Solutions (MITP Amplitude Games)

Erik Panzer

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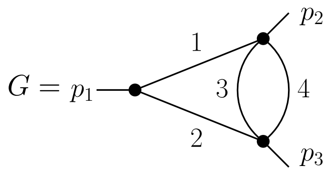
1 Power counting and factorization

Consider the Feynman integral $I_G(D, n, z)$ of the graph



1. Compute the graph polynomials $\mathcal{U}$ and $\mathcal{F}$.

   Solution:

$$
\begin{aligned}
\mathcal{U} &= x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 \\
\mathcal{F} &= -p_1^2 x_1x_2(x_3+x_4) - p_2^2 x_1x_3x_4 - p_3^2 x_2x_3x_4 + \mathcal{U}(m_1^2x_1 + m_2^2x_2 + m_3^2x_3 + m_4^2x_4)
\end{aligned}
$$

2. Determine the two singular hyperplanes that contain the point $(D, n) = (4, 1, 1, 1, 1)$.

   Solution:

   - overall divergence: $\omega = 0$ where $\omega = n_1 + n_2 + n_3 + n_4 - D$
   - subdivergence $\{3,4\}$: $\omega(\{3,4\}) = 0$ where $\omega(\{3,4\}) = n_3 + n_4 - D/2$

3. Show that $\mathcal{U}$ and $\mathcal{F}$ factorize to leading order on the subdivergence, and conclude that the leading order of the ε-expansion is

$$
I_G(4-2\varepsilon, 1, 1, 1, 1, z) = \frac{1}{2\varepsilon^2} + \mathcal{O}\left(\varepsilon^{-1}\right).
$$

   Solution: With $x_3 \to x_3\rho$ and $x_4 \to x_4$, the leading orders as $\rho \to 0$ are

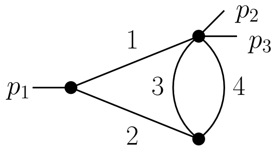
   - $\mathcal{U} \to \rho(x_3+x_4)(x_1+x_2) + \mathcal{O}\left(\rho^2\right)$
   - $\mathcal{F} \to \rho(x_3+x_4)\left[-p_1^2x_1x_2 + (x_1+x_2)(m_1^2x_1 + m_2^2x_2)\right] + \mathcal{O}\left(\rho^2\right)$

   By factorization, the coefficient of the double pole $1/[\omega \cdot \omega(\{3,4\})]$ is

$$
\operatorname*{Res}_{\omega(\{3,4\})=0} \operatorname*{Res}_{\omega=0} I_G = \operatorname*{Res}_{\omega(\{3,4\})=0} P_G = P_{\{3,4\}} \cdot P_{G/\{3,4\}} = 1 \cdot 1 = 1
$$

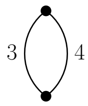
   is the square of the period of the bubble (note both the subgraph $\{3,4\}$, and the quotent graph $G/\{3,4\}$, are bubble graphs). Hence the double pole is $1/[\omega \cdot \omega(\{3,4\})] = 1/[2\varepsilon \cdot \varepsilon]$.

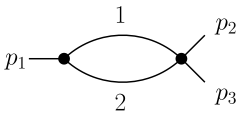
4. Show that $I_G - I_{G'}$ is finite at $(D, n) = (4, 1, 1, 1, 1)$, where G' is the graph

$G' =$ 

Hint: Compute both residues.

Solution: The graph G' has the same overall power counting, and also the same subdivergence $\omega(\{3,4\}) = 0$. The corresