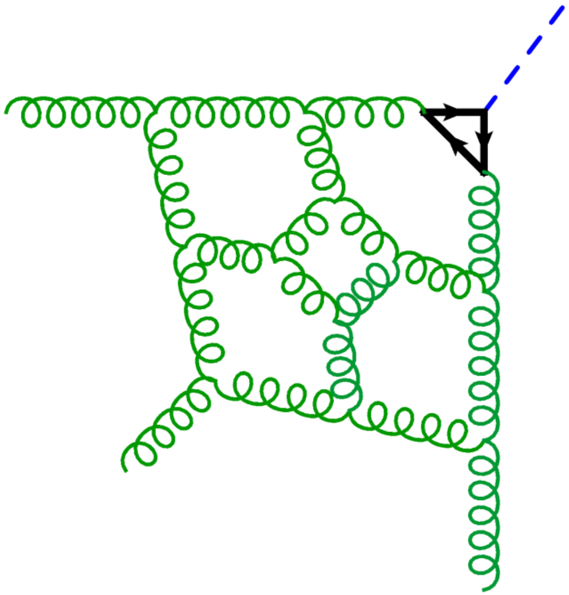
[Amplitude] Bootstrap



Lance Dixon (SLAC)

"The Amplitude Games"
Mainz Institute for Theoretical Physics
19-20 July, 2021

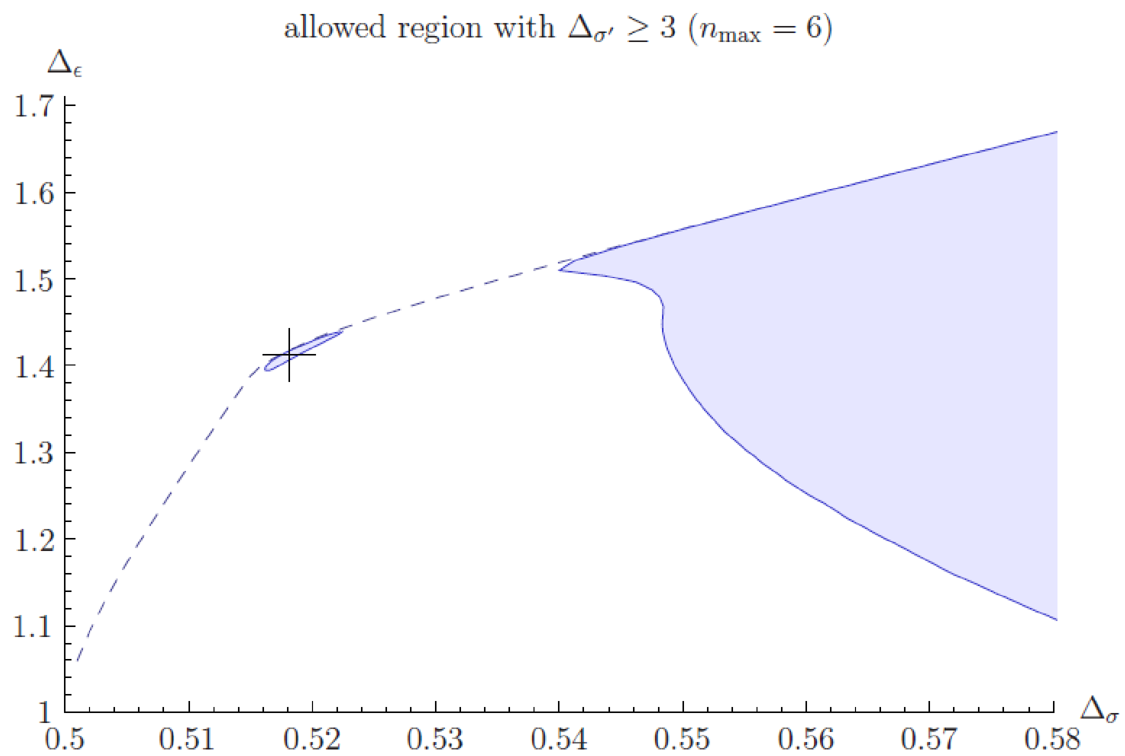
SLAC NATIONAL ACCELERATOR LABORATORY

Those who explore an unknown world are travelers without a map: the map is the result of the exploration. The position of their destination is not known to them, and the direct path that leads to it is not yet made.

Hideki Yukawa

Not the conformal bootstrap

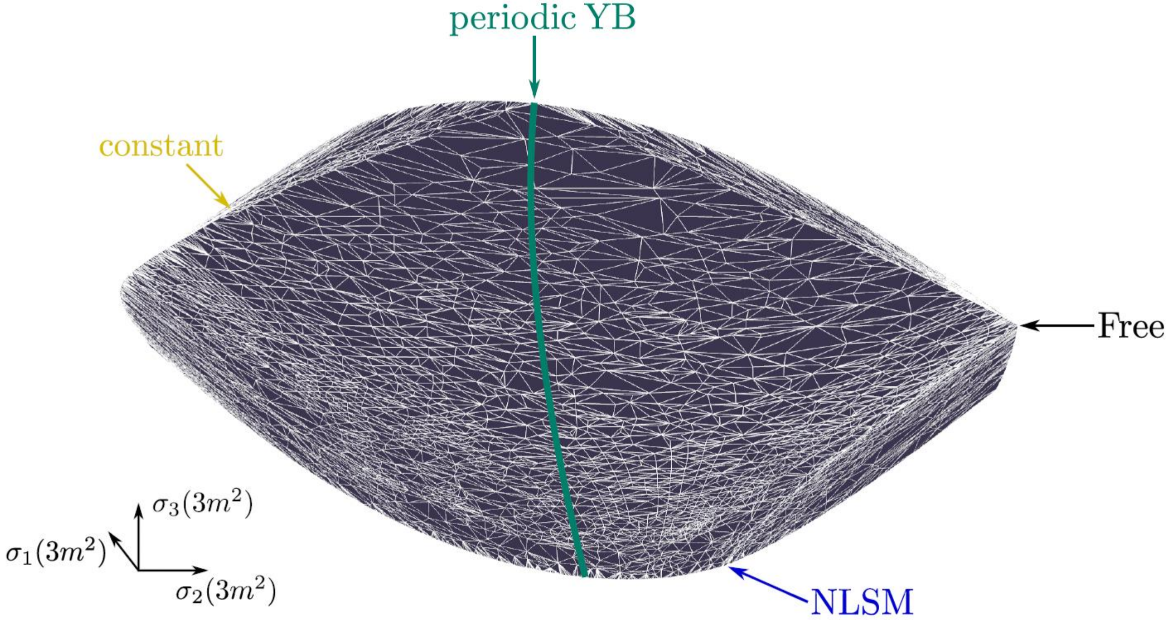
- Constraining allowed CFTs using crossing symmetry, positivity,…



Kos, Poland, Simmons-Duffin, 1406.4858

Not the S-matrix bootstrap

- Constraining allowed S-matrices for massive theories using unitarity, crossing symmetry,…



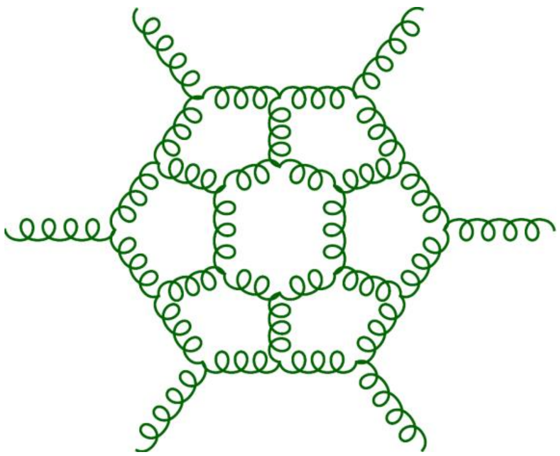
Córdova, He, Kruczenski, Vieira, 1909.06495

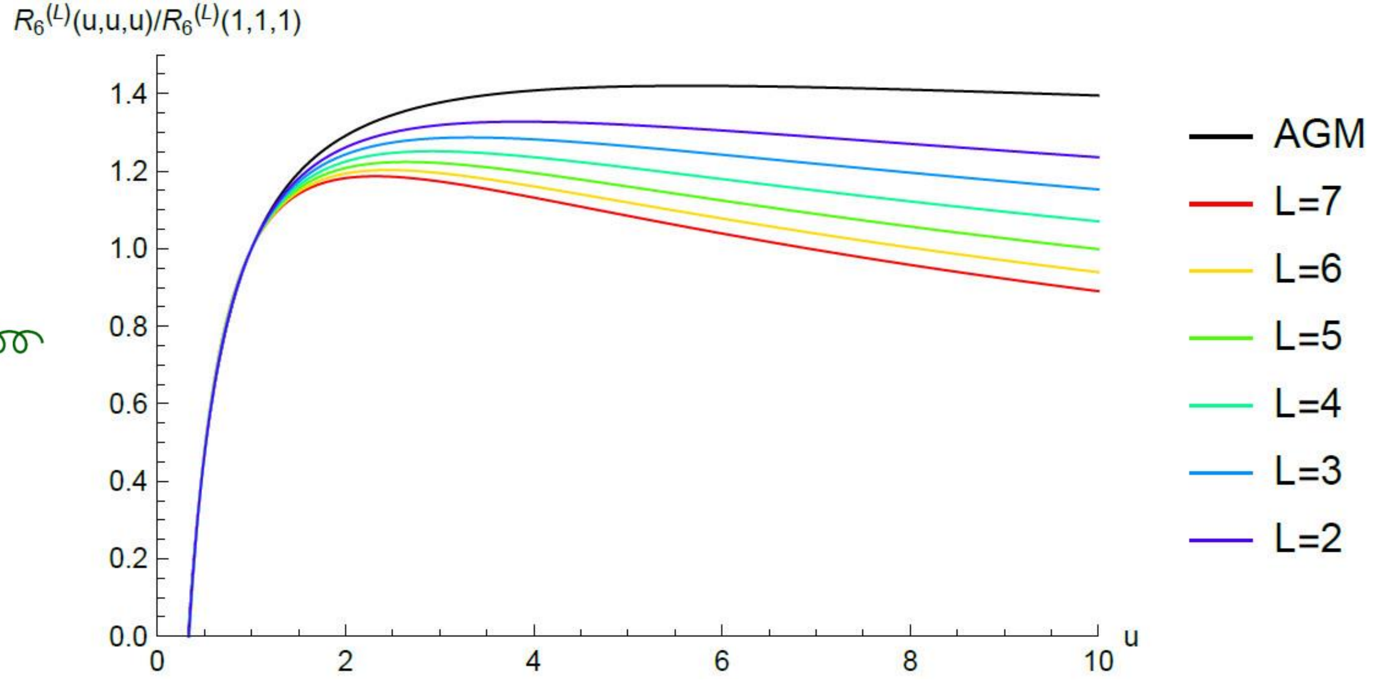
Amplitude bootstrap

- Pick a specific theory (often planar N=4 SYM!)
- Explore its nontrivial scattering amplitudes in great detail (high loop order) by:

1. making assumptions about their functional dependence
2. writing an ansatz (guess) as a linear combination of such functions
3. constraining the ansatz with known (or suspected) properties until it’s uniquely determined
4. checking the answer further with known constraints
5. computing the amplitude explicitly wherever possible

e.g. 6 gluon scattering to 7 loops





Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou, 1903.10890

Rich theoretical “data” mine

- Rare to have perturbative results to 7 loops
- Usually high loop order → single numbers such as β functions or anomalous dimensions
- Here we have analytic functions of 2,3,6 variables
- Many limits to study (and exploit)

Outline

Lecture 1:

1. General properties of N=4 SYM amplitudes
2. A new bootstrapping arena, form factors

Lecture 2:

1. Generalized polylogarithms and symbols
2. IR divergences and BDS(-like) normalizations
3. Branch cuts, pair relations and space construction

Lecture 3:

1. Finding the amplitude in the space with the aid of (multiple) final entry conditions, and the FFOPE
2. Form factor results

Lecture 4:

1. 6 gluon scattering and hexagon functions
2. 7 gluon scattering and heptagon functions

Lecture 1

Different routes to perturbative amplitudes

Draw all Feynman graphs G_i

↓

Evaluate all Feynman rules: I_i

↓

Perform all loop integrations: A_i

↓

$$A = \sum_i A_i$$

Evaluate all unitarity cuts C_α

↓

Construct **local**: integrand I

↓

Perform all loop integrations: A_α

↓

$$A = \sum_\alpha A_\alpha$$

Bootstrap: Guess $A = \sum_m c_m F_m$

F_m known functions

$c_m \in \mathbb{Q}$ unknown constants

↓

Solve constraints, linear equations for $c_m \to r_m$

↓

$$A = \sum_m r_m F_m$$

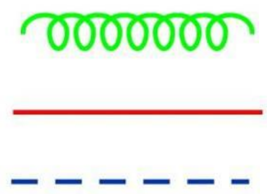
N=4 SYM particle content

Brink, Schwarz, Scherk; Gliozzi, Scherk, Olive (1977)

massless spin 1 gluon

4 massless spin 1/2 gluinos

6 massless spin 0 scalars



Gauge group often:
$G = SU(N_c)$

SUSY
Q_a, a=1,2,3,4
shifts helicity
by 1/2 $\longleftrightarrow$

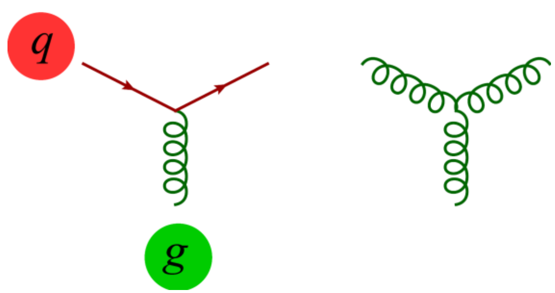
$\mathcal{N}=4$	1	$\longleftrightarrow$	4	$\longleftrightarrow$	6	$\longleftrightarrow$	4	$\longleftrightarrow$	1
	g^-		$\lambda^-_{\bar{i}}$		$\bar{\phi}_{\bar{i}\bar{j}}, \phi_{ij}$		λ^+_i		g^+
helicity	-1		$-\frac{1}{2}$		0		$\frac{1}{2}$		1

all in adjoint representation of G

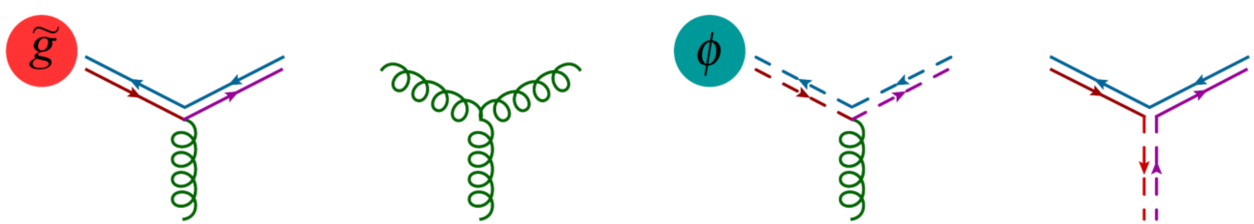
QCD vs. N=4 SYM

- QCD has gluons and quarks in fundamental rep. of $SU(N_c)$
- Replace quarks with 4 copies of fermions in adjoint rep. (gluinos) and add 6 real adjoint scalars
- Feynman vertices:

QCD



N=4 SYM

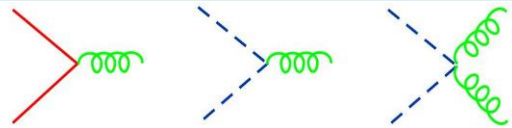


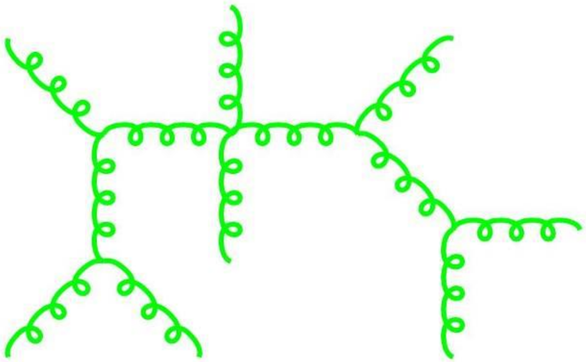
QCD vs. N=4 SYM at tree level

At tree-level essentially identical

Consider a tree amplitude for *n* gluons.

Fermions and scalars cannot appear because they are produced in pairs



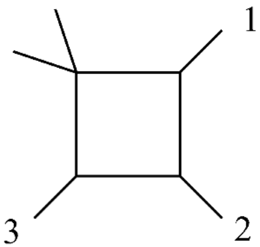


Hence the amplitude is the same in QCD and N=4 SYM.
The QCD tree amplitude "secretly" obeys all identities of N=4 supersymmetry:

No longer true at quantum (loop) level

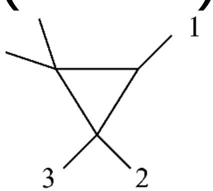
N=4 SYM very special

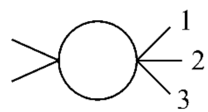
- At one loop, cancellation of loop momenta in numerator → only scalar box integrals
  Bern, LD, Dunbar, Kosower, hep-ph/9403226
- Weight 2 functions – dilogs. For example,



$$= \mathrm{Li}_2\left(1-\frac{s_{123}}{s_{12}}\right)+\mathrm{Li}_2\left(1-\frac{s_{123}}{s_{23}}\right)+\frac{1}{2}\ln^2\left(\frac{s_{12}}{s_{23}}\right)+\cdots$$

- QCD results also contain single log's and rational parts from (tensor) triangle + bubble integrals





$$=\frac{1}{\epsilon}-\ln(s_{123})$$

Higher loops

- Much evidence that N=4 SYM amplitudes have "uniform weight (transcendentality)" $2L$ at loop order L.

→*A fundamental bootstrapping assumption*

- Weight ~ number of integrations, e.g.

$$\ln(s) = \int_1^s \frac{dt}{t} = \int_1^s d\ln t \qquad 1$$

$$\mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1-t) = \int_0^x d\ln t \cdot [-\ln(1-t)] \qquad 2$$

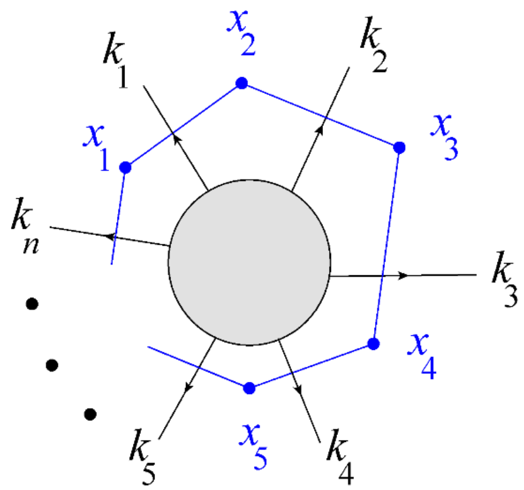
$$\mathrm{Li}_n(x) = \int_0^x \frac{dt}{t} \mathrm{Li}_{n-1}(t) \qquad n$$

Planar limit

- In the large N_c limit, $N_c \to \infty$ for gauge group $G = SU(N_c)$, only planar Feynman diagrams contribute
- Therefore the only important kinematic invariants are color-adjacent, can be written in terms of dual variables x_i^μ:

$$(k_i + k_{i+1} + \cdots + k_{j-1})^2 = (x_i - x_j)^2 \equiv x_{ij}^2$$

- More amazingly, planar N=4 SYM has a **dual conformal symmetry**, generated by translations + boosts + dual inversion:

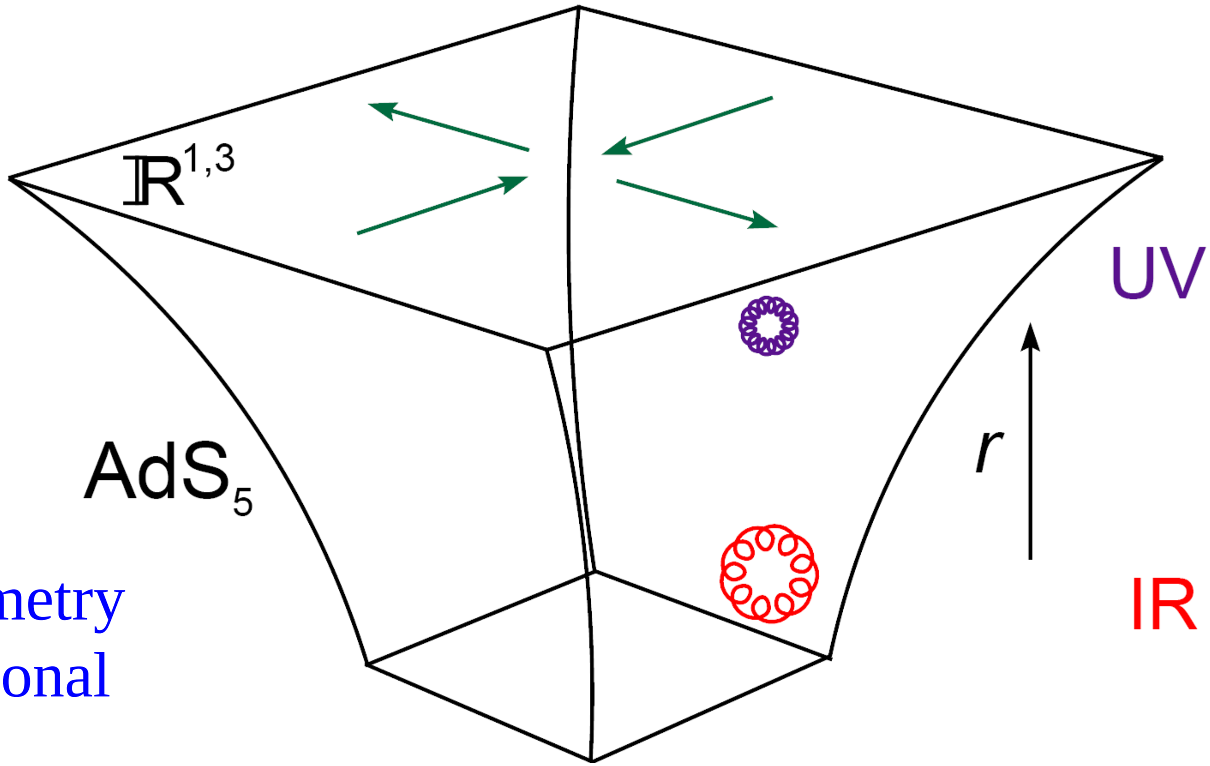


$$x_i^\mu \to \frac{x_i^\mu}{x_i^2} \qquad \Rightarrow \qquad x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2}$$

- **Exercise**: verify " $\Rightarrow$ "

Dual conformal invariance from AdS/CFT + T-duality

Alday, Maldacena, 0705.0303



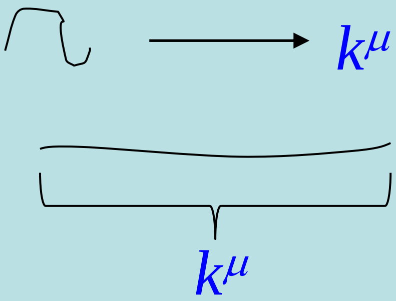
SO(4,2) isometry of 5 dimensional space-time ←→ 4d conformal symmetry

T-duality symmetry of string theory

Alday, Maldacena, 0705.0303

- Exchanges string world-sheet variables σ, τ
- $X^\mu(\tau,\sigma) = x^\mu + k^\mu \tau$ + oscillators

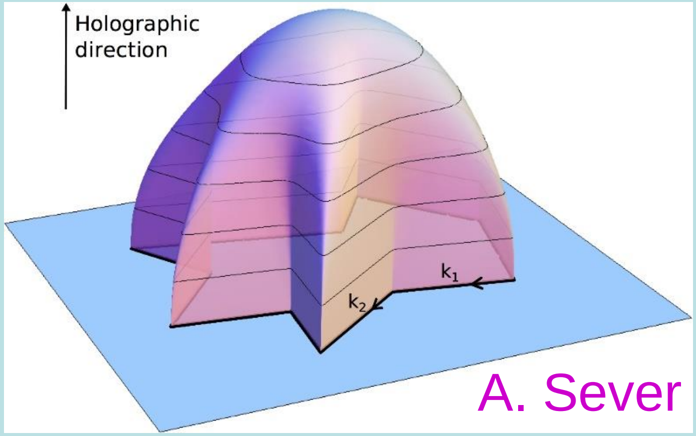
$\rightarrow$ $X^\mu(\tau,\sigma) = x^\mu + k^\mu \sigma$ + oscillators



- Strong coupling limit of planar N=4 SYM

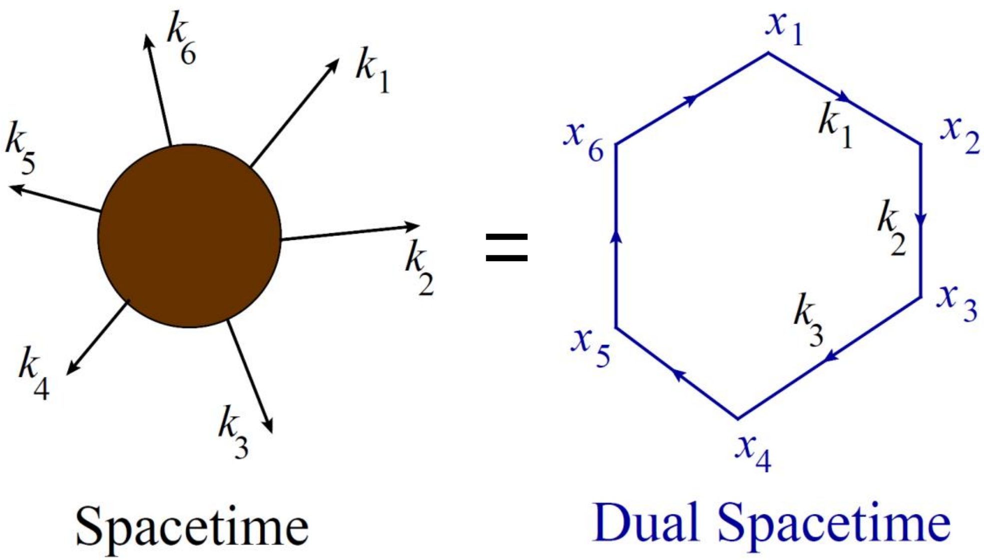
is semi-classical limit of string theory: world-sheet stretches tight around minimal area surface in AdS.

- Boundary determined by momenta of external states: light-like polygon with null edges = momenta k^μ



A. Sever

Amplitudes = Wilson loops



- Polygon vertices x_i are not positions but dual momenta, $x_i - x_{i+1} = k_i$
- Transform like positions under dual conformal symmetry

Alday, Maldacena, 0705.0303
Drummond, Korchemsky, Sokatchev, 0707.0243
Brandhuber, Heslop, Travaglini, 0707.1153
Drummond, Henn, Korchemsky, Sokatchev, 0709.2368, 0712.1223, 0803.1466;
Bern, LD, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1465

Duality verified to hold at weak coupling too

The [Dual] Conformal Group

SO(4,2) $\supset$ SO(3,1) [rotations+boosts] + translations+dilatations + special-conformal

15 = 3 + 3 + 4 + 1 + 4

- The nontrivial generators are special conformal K^μ
- Correspond to inversion · translation · inversion
- To obtain a [dual] conformally invariant function $f(x_{ij}^2)$ just have to check invariance under inversion,

$$x_i^\mu \to x_i^\mu / x_i^2$$

Dual conformal invariance

- Wilson n-gon invariant under inversion: $x_i^\mu \to \frac{x_i^\mu}{x_i^2}, \quad x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2}$

$$x_{ij}^2 = (k_i + k_{i+1} + \cdots + k_{j-1})^2 \equiv s_{i,i+1,\cdots,j-1}$$

- Fixed, up to functions of invariant cross ratios:

$$u_{ijkl} \equiv \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$$

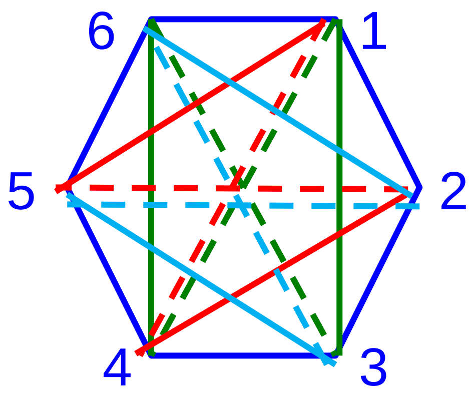
- $x_{i,i+1}^2 = k_i^2 = 0$ → no such variables for $n = 4,5$

