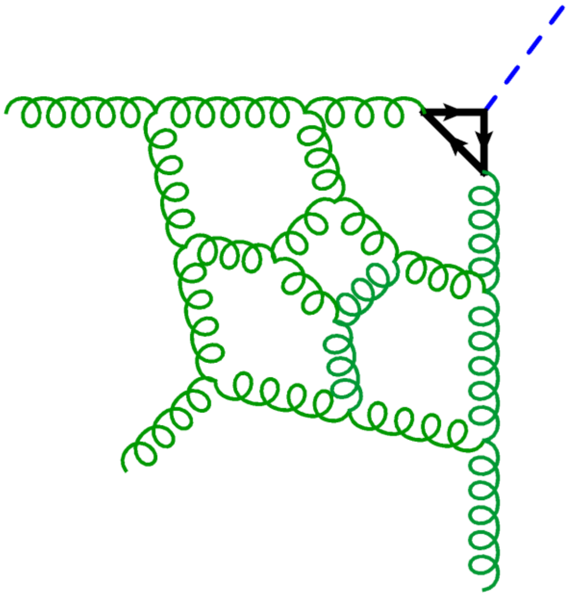
[Amplitude] Bootstrap
Lecture 2



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"The Amplitude Games"
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SLAC
NATIONAL ACCELERATOR LABORATORY

2d HPLs

Gehrmann, Remiddi, hep-ph/0008287

Space graded by weight n. Every function F obeys:

$$\boxed{\begin{aligned}\frac{\partial F(u,v)}{\partial u} &= \frac{F^u}{u} - \frac{F^w}{1-u-v} - \frac{F^{1-u}}{1-u} + \frac{F^{1-w}}{u+v}\\[2ex] \frac{\partial F(u,v)}{\partial v} &= \frac{F^v}{v} - \frac{F^w}{1-u-v} - \frac{F^{1-v}}{1-v} + \frac{F^{1-w}}{u+v}\end{aligned}}$$

$$w = 1 - u - v$$

where $F^u, F^v, F^w, F^{1-u}, F^{1-v}, F^{1-w}$ are weight n-1 2d HPLs.

To bootstrap *Hggg* amplitude beyond 2 loops, find as small a subspace of 2d HPLs as possible, construct it to high weight.

Generalized polylogarithms

Chen, Goncharov, Brown,…

- Can be defined as iterated integrals, e.g.

$$\boxed{G(a_1, a_2, \ldots, a_n, x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n, t)}$$

- Or define differentially: $\boxed{dF = \sum_{s_k \in \mathcal{S}} F^{s_k} \, d \ln s_k}$

- There is a Hopf algebra that "co-acts" on the space of polylogarithms, $\Delta: F \to F \otimes F$
- The derivative dF is one piece of Δ: $\Delta_{n-1,1} F = \sum_{s_k \in \mathcal{S}} F^{s_k} \otimes \ln s_k$
- so we refer to F^{s_k} as a $\{n\text{-}1,1\}$ coproduct of F
- $\boldsymbol{s_k}$ **are letters in the symbol alphabet** $\mathcal{S}$

Generalized polylogarithms (cont.)

- The {n-1,1} coaction can be applied iteratively.
- Define the $\{n\text{-}2,1,1\}$ "double" coproducts, $F^{s_k,,s_j}$,

via the derivatives of the $\{n\text{-}1,1\}$ single coproducts F^{s_j}:

$$\boxed{dF^{s_j} \equiv \sum_{s_k\in\mathcal{S}} F^{s_k,,s_j}\, d\ln s_k}$$

- And so on for the $\{n\text{-}m,1,\ldots,1\}$ m^{th} coproducts of *F*.
- The maximal iteration, *n* times for a weight *n* function, is

the symbol,

$$\mathcal{S}[F] = \sum_{s_{i_1},\ldots,s_{i_n}\in\mathcal{S}} F^{s_{i_1},\ldots,s_{i_n}}\, d\ln s_{i_1} \ldots d\ln s_{i_n} \quad \equiv \sum_{s_{i_1},\ldots,s_{i_n}\in\mathcal{S}} F^{s_{i_1},\ldots,s_{i_n}}\, s_{i_1}\otimes\ldots\otimes s_{i_n}$$

where now $F^{s_{i_1},\ldots,s_{i_n}}$ are just rational numbers

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

Symbol alphabet for *Hggg*

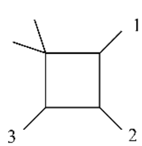
Gehrmann, Remiddi, hep-ph/0008287

- Comparing
$$\frac{\partial F(u,v)}{\partial u} = \frac{F^u}{u} - \frac{F^w}{1-u-v} - \frac{F^{1-u}}{1-u} + \frac{F^{1-w}}{u+v}$$
$$\frac{\partial F(u,v)}{\partial v} = \frac{F^v}{v} - \frac{F^w}{1-u-v} - \frac{F^{1-v}}{1-v} + \frac{F^{1-w}}{u+v}$$

with $dF = \sum_{s_k \in \mathcal{S}} F^{s_k}\, d\ln s_k$

we see that $\mathcal{S} = \{u, v, w, 1-u, 1-v, 1-w\}$ $\quad w = 1-u-v$

Exercise: Verify that all 3 dihedral (cyclic) permutations of the (finite part of the) box integral are in this space.



$$= \mathrm{Li}_2\left(1-\frac{1}{u}\right) + \mathrm{Li}_2\left(1-\frac{1}{v}\right) + \frac{1}{2}\ln^2\left(\frac{u}{v}\right) + \cdots$$

Iterative construction and "integrability"

- Suppose we know all functions F at weight n-1, and the dimension of this space is d_{n-1}.
- We can use the differential definition to construct all functions at the next weight up, *n*.
- In the 2dHPL case, naively we get $d_n = 6d_{n-1}$ weight n functions, given that there are 6 {n-1,1} coproducts $F^u, F^v, F^w, F^{1-u}, F^{1-v}, F^{1-w}$
- But there is an integrability constraint, that the mixed partial derivatives must be equal:

$$\boxed{\frac{\partial^2 F(u,v)}{\partial u \partial v} = \frac{\partial^2 F(u,v)}{\partial v \partial u}}$$

Homework

- Use equality of the mixed partial derivatives to derive a set of linear constraints on the $\{n\text{-}2,1,1\}$ "double" coproducts, $F^{s_{k,},s_j}$
- After looking at what multiplies the independent rational functions of u and v, you should find 9 independent relations:

$F^{1-u,1-v} - F^{1-v,1-u} = 0,$ and dihedral images (3 equations)

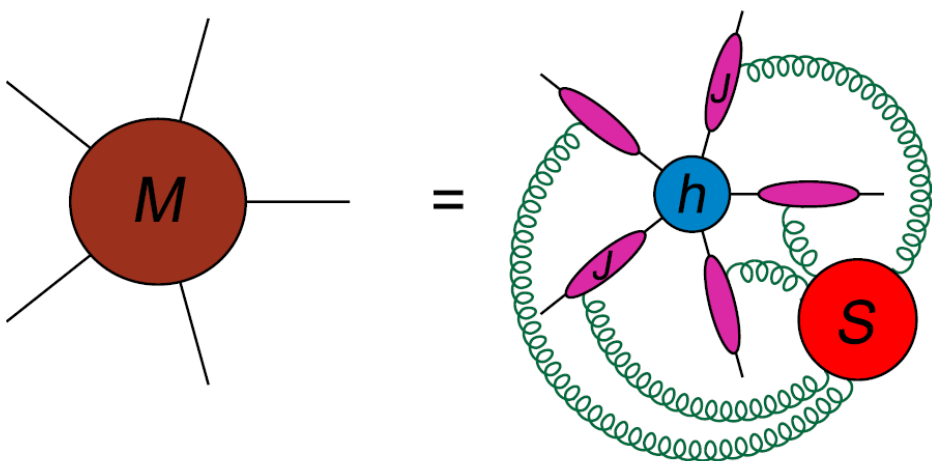
$F^{u,v} - F^{v,u} + F^{1-w,v} - F^{v,1-w} = 0,$ and dihedral images (6 eqns)

- Note that you might initially find linear combinations of these relations, but they can be rearranged into this form.

Infrared divergences

1980s QCD factorization: Collins, Soper, Sterman, Mueller, Sen, Magnea, Korchemsky, …

- All on-shell amplitudes in massless gauge theory are infrared divergent due to soft gluon exchange, and virtual collinear splitting.

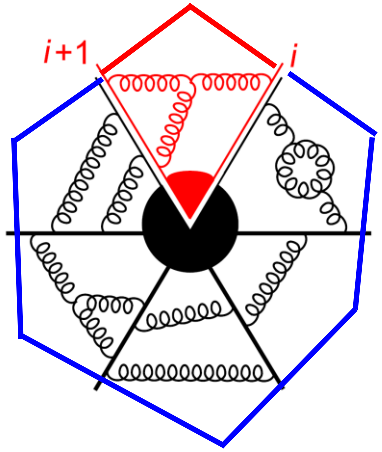


- Soft divergences quite complicated "web" for n-point amplitude because soft gluons can "see" all hard particles

Planar IR divergences

e.g. Bern, LD, Smirnov, hep-th/0505205

- IR divergences simplify drastically in the planar limit:
- The n (adjoint) hard particles generically have very different color indices on each side of their double-edged ('t Hooft) color lines, defining n *"wedges"*:



- Each soft gluon can only be emitted and absorbed within a single wedge.
- Collinear virtual emission can be assigned to wedges too.
- Each wedge is very simple kinematically, depending only on $s_{i,i+1}$
- It is the square root of a 2-point "Sudakov" form factor
- Furthermore, it is dual to a piece of the n-gon, containing a single vertex, or cusp.
- Leading behavior ~ cusp anomalous dimension.

Magnea, Sterman

Korchemsky, Radyushkin

BDS Ansatz

Bern, LD, Smirnov, hep-th/0505205

For planar N=4 SYM n-point MHV amplitudes:

$$\boxed{\frac{\mathcal{A}_n^{\rm BDS}}{\mathcal{A}_n^{\rm tree}} \equiv \mathcal{M}_n^{\rm BDS} = \exp\Big[\sum_{l=1}^{\infty}\Big[\frac{\lambda}{8\pi^2}\Big]^l \Big(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + \mathcal{O}(\epsilon)\Big)\Big]}$$

- $\lambda = N_c g_{YM}^2$ is the ‘t Hooft coupling
- $M_n^{(1)}(l\epsilon)$ is 1-loop MHV amplitude, in dimensional regularization with $D = 4 - 2l\epsilon$
- Unusual dimension and $f^{(l)}(\epsilon) \sim \Gamma_{cusp} + \mathcal{O}(\epsilon)$ designed to reproduce Sudakov form factor for each wedge, Magnea, Sterman capture all IR divergences. (Also proper collinear limits.)
- $C^{(l)}$ is a constant for $n = 4{,}5$, as a consequence of

dual conformal invariance: no cross ratios on which to depend!

Remainder function

Bern, LD, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1465;
Drummond, Henn, Korchemsky, Sokatchev, 0803.1466

- Starting at $n = 6$, BDS ansatz for MHV amplitudes needs correction, define a remainder function:

$$\boxed{\lim_{\epsilon \to 0} \frac{\mathcal{A}_n^{\rm MHV}(s_{i,i+1}, \epsilon)}{\mathcal{A}_n^{\rm BDS}(s_{i,i+1}, \epsilon)} \equiv \exp[R_n(u_i)]}$$

- Dual conformal invariance

→ R_n only depends on $3n$-15 cross ratios u_i

- Collinear properties of $\mathcal{A}_n^{\rm BDS}$ are "correct"

→ $R_n \to R_{n-1}$ smoothly as any two gluons become collinear

- In particular, $R_6 \to R_5 = 0$. Similarly for *Hggg* remainder R

Remainder function issues

- Despite the nice collinear properties of the remainder function, it does not have the nicest causal (branch cut) properties, leading it to live in "too big" a space of functions.

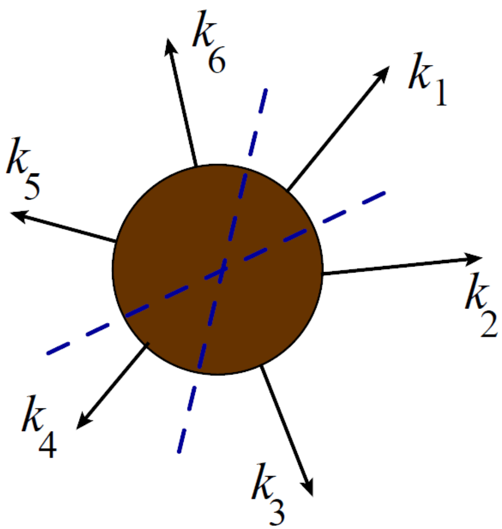
- Not good for bootstrapping.

- Problem is that BDS ansatz exponentiates the full one-loop amplitude. The perturbative expansion of the exponential at two loops includes $[M_n^{(1)}]^2$, and this quantity violates certain causal "Steinmann" relations

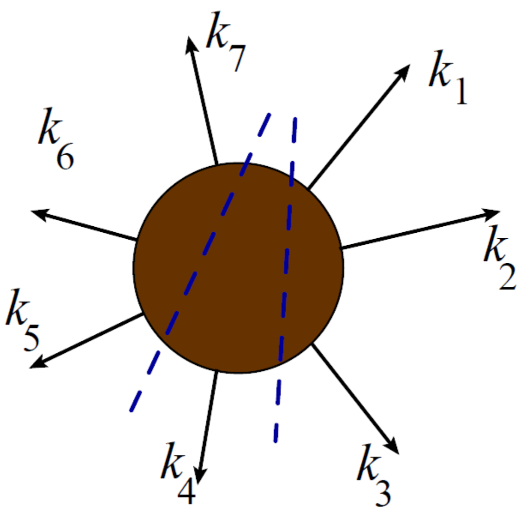
Steinmann relations

Steinmann, Helv. Phys. Acta (1960) Bartels, Lipatov, Sabio Vera, 0802.2065

- Amplitudes should not have overlapping branch cuts:



Not Allowed



Allowed

$$\mathrm{Disc}_{s_{234}}\left[\mathrm{Disc}_{s_{123}}\mathcal{A}_6\right] = 0$$

can't apply to 2 particle cuts in massless case because they are not independent

BDS-like normalization

- Inspecting the 1-loop 6-gluon amplitude, it splits into a part with no 3-particle cuts (only $\ln s_{i,i+1}$), and a finite dual conformal part:

i index is mod 6

$$M_6^{(1)}(\epsilon) = \widehat{M}_6^{(1)}(\epsilon) + \mathcal{E}_6^{(1)}(u,v,w)$$

$$\widehat{M}_6^{(1)}(\epsilon) = \sum_{i=1}^{6}\Big[-\frac{1}{\epsilon^2}(1-\epsilon\ln s_{i,i+1}) - \ln s_{i,i+1}\ln s_{i+1,i+2} + \frac{1}{2}\ln s_{i,i+1}\ln s_{i+3,i+4} + \zeta_2\Big]$$

$$\mathcal{E}_6^{(1)} = \mathrm{Li}_2\left(1-\frac{1}{u}\right) + \mathrm{Li}_2\left(1-\frac{1}{v}\right) + \mathrm{Li}_2\left(1-\frac{1}{w}\right)$$

- So we can normalize by the more minimal ansatz,

$$\mathcal{A}_6^{\mathrm{BDS-like}} = \mathcal{A}_6^{\mathrm{tree}}\exp\Big[\sum_{l=1}^{\infty}\Big[\frac{\lambda}{8\pi^2}\Big]^2\big(f^{(l)}(\epsilon)\widehat{M}_6^{(1)}(l\epsilon) + C^{(l)}\big)\Big]$$

Alday, Gaiotto, Maldacena, 0911.4708

BDS-like normalized amplitude $\mathcal{E}$

Caron-Huot, LD, von Hippel, McLeod, 1609.00669

- For 6 gluons, best to work with:

from $f^{(l)}(\epsilon \to 0)$

$$\mathcal{E}_6(u_i) = \lim_{\epsilon\to 0} \frac{\mathcal{A}_6(s_{i,i+1},\epsilon)}{\mathcal{A}_6^{\mathrm{BDS-like}}(s_{i,i+1},\epsilon)} = \exp[\frac{\Gamma_{\mathrm{cusp}}}{4}\mathcal{E}_6^{(1)} + R_6]$$

because it obeys

$$\boxed{\mathrm{Disc}_{s_{234}}\left[\mathrm{Disc}_{s_{123}}\mathcal{E}_6\right] = 0}$$

- There's also a Steinmann-preserving 7-gluon normalization.
  LD, Drummond, Harrington, McLeod, Papathanasiou, Spradlin, 1612.08976
- For 3-point "*Hggg*" form factor $\mathcal{F}_3$, there are no Steinmann relations to preserve, but we can still expect a more minimal normalization to simplify things.

BDS-like normalization for $\mathcal{F}_3$

- Inspecting the 1-loop amplitude,

i index is mod 3

$$M_3^{(1)}(\epsilon) = \widehat{M}_3^{(1)}(\epsilon) + \mathcal{E}^{(1)}(u,v,w)$$

$$\widehat{M}_3^{(1)}(\epsilon) = \sum_{i=1}^{3} [-\frac{1}{\epsilon^2}(1-\epsilon \ln s_{i,i+1}) - \frac{1}{2}\ln^2 s_{i,i+1}] + \frac{9}{2}\zeta_2 + \sum_{i=1}^{3}[\ln^2 u_i - \ln u_i \ln u_{i+1}]$$

$$\mathcal{E}^{(1)} = 2\left[\, \mathrm{Li}_2\left(1-\frac{1}{u}\right) + \mathrm{Li}_2\left(1-\frac{1}{v}\right) + \mathrm{Li}_2\left(1-\frac{1}{w}\right) \right]$$

and we normalize by,

$$\mathcal{F}_3^{\mathrm{BDS-like}} = \mathcal{F}_3^{\mathrm{tree}} \exp[\sum_{l=1}^{\infty} [\frac{\lambda}{8\pi^2}]^2 (f^{(l)}(\epsilon) \widehat{M}_3^{(1)}(l\epsilon) + C^{(l)})]$$

$$\mathcal{E}(u_i) = \lim_{\epsilon\to 0} \frac{\mathcal{F}_3\,(s_{i,i+1},\epsilon)}{\mathcal{F}_3^{\mathrm{BDS-like}}(s_{i,i+1},\epsilon)} = \exp[\frac{\Gamma_{\mathrm{cusp}}}{4}\mathcal{E}^{(1)} + R]$$

Relation between R, $\mathcal{E}$

$$\mathcal{E}(u_i) = \exp[\frac{\Gamma_{\rm cusp}}{4}\mathcal{E}^{(1)} + R]$$

- In collinear limit, $\mathcal{E}(u_i) \to \exp[\frac{\Gamma_{\rm cusp}}{4}\mathcal{E}^{(1)}]$
- Cusp anomalous dimension known to all loop orders (by inverting a semi-infinite matrix $\mathbb{K}$, not in closed form)

  Beisert, Eden, Staudacher, hep-th/0610251

$$\frac{\Gamma_{\rm cusp}}{4} = g^2 - 2\zeta_2 g^4 + 22\zeta_4 g^6 - (219\zeta_6 + 8\zeta_3^2)g^8 + \cdots$$

where $g^2 \equiv \frac{\lambda}{16\pi^2}$

- At symbol level, all zeta values vanish,

$$\mathcal{E}(u_i) = \exp\left[g^2\mathcal{E}^{(1)} + R\right] = 1 + g^2\,\mathcal{E}^{(1)} + g^4\left[R^{(2)} + \frac{1}{2}\left(\mathcal{E}^{(1)}\right)^2\right] + \cdots$$

- So $\mathcal{E}^{(2)} = R^{(2)} + \frac{1}{2}[\mathcal{E}^{(1)}]^2$, etc.

Inspecting the 2 loop symbol

- Symbol of the infrared finite "remainder function" from Brandhuber, Travaglini, Yang, 1201.4170

$$
\begin{aligned}
\mathcal{S}\left(R_3^{(2)}\right) = 4\bigg[& -2u \otimes (1-u) \otimes (1-u) \otimes \frac{1-u}{u} + u \otimes (1-u) \otimes u \otimes \frac{1-u}{u} \\
& - u \otimes (1-u) \otimes v \otimes \frac{1-v}{v} - u \otimes (1-u) \otimes w \otimes \frac{1-w}{w} \\
& - u \otimes v \otimes (1-u) \otimes \frac{1-v}{v} - u \otimes v \otimes (1-v) \otimes \frac{1-u}{u} \\
& + u \otimes v \otimes w \otimes \frac{1-u}{u} + u \otimes v \otimes w \otimes \frac{1-v}{v} \\
& + u \otimes v \otimes w \otimes \frac{1-w}{w} - u \otimes w \otimes (1-u) \otimes \frac{1-w}{w} \\
& + u \otimes w \otimes v \otimes \frac{1-u}{u} + u \otimes w \otimes v \otimes \frac{1-v}{v} \\
& + u \otimes w \otimes v \otimes \frac{1-w}{w} - u \otimes w \otimes (1-w) \otimes \frac{1-u}{u} \bigg] \; + \; \text{cyclic}
\end{aligned}
$$

- It is not that complicated (84 terms), but every pair of adjacent letters appears.
- So let's convert to $\mathcal{E}^{(2)} = R^{(2)} + \frac{1}{2}[\mathcal{E}^{(1)}]^2$

A better alphabet

- Motivated by a similar change of variables in the 6 gluon case Caron-Huot, LD, von Hippel, McLeod, 1609.00669 (which exposes the Steinmann relations there), we also switch to the alphabet

$$\mathcal{S}' = \{\, a = \frac{u}{vw}, b = \frac{v}{wu}, c = \frac{w}{uv}, d = \frac{1-u}{u}, e = \frac{1-v}{v}, f = \frac{1-w}{w} \}$$

- We find that the symbol of the one- and two-loop amplitudes simplify remarkably, down to just 1 and 2 terms, plus dihedral images(!!!):

$$\mathcal{S}\left[\mathcal{E}^{(1)}\right] = (-1)\, b \otimes d + \text{dihedral}$$

$$\mathcal{S}\left[\mathcal{E}^{(2)}\right] = 4\, b \otimes d \otimes d \otimes d + 2\, b \otimes b \otimes b \otimes d + \text{dihedral}$$

Exercise(s)

- From $$\mathcal{E}^{(1)}(u,v,w) \;=\; 2\Big[\mathrm{Li}_2\Big(1-\frac{1}{u}\Big) + \mathrm{Li}_2\Big(1-\frac{1}{v}\Big) + \mathrm{Li}_2\Big(1-\frac{1}{w}\Big)\Big]$$

compute the symbol by taking derivatives, first in the alphabet

$$\mathcal{S} = \{u, v, w, 1-u, 1-v, 1-w\}$$

and then convert it to $\mathcal{S}' = \{a, b, c, d, e, f\}$

- You can also derive relations between coproducts in different alphabets using the chain rule:

$$F^{u} = F^{a} - F^{b} - F^{c} - F^{d} \qquad\qquad F^{1-u} = F^{d}$$

+ dihedral images

Many adjacency constraints!

$$\boxed{F^{d,e} = F^{e,d} = F^{e,f} = F^{f,e} = F^{f,d} = F^{d,f} = 0}$$

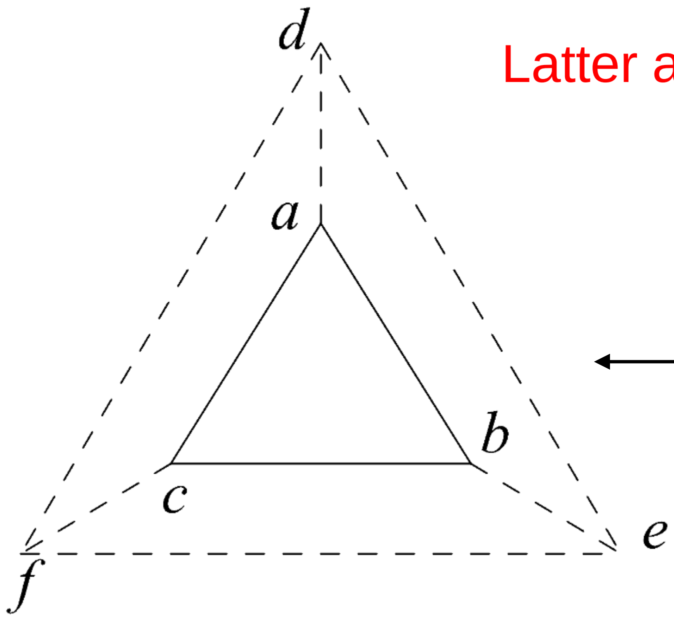
Hold for 2 loop QCD amplitudes too, planar and nonplanar!

LD, Mcleod, Wilhelm, 2012.12286

$$\boxed{F^{a,d} = F^{d,a} = F^{b,e} = F^{e,b} = F^{c,f} = F^{f,c} = 0}$$

Latter are NEW: Hold for planar N=4 SYM to 8 loops!

LD, Gürdoğan, Mcleod, Wilhelm, to appear



Mnemonic for dihedral symmetry;
6 dashed lines indicate 12 forbidden pairs.

Pair recap for planar N=4 SYM

- There are 9 integrability relations. In the alphabet $\mathcal{S}'$,

6 of them become the antisymmetric parts of the 12 relations

$$F^{d,e} = F^{e,d} = F^{e,f} = F^{f,e} = F^{f,d} = F^{d,f} = 0$$

$$F^{a,d} = F^{d,a} = F^{b,e} = F^{e,b} = F^{c,f} = F^{f,c} = 0$$

- The 3 remaining integrability relations are a bit longer:

$$F^{a,b} - F^{b,a} + F^{a,c} - F^{c,a} = 0$$
$$F^{b,c} - F^{c,b} + F^{a,c} - F^{c,a} = 0$$

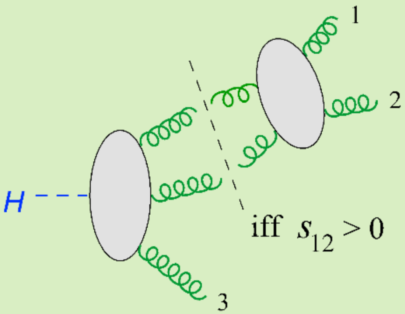
$$F^{d,b} - F^{b,d} + F^{c,d} - F^{d,c} + F^{e,c} - F^{c,e} + F^{a,e} - F^{e,a}$$
$$+ F^{f,a} - F^{a,f} + F^{b,f} - F^{f,b} + 4(F^{c,b} - F^{b,c}) = 0$$

- Number of allowed pairs is 36 – 6 – 9 = 21

(This number is saturated by 3 loops.)

Branch cut conditions

All massless particles
→ all branch cuts start at origin in $s_{i,i+1}$, s_{123}



→ Branch cuts all start from 0 or ∞ in $u = \frac{s_{12}}{s_{123}}$ or v or w

→ Only 3 weight 1 functions, not 6: $\{ \ln u, \ln v, \ln w \}$ $\cancel{\ln(1-u)}$

- Quite generally, derivatives commute with branch cuts:

$$\frac{\partial}{\partial z} \mathrm{Disc}_{z=z_0} f(z) = \mathrm{Disc}_{z=z_0} \frac{\partial f}{\partial z}$$

- Derivatives of higher weight functions must obey branch-cut condition too. At symbol level this means their first entries can only be $\{u, v, w\}$, or $\{a, b, c\}$

Branch cut conditions

- In the form factor case, requiring no branch cut as $u \to 1$ for arbitrary $v, w \to 0$ gives the condition (+ dihedral images):

$$\boxed{F^{1-u}(1, v, w)|_{v,w\to 0} = 0}$$

- This condition is stronger than the first-entry condition. At symbol level it says that a $1 - u$ anywhere in the symbol must be preceded by $u, 1 - v, or\ 1 - w$

- For 6 and 7 gluons, the appropriate first entry condition is enough, at symbol level.

Heuristic view of space

weight

…

$\int\int\int\int\int\int$

derivatives

$\int\int\int\int\int$

4

$\int\int\int\int$

3

$\mathrm{Li}_3(1\text{-}1/u_i)$, true 2D HPLs, …

2

$\mathrm{Li}_2(1\text{-}1/u_i)$ $\ln^2 u_i$ $\ln u_i \ln u_{i+1} - \zeta_2$

1

$\ln u$ $\ln v$ $\ln w$

0

1

$$\boxed{\frac{\partial F(u,v)}{\partial u} = \frac{F^u}{u} - \frac{F^w}{1-u-v} - \frac{F^{1-u}}{1-u} + \frac{F^{1-w}}{u+v}}$$

Construction of space

- Weight 2: Must start with $\{a, b, c\}$. 6 choices of letters in second slot before applying pair constraints.
- First suppose second letter is also $\{a, b, c\}$. Then "longest" pair constraint collapses to $F^{b,c} = F^{c,b}$. Combining it with the other two long ones gives the dihedral images, $F^{a,b} = F^{b,a}$, $F^{c,a} = F^{a,c}$.
- Solved by $$a \otimes a = S\left[\frac{1}{2}\ln^2 a\right] \qquad + \text{cyclic} \qquad (3)$$ $$a \otimes b + b \otimes a = S[\ln a \ln b] \qquad + \text{cyclic} \qquad (3)$$
- If last letter is d, "longest" pair constraint → $F^{b,d} = F^{c,d}$

→ $$b \otimes d + c \otimes d = -2u \otimes \frac{1-u}{u} = S\left[2\mathrm{Li}_2\left(1 - \frac{1}{u}\right)\right] \qquad + \text{cyclic} \quad (3)$$

- Weight 2 space is 9 dimensional.

