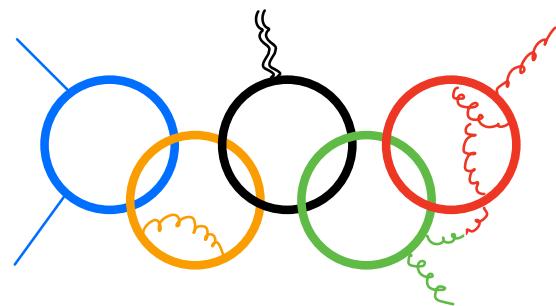


The Amplitude Games



MIT P School 2021

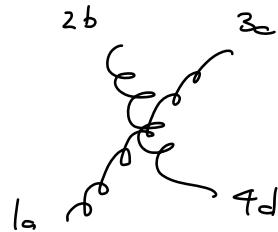
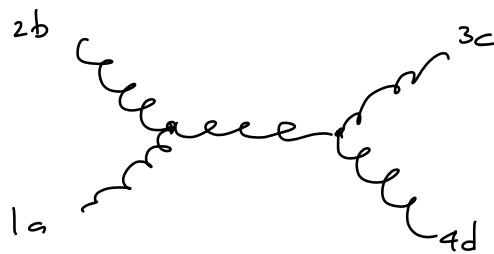
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Homework Solutions

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①) Let's construct the s-channel numerator, which receives contributions from the diagrams



From the Feynman rules, I find

$$\text{Diagram} = -ig^2 f^{abc} f^{ade} \left[\epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 - \epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3 \right] + \text{(other color structures)}$$

\Rightarrow contribution $(\epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 - \epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3) s$ to n_s (up to overall scale.)

$$\text{Diagram} = g^2 f^{abc} f^{ade} \frac{-i}{s} \left[(p_1 - p_2)^\mu \epsilon_1 \cdot \epsilon_2 + (p_2 - q)^\mu \epsilon_1 \cdot \epsilon_2 + (q - p_1)^\mu \epsilon_2 \cdot \epsilon_1 \right] \times \left[(p_3 - p_4)_\mu \epsilon_3 \cdot \epsilon_4 + (p_4 + q)_\mu \epsilon_3 \cdot \epsilon_{4\mu} + (-q - p_3)_\mu \epsilon_4 \cdot \epsilon_{3\mu} \right]$$

(2)

Here,

$$n_s = [(p_1 - p_2) \cdot \epsilon_1 \cdot \epsilon_2 + (p_2 - q) \cdot \epsilon_1 \cdot \epsilon_2 + (q - p_1) \cdot \epsilon_2 \cdot \epsilon_1] [(p_3 - p_4) \cdot \epsilon_3 \cdot \epsilon_4 + (p_4 - q) \cdot \epsilon_3 \cdot \epsilon_4 - (q - p_3) \cdot \epsilon_4 \cdot \epsilon_3]$$

$$+ (\epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 - \epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3) s$$

$$= \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 (p_1 - p_2) \cdot (p_3 - p_4) + (\epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 - \epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3) s \\ + (\epsilon_1 \cdot \epsilon_2 p_1 \cdot p_2 \in \text{terms}) \quad \left\{ \begin{array}{l} s = 34 \\ t = 14 \\ u = 24 \end{array} \right.$$

$$\therefore n_{s,A} = \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 (u - t) + (\epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 - \epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3) s$$

Can construct n_t, n_u by cyclic permutation $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$

$$n_{t,A} = \epsilon_2 \cdot \epsilon_3 \epsilon_1 \cdot \epsilon_4 (s - u) + (\epsilon_2 \cdot \epsilon_1 \epsilon_3 \cdot \epsilon_4 - \epsilon_2 \cdot \epsilon_4 \epsilon_3 \cdot \epsilon_1) t$$

$$n_{u,A} = \epsilon_3 \cdot \epsilon_1 \epsilon_2 \cdot \epsilon_4 (t - s) + (\epsilon_3 \cdot \epsilon_2 \epsilon_1 \cdot \epsilon_4 - \epsilon_3 \cdot \epsilon_4 \epsilon_1 \cdot \epsilon_2) u$$