$n = 6$ → precisely 3 ratios:

$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12}s_{45}}{s_{123}s_{345}}$$

$$v = \frac{s_{23}s_{56}}{s_{234}s_{123}}$$

$$w = \frac{s_{34}s_{61}}{s_{345}s_{234}}$$

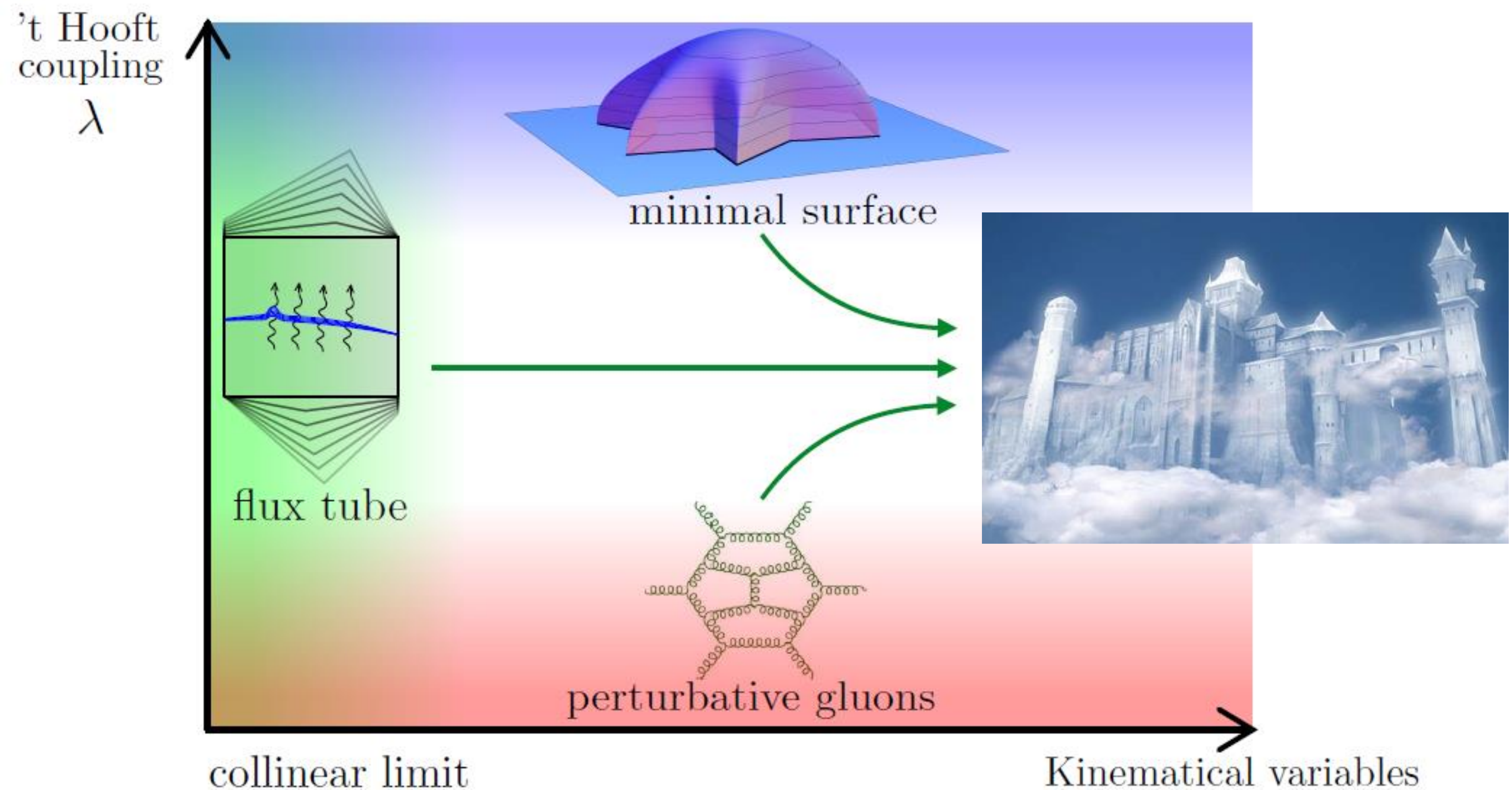


$n = 7$ → 6 ratios.

In general, $3n$-15 ratios.

