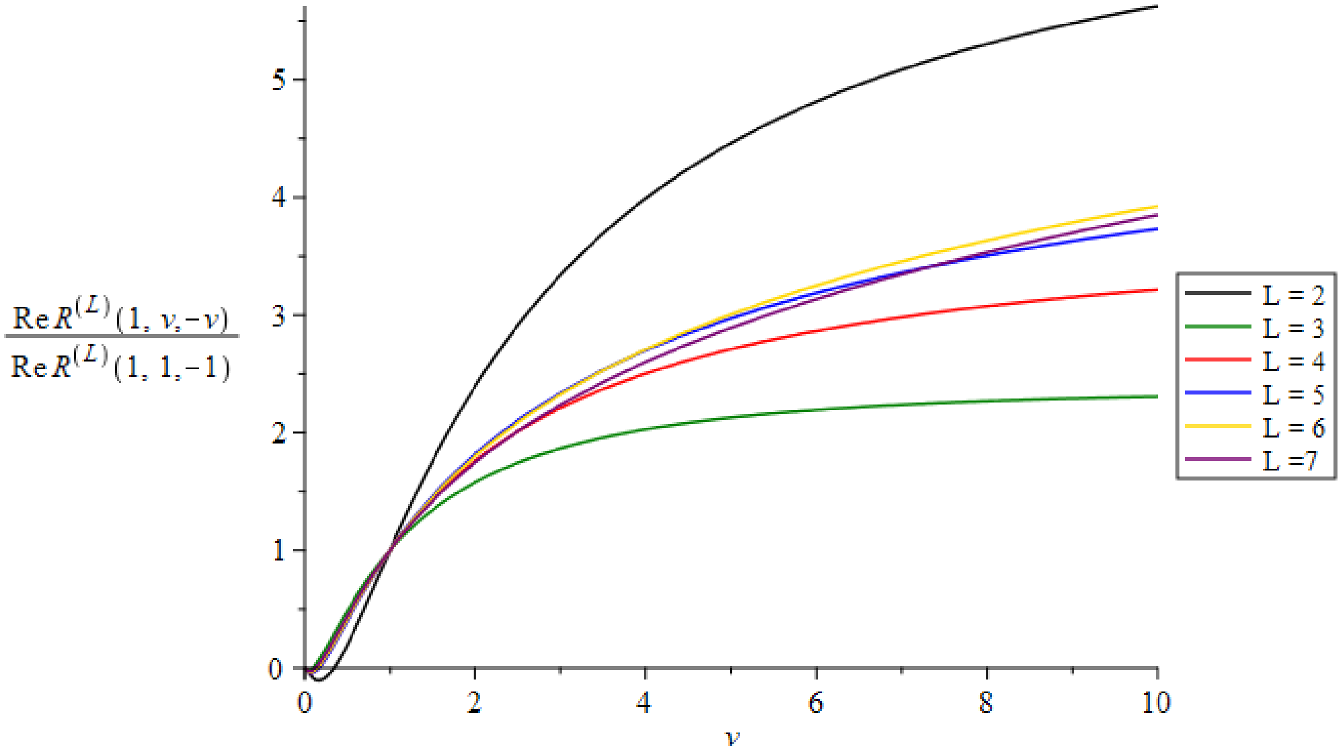
- Planar N=4 has no renormalons or instantons → perturbation theory has finite radius of convergence, = 1/16 for cusp anomalous dimension
- Expect successive loop-order ratios to go to -16 eventually.
- Bit of overshoot here for 6/5 and 7/6 …

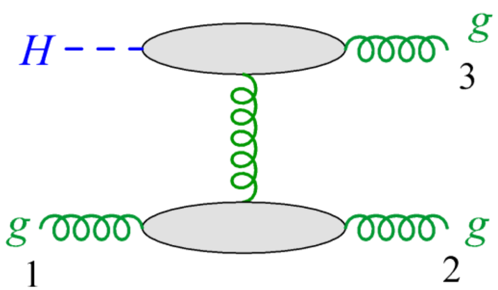
$gg \rightarrow Hg$ kinematics ($m_H^2 > 0$)



Imaginary parts open up in physical scattering region, and are bigger than real part at large loop orders

Space-like Higgs, e.g. $Hg \to gg$

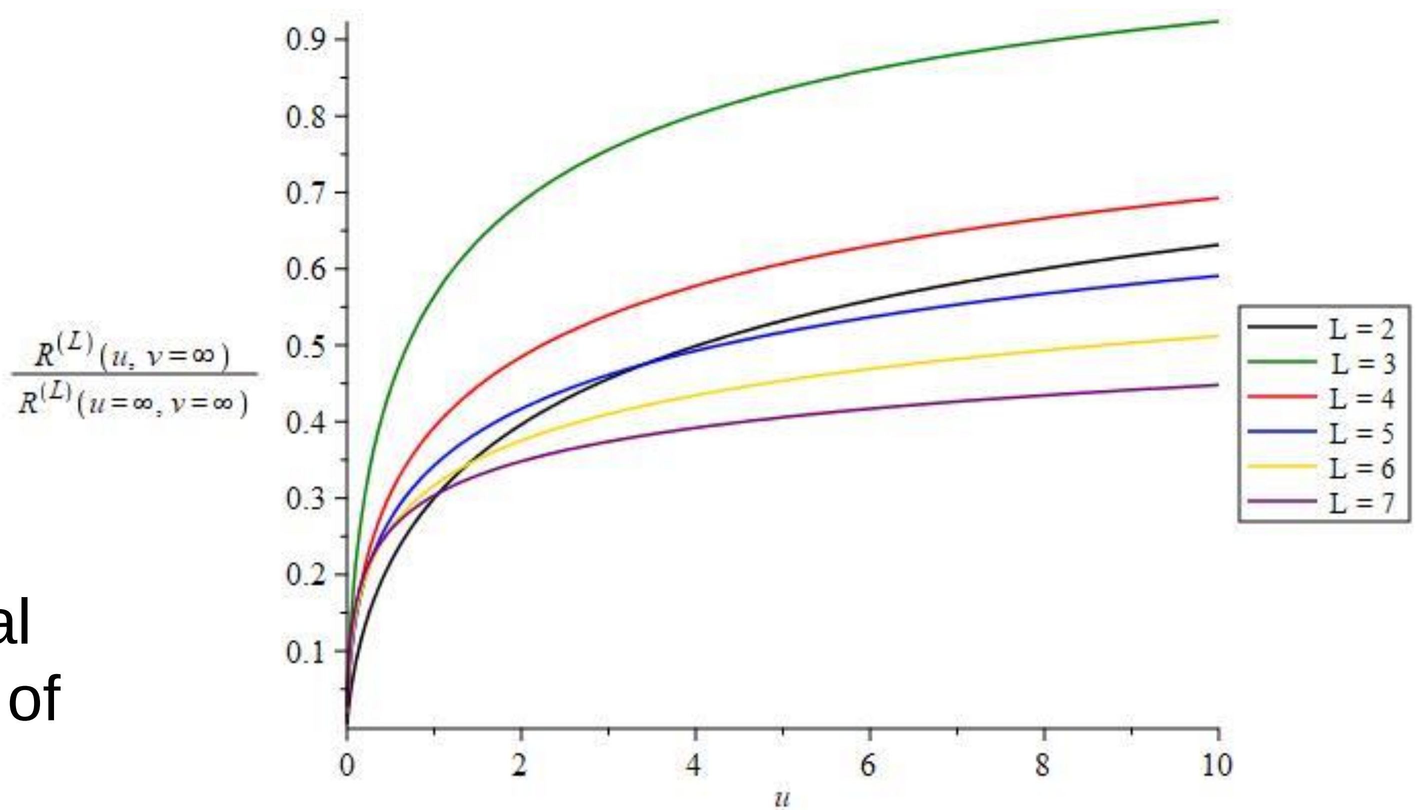




Approaches forward/Regge limit as $\frac{s_{23}}{m_H^2} = v \to \infty$

for fixed $u = \frac{s_{12}}{m_H^2}$, but no large logs in R, only in BDS-like ansatz.

Real “impact factor” appears in space-like Regge limit, $v \to \infty$



Nontrivial function of $u = \frac{s_{12}}{m_H^2}$

Hggg Form Factor Conclusions

- The simplest bootstrapping problem, given the FFOPE data; feasible even to 8 loops
- Would be great to see if

N=4 result = [max. transcendental part of QCD]

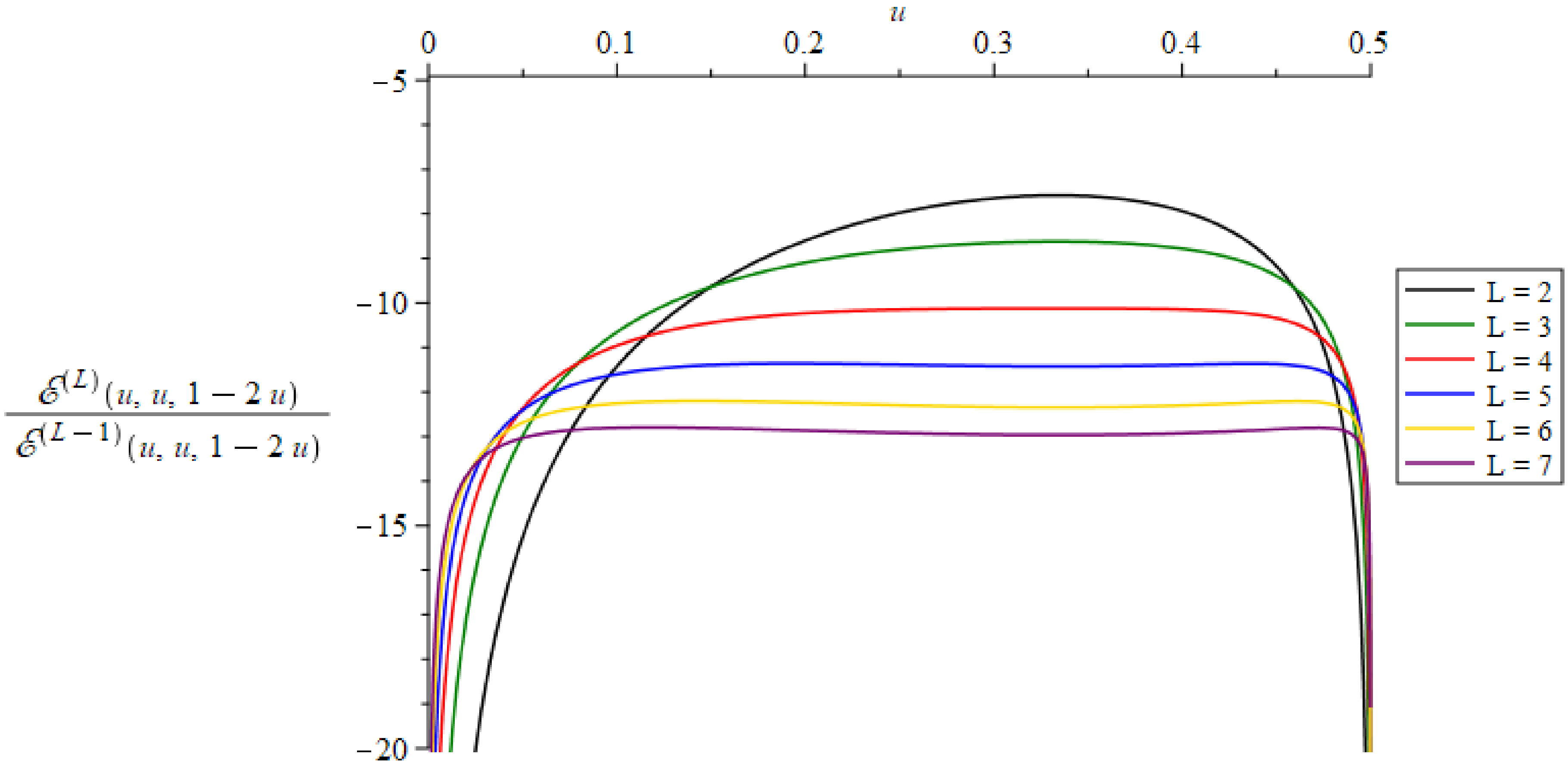
at 3 loops too!

- Illustrates general bootstrapping strategy:

an Amplitude Game that is literally a Word Game!

Extra Slides

Successive loop ratios on Euclidean symmetric line



Example 1: Harmonic Polylogarithms of one variable (HPLs {0,1})

Remiddi, Vermaseren, hep-ph/9905237

- Subsector of hexagon functions.
- Gen'lize classical polylogs: $\mathrm{Li}_n(u) = \int_0^u \frac{dt}{t} \mathrm{Li}_{n-1}(t), \quad \mathrm{Li}_1(t) = -\ln(1-t)$
- Define HPLs by iterated integration:

$$H_{0,\vec{w}}(u) = \int_0^u \frac{dt}{t} H_{\vec{w}}(t), \quad H_{1,\vec{w}}(u) = \int_0^u \frac{dt}{1-t} H_{\vec{w}}(t)$$

- Or by derivatives

$$dH_{0,\vec{w}}(u) = H_{\vec{w}}(u)\ d\ln u \quad dH_{1,\vec{w}}(u) = -H_{\vec{w}}(u) d\ln(1-u)$$

- Symbol letters: $\mathcal{S} = \{u, 1-u\}$
- Weight n = length of binary string $\vec{w}$
- Number of functions at weight $n = 2L$: 2^{2L}

Values of HPLs {0,1} at $u = 1$

- Classical polylogs

evaluate to Riemann zeta values

$$\mathrm{Li}_n(u) = \int_0^u \frac{dt}{t}\mathrm{Li}_{n-1}(t) = \sum_{k=1}^{\infty} \frac{u^k}{k^n}$$

$$\mathrm{Li}_n(1) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \zeta(n) \equiv \zeta_n$$

- HPL's evaluate to nested sums called multiple zeta values (MZVs):

$$\zeta_{n_1,n_2,\ldots,n_m} = \sum_{k_1>k_2>\cdots>k_m>0}^{\infty} \frac{1}{k_1^{n_1} k_2^{n_2} \cdots k_m^{n_m}}$$

Weight $n = n_1 + n_1 + \ldots + n_m$

- MZV's obey many identities, e.g. stuffle

$$\zeta_{n_1}\zeta_{n_2} = \zeta_{n_1,n_2} + \zeta_{n_2,n_1} + \zeta_{n_1+n_2}$$

- All reducible to Riemann zeta values until weight 8.
  Irreducible MZVs: $\zeta_{5,3},\ \zeta_{7,3},\ \zeta_{5,3,3},\ \zeta_{9,3},\ \zeta_{6,4,1,1},\ \ldots$