$$\text{Now } n_{s,A} + n_{t,A} + n_{u,A} =$$

$$\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 (u - t) + (\epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 - \epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3) s \\ + \epsilon_2 \cdot \epsilon_3 \epsilon_1 \cdot \epsilon_4 (s - u) + (\epsilon_2 \cdot \epsilon_1 \epsilon_3 \cdot \epsilon_4 - \epsilon_2 \cdot \epsilon_4 \epsilon_3 \cdot \epsilon_1) t \\ + \epsilon_3 \cdot \epsilon_1 \epsilon_2 \cdot \epsilon_4 (t - s) + (\epsilon_3 \cdot \epsilon_2 \epsilon_1 \cdot \epsilon_4 - \epsilon_3 \cdot \epsilon_4 \epsilon_1 \cdot \epsilon_2) u$$

(3)

$$\Rightarrow n_{s,A} + n_{t,A} + n_{u,A} = E_1 \cdot E_2 \cdot E_3 \cdot E_4 \underbrace{(u-t+t-u)}_0 + \text{etc}$$

$$= 0.$$

2) The Lorentz force law is

(4)

$$\frac{dp_1}{dt} = \frac{Q_1 e}{m_1} F^{\mu\nu}(r, t) u_{1\nu}(t)$$

At LO, $r(t) \approx b + u_1 t$, $p_{1\nu} \approx m u_1$

$$\frac{dp_1}{dt} = Q_1 e F^{\mu\nu}(b + u_1 t, t) u_{1\nu}$$

Now we need a useful expression for $F^{\mu\nu}$. This is the field set up by particle 2. In its rest frame we know

$$A^\nu(x) = \frac{Q_2 e}{4\pi r} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{Q_2 e}{4\pi r} u_2^\nu$$

where $r = \sqrt{x^2 + y^2 + z^2} = \sqrt{-x \cdot x + (u_2 \cdot x)^2}$
 $= \sqrt{\eta_{\mu\nu} x^\mu x^\nu}$

$$\therefore A^\nu(x) = \frac{1}{4\pi} \frac{Q_2 e}{\sqrt{(u_2 \cdot x)^2 - x^2}} u_2^\nu$$

To construct $F^{\mu\nu}$ we need

$$\partial_\nu \sqrt{(u_2 \cdot x)^2 - x^2} = \frac{1}{\sqrt{(u_2 \cdot x)^2 - x^2}} (u_{2\nu} - x_\nu)$$

(5)

$$\Rightarrow F^{\infty}(x) = + \frac{1}{4\pi} \frac{Q_1 Q_2 e^2}{[(u_1 \cdot x)^2 - x^2]^{\frac{3}{2}}} (x^t u_2^v - x^v u_2^t)$$

Thus

$$\begin{aligned} \frac{dp_1^t}{dx} &= \frac{1}{4\pi} \frac{Q_1 Q_2 e^2}{[(u_1 \cdot u_2 x)^2 - (b+u_1 x)^2]^{\frac{3}{2}}} (b^t u_2^v - b^v u_2^t + c(u_1^t u_2^v - u_1^v u_2^t)) u_{12} \\ &= \frac{1}{4\pi} \frac{Q_1 Q_2 e^2}{[-b^2 + c^2(\gamma^2 - 1)]^{\frac{3}{2}}} [\gamma b^t + c(\gamma u_1^t - u_2^t)] \end{aligned}$$

The LO impulse is then

$$\begin{aligned} \Delta p_1^t &= \frac{Q_1 Q_2 e^2}{4\pi} \int_{-\infty}^{\infty} dx \frac{\gamma b^t + c(\gamma u_1^t - u_2^t)}{[-b^2 + c^2(\gamma^2 - 1)]^{\frac{3}{2}}} \\ &= \frac{Q_1 Q_2 e^2}{4\pi} \int_{-\infty}^{\infty} dx \frac{\gamma b^t}{[-b^2 + c^2(\gamma^2 - 1)]^{\frac{3}{2}}} \\ &= \frac{Q_1 Q_2 e^2}{4\pi} \gamma b^t \int_{-\infty}^{\infty} \frac{d(\sqrt{\gamma^2 - 1} \xi)}{\sqrt{\gamma^2 - 1}} \frac{1}{[-b^2 + (\xi \sqrt{\gamma^2 - 1})^2]^{\frac{3}{2}}} \\ &= \frac{Q_1 Q_2 e^2}{4\pi} \frac{\gamma}{\sqrt{\gamma^2 - 1}} b^t \int_{-\infty}^{\infty} d\xi \frac{1}{[-b^2 + \xi^2]^{\frac{3}{2}}} \end{aligned}$$

(6)

$$\therefore \Delta_p = \frac{Q_1 Q_2 e^2}{2\pi} \frac{\gamma}{\sqrt{\gamma^2 - 1}} \frac{b^5}{-b^2}$$

(Note $b^2 = -|\underline{b}|^2 < 0.$)

This agrees with the expression from lecture 3.

3. The Schwarzschild metric is

(7)

$$ds^2 = \left(1 - \frac{2Gm}{r}\right) dt'^2 - \left(1 - \frac{2Gm}{r}\right)^{-1} dr^2 - r^2 d\Omega^2$$

We seek a coordinate change to the form

$$ds^2 = dt^2 - dr^2 - r^2 d\Omega^2 - \frac{2Gm}{r} (dt + dr)^2.$$

Since the $r^2 d\Omega^2$ term appears in both forms of the metric, leave r and angles untouched.

Simplest option is $t' = t + f(r)$.

$$\Rightarrow dt' = dt + f'(r) dr \quad f'(r) = \frac{df}{dr}$$

We can just focus on dt^2 , dr^2 parts of ds^2 :

$$\begin{aligned} & \left(1 - \frac{2Gm}{r}\right) (dt + f'(r) dr)^2 - \left(1 - \frac{2Gm}{r}\right)^{-1} dr^2 \\ &= dt^2 - dr^2 - \frac{2Gm}{r} (dt + 2 dt dr + dr^2) \end{aligned} \quad \left. \begin{array}{l} \text{to match} \\ \text{metrics.} \end{array} \right\}$$

First expand LHS:

$$\left(1 - \frac{2Gm}{r}\right) \left(dt^2 + 2f'(r)dt dr + f'(r)^2 dr^2\right) \quad (8)$$

$$- \left(1 - \frac{2Gm}{r}\right)^{-1} dr^2$$

$$= \left(1 - \frac{2Gm}{r}\right) dt^2 + f'(r) \left(1 - \frac{2Gm}{r}\right) 2 dt dr \\ + \left[\frac{r - 2Gm}{r} f'(r)^2 - \frac{r}{r - 2Gm} \right] dr^2$$

RHS is

$$= \left(1 - \frac{2Gm}{r}\right) dt^2 + \left[-\frac{2Gm}{r}\right] 2 dt dr + \left[-1 - \frac{2Gm}{r}\right] dr^2$$

Comparing, there are two equations:

$$f'(r) \left(1 - \frac{2Gm}{r}\right) = -\frac{2Gm}{r} \quad (1)$$

$$\frac{r - 2Gm}{r} f'(r)^2 - \frac{r}{r - 2Gm} = -\frac{r + 2Gm}{r} \quad (2)$$

Solve (1) :

$$f'(r) = -\frac{2Gm}{r - 2Gm} \Rightarrow f(r) = -2Gm \log(r - 2Gm) \\ + (\text{irrelevant constant})$$

Check (2) :

(7)

$$\begin{aligned} \frac{r-2Gm}{r} - \frac{4G^2m^2}{(r-2Gm)^2} &= \frac{r}{r-2Gm} \\ = \frac{4G^2m^2 - r^2}{r(r-2Gm)} &= \frac{(2Gm-r)(2Gm+r)}{r(r-2Gm)} = -\frac{r+2Gm}{r} \quad \checkmark \end{aligned}$$

Thus

$$t' = t - 2Gm \log(r-2Gm)$$

is the coordinate transformation.