Solving for Planar N=4 SYM Amplitudes

Images: A. Sever, N. Arkani-Hamed

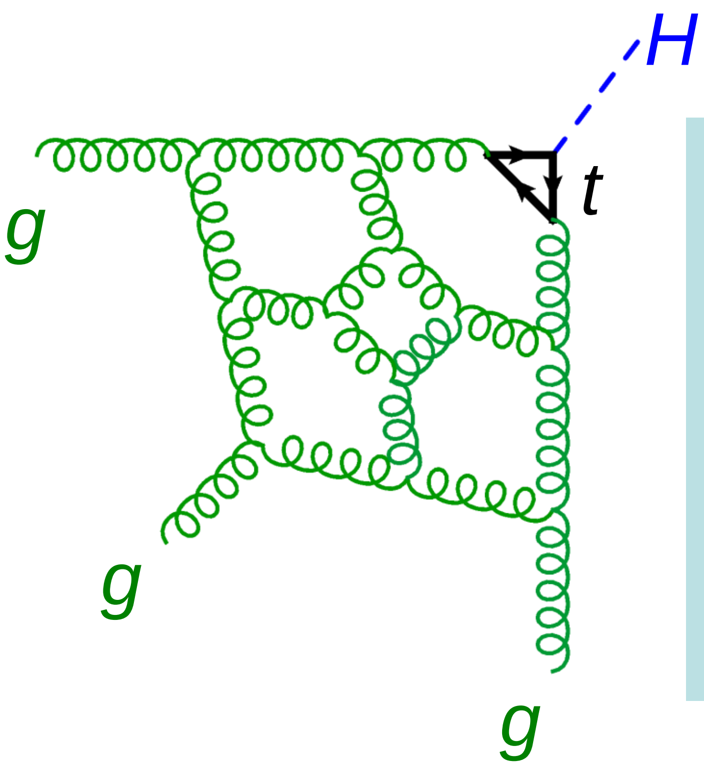


Bootstrapping arenas

- It has proved possible to bootstrap, so far:
  6 gluon amplitudes through 7 loops, and
  7 gluon amplitudes through 4 loops.
- The number of variables is different in QCD and in planar N=4 SYM due to dual conformal symmetry.
- Before describing how to bootstrap all-gluon amplitudes, we will introduce a simpler kinematic setup with even fewer kinematic variables, where QCD and planar N=4 SYM are even more closely related.

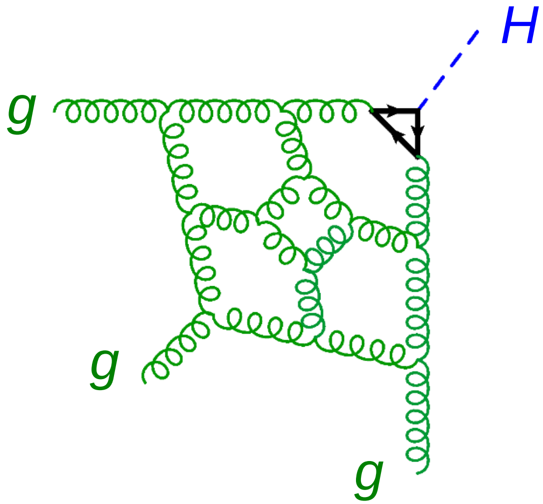
"Higgs" amplitudes and N=4 form factors

LD, A. McLeod, M. Wilhelm, 2012.12286
+ in progress also with Ö. Gürdoğan



- Higgs production at LHC is dominantly via gluon fusion, mediated by a top quark loop.
- Since $\boldsymbol{2m_{top}} = 350\ \mathrm{GeV}$
  $\gg \boldsymbol{m_{Higgs}} = 125\ \mathrm{GeV}$,

we can integrate out the top quark to get a leading operator $H\,G^{a}_{\mu\nu}G^{\mu\nu\,a}$



"Higgs" amplitudes and N=4 form factors

- Higgs is a scalar, color singlet. In QCD its amplitudes with gluons are matrix elements of $G_{\mu\nu}^{a} G^{\mu\nu\, a}$ with on-shell gluons: "form factors"
- In N=4, this operator is part of the (BPS-protected) stress tensor supermultiplet, which also includes for example $\phi_1^\dagger \phi_1 - \phi_2^\dagger \phi_2$ ($\in \mathbf{20}$ of $SU(4)_R$)
- *Hgg* "Sudakov" form factor is "too simple" to bootstrap; it has no kinematic dependence beyond overall $(-s_{12})^{-L\epsilon}$
- *Hggg* is "just right", depends on 2 dimensionless ratios

$\textit{Hggg}$ kinematics is two-dimensional

$$k_1 + k_2 + k_3 = -k_H$$

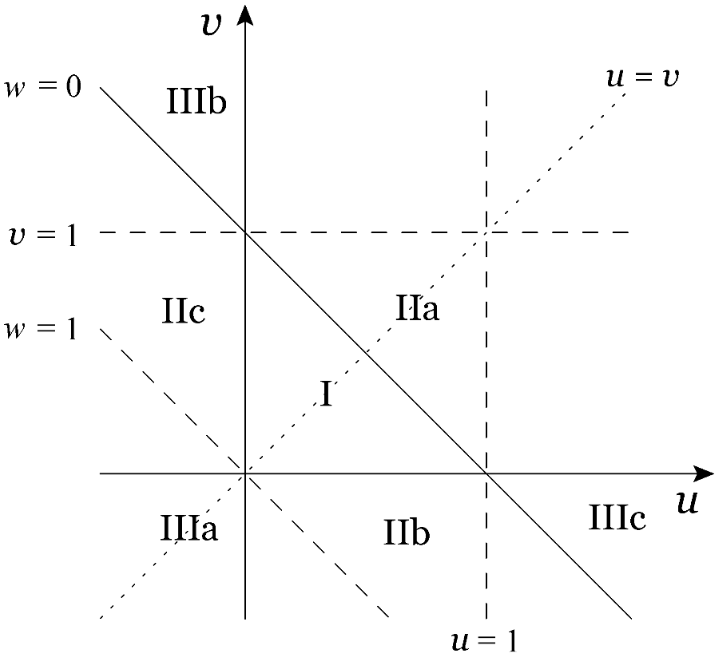
$$s_{123} = s_{12} + s_{23} + s_{31} = m_H^2$$

$$s_{ij} = (k_i + k_j)^2 \qquad k_i^2 = 0$$

$$u = \frac{s_{12}}{s_{123}} \qquad v = \frac{s_{23}}{s_{123}} \qquad w = \frac{s_{31}}{s_{123}}$$

not cross ratios!

