The Amplitudes Games

The double copy

1. Consider the 4-gluon tree amplitude in pure Yang-Mills theory, expressed in BCJ form:

$$\mathcal{A} = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u} \,. \tag{1}$$

The Feynman rules (in Feynman gauge) lead to numerators of the form

$$n_i = n_{i,A} + n_{i,B} \,, \tag{2}$$

where $n_{i,A}$ is a linear combination of dot products of *two* pairs of polarisation vectors time Mandelstam invariants, while $n_{i,B}$ is a linear combination of dot products of *one* pair of polarisation vectors, times dot products of polarisation vectors and momenta, times Mandelstams.

Compute the $n_{i,A}$ for $i = s, t, u$ and show that

$$n_{s,A} + n_{t,A} + n_{u,A} = 0 \,. \tag{3}$$

(It's also true that $n_{s,B} + n_{t,B} + n_{u,B} = 0$, but the algebra is messier. If you want to check this, it can be helpful to choose a convenient gauge.)

2. Two charges $Q_1 e$ and $Q_2 e$ scatter electromagnetically. Compute the leading order impulse on particle 1 from the Lorentz force law

$$\frac{\mathrm{d}p_1^\mu}{\mathrm{d}\tau} = \frac{Q_1 e}{m_1} F^{\mu\nu}(r_1(\tau)) p_{1\,\nu}(\tau) \,, \tag{4}$$

where m_1 is the mass or particle 1, and its trajectory is $r_1(\tau)$. To do so, assume that the trajectory can be taken as a straight line with impact parameter b at zeroth order:

$$r_1(\tau) = b + u_1 \tau + (O)(e^2) \,. \tag{5}$$

(Recall that the impact parameter satisfies $b \cdot u_1 = 0 = b \cdot u_2$.) Compare your result to the expression derived in lecture 3 from scattering amplitudes.

3. The Schwarzschild metric, in the usual Schwarzschild coordinates, is

$$\mathrm{d}s^2 = \left(1 - \frac{2Gm}{r}\right) \mathrm{d}t'^2 - \left(1 - \frac{2Gm}{r}\right)^{-1} \mathrm{d}r^2 - r^2 \mathrm{d}\Omega^2 \,. \tag{6}$$

Construct an explicit change of coordinates to bring the metric into the Kerr-Schild form

$$\begin{aligned} \mathrm{d}s^2 &= \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu - \frac{2Gm}{r} (\mathrm{d}t + \mathrm{d}r)^2 \\ &= \mathrm{d}t^2 - \mathrm{d}r^2 - r^2 \mathrm{d}\Omega^2 - \frac{2Gm}{r} (\mathrm{d}t + \mathrm{d}r)^2 \,. \end{aligned} \tag{7}$$