Construction of space (cont.)

- Weight 3: Start with the 9 weight 2 symbol, and tack on all 6 choices of letters in 3rd slot.
- Apply pair constraints.
- Also apply branch cut constraints. (Satisfied at weight 2.)
- Finally there are some triple constraints.
- Solve on computer → weight 3 is 21 dimensional
- Promote to functions:

At low weights, most are logs and 1d HPLs:

$$H_{0,\vec{w}}(x) = \int_0^x \frac{dt}{t} H_{\vec{w}}(t), \qquad H_{1,\vec{w}}(x) = \int_0^x \frac{dt}{1-t} H_{\vec{w}}(t)$$

$$dH_{0,\vec{w}}(x) = H_{\vec{w}}(x)\ d\ln x \qquad dH_{1,\vec{w}}(x) = -H_{\vec{w}}(x) d\ln(1-x)$$

Symbol alphabet for $H_{\vec{w}}(x)$: $\{x, 1-x\}$

Construction of space (cont.)

Let $u = u_1, v = u_2, w = u_3$

- 3 weight 1 functions: $\ln u_i$
- 9 weight 2 functions:

$$\frac{1}{2}\ln^2\left(\frac{u_i}{u_{i+1}}\right) + \zeta_2\ , \quad \frac{1}{2}\ln^2\left(\frac{u_i u_{i+1}}{u_{i+2}}\right) + \zeta_2\ , \ \ H_{0,1}(1-u_i)$$

- 22 weight 2 functions:

$$\left\{\frac{1}{6}\ln^3\left(\frac{u_i}{u_{i+1}}\right) + \zeta_2 \ln\left(\frac{u_i}{u_{i+1}}\right), \ \ \frac{1}{6}\ln^3\left(\frac{u_i u_{i+1}}{u_{i+2}}\right) + \zeta_2 \ln\left(\frac{u_i u_{i+1}}{u_{i+2}}\right), \right.$$
$$\frac{1}{6}\ln^3 u_i - H_{0,1,1}(1-u_i)\,, \quad \frac{1}{6}\ln^3 u_i + \ln u_i\, H_{0,1}(1-u_i) + 2\, H_{0,1,1}(1-u_i)$$
$$\left. H_{0,0,1}(1-u_i) + H_{0,1,1}(1-u_i) \right\},$$

[five 3-orbits]

+ two more complicated 3-orbits

+ ζ_3 [need first at 6 loops!]

Symbol is too verbose → Nested representation better

loop order L	1	2	3	4	5	6	7	8
terms in $S[\mathcal{E}^{(L)}]$	6	12	636	11,208	263,880	4,916,466	97,594,968	???

- Define every function by its $\{n-1,1\}$ coproducts, i.e. its first derivatives.
- Also need to specify constants of integration at one point, e.g. $(u,v,w) = (1,0,0)$

Three-index tensors

- Function space defined by

$dF = \sum_{s_\alpha \in \mathcal{S}} F^{s_\alpha}\, d\ln s_\alpha$ or $\Delta_{n-1,1}F = \sum_{s_\alpha \in \mathcal{S}} F^{s_\alpha} \otimes \ln s_\alpha$

- Let $\{f_i^{(n)}\}$ be a basis for the d_n functions at weight n, and expand the derivatives of $f_i^{(n)}$ in terms of $f_i^{(n-1)}$:

$$\boxed{df_i^{(n)} = \sum_{j,\alpha} T_{ij\alpha}^{(n)} f_j^{(n-1)} d\ln s_\alpha}$$

- Tensors $T_{ij\alpha}^{(n)} \in \mathbb{Q}$ have dimension $d_n \times d_{n-1} \times |\mathcal{S}|$, fully characterize space at symbol level, and up to constants at function level.