$$u + v + w = 1$$



I = decay / Euclidean

IIa,b,c = scattering / spacelike operator

IIIa,b,c = scattering / timelike operator

N=4 amplitude is S_3 invariant

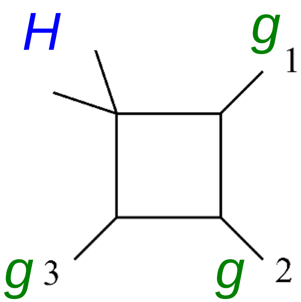
S_3 dihedral symmetry generated by:

a. cycle: $i \to i + 1 \pmod 3$, or

$$u \to v \to w \to u$$

a. flip, e.g. $u \leftrightarrow v$

One loop integrals/amplitudes



$$= \mathrm{Li}_2\left(1-\frac{s_{123}}{s_{12}}\right)+\mathrm{Li}_2\left(1-\frac{s_{123}}{s_{23}}\right)+\frac{1}{2}\ln^2\left(\frac{s_{12}}{s_{23}}\right)+\cdots$$

$$= \mathrm{Li}_2\left(1-\frac{1}{u}\right)+\mathrm{Li}_2\left(1-\frac{1}{v}\right)+\frac{1}{2}\ln^2\left(\frac{u}{v}\right)+\cdots$$

A two-loop story

- Gehrmann et al. computed *H*ggg in QCD at 2 loops

  Gehrmann, Jaquier, Glover, Koukoutsakis, 1112.3554

- Soon after, Brandhuber et al. computed stress tensor 3-point form factor $\mathcal{F}_3$ in N=4 SYM,

  Brandhuber, Travaglini, Yang, 1201.4170

  saw that "maximally transcendental part" of QCD

result (both (+++) and (-++)) was **same as N=4 result**

- This "principle of maximal transcendentality"

  Kotikov, Lipatov, Velizhanin, hep-ph/0301021, hep-ph/0611204

works for DGLAP and BFKL anomalous dimensions, but not for generic scattering amplitudes.

[Why does it work for this one??]

2d HPLs

Gehrmann, Remiddi, hep-ph/0008287

Space graded by weight n. Every function F obeys:

$$\frac{\partial F(u,v)}{\partial u} = \frac{F^u}{u} - \frac{F^w}{1-u-v} - \frac{F^{1-u}}{1-u} + \frac{F^{1-w}}{u+v}$$

$$\frac{\partial F(u,v)}{\partial v} = \frac{F^v}{v} - \frac{F^w}{1-u-v} - \frac{F^{1-v}}{1-v} + \frac{F^{1-w}}{u+v}$$

$$w = 1 - u - v$$

where $F^u, F^v, F^w, F^{1-u}, F^{1-v}, F^{1-w}$ are weight n-1 2d HPLs.

To bootstrap *Hggg* amplitude beyond 2 loops, find as small a subspace of 2d HPLs as possible, construct it to high weight.

Extra Slides

Example 1: Harmonic Polylogarithms of one variable (HPLs {0,1})

Remiddi, Vermaseren, hep-ph/9905237

- Subsector of hexagon functions.
- Gen'lize classical polylogs: $\mathrm{Li}_n(u) = \int_0^u \frac{dt}{t} \mathrm{Li}_{n-1}(t), \quad \mathrm{Li}_1(t) = -\ln(1-t)$
- Define HPLs by iterated integration:

$$H_{0,\vec{w}}(u) = \int_0^u \frac{dt}{t} H_{\vec{w}}(t), \quad H_{1,\vec{w}}(u) = \int_0^u \frac{dt}{1-t} H_{\vec{w}}(t)$$

- Or by derivatives

$$dH_{0,\vec{w}}(u) = H_{\vec{w}}(u)\ d\ln u \quad dH_{1,\vec{w}}(u) = -H_{\vec{w}}(u) d\ln(1-u)$$

- Symbol letters: $\mathcal{S} = \{u, 1-u\}$
- Weight n = length of binary string $\vec{w}$
- Number of functions at weight $n = 2L$: 2^{2L}

Values of HPLs {0,1} at $u = 1$

- Classical polylogs

evaluate to Riemann zeta values

$$\mathrm{Li}_n(u) = \int_0^u \frac{dt}{t}\mathrm{Li}_{n-1}(t) = \sum_{k=1}^{\infty} \frac{u^k}{k^n}$$

$$\mathrm{Li}_n(1) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \zeta(n) \equiv \zeta_n$$

- HPL's evaluate to nested sums called multiple zeta values (MZVs):

$$\zeta_{n_1,n_2,\ldots,n_m} = \sum_{k_1>k_2>\cdots>k_m>0}^{\infty} \frac{1}{k_1^{n_1} k_2^{n_2} \cdots k_m^{n_m}}$$

Weight $n = n_1 + n_1 + \ldots + n_m$

- MZV's obey many identities, e.g. stuffle

$$\zeta_{n_1}\zeta_{n_2} = \zeta_{n_1,n_2} + \zeta_{n_2,n_1} + \zeta_{n_1+n_2}$$

- All reducible to Riemann zeta values until weight 8. Irreducible MZVs: $\zeta_{5,3},\ \zeta_{7,3},\ \zeta_{5,3,3},\ \zeta_{9,3},\ \zeta_{6,4,1,1},\ \ldots$

Example 2: Single-valued harmonic polylogarithms of one complex variable

Brown, C. R. Acad. Sci. Paris, Ser. I 338 (2004) 527

- Also a subsector of hexagon functions, in the "multi-Regge limit"
- 4 symbol letters: $\mathcal{S} = \{z, 1-z, \bar{z}, 1-\bar{z}\}$
- But also require function to be real analytic in
  $(z, \bar{z}) \in \mathbb{C} - \{0, 1\}$
- Constrains the first entry of the symbol to be
  $z\bar{z} \leftrightarrow \ln|z|^2$ or $(1-z)(1-\bar{z}) \leftrightarrow \ln|1-z|^2$
- Brown: One SVHPL for each HPL
- Powerful constraint: $4^{2L} \rightarrow 2^{2L}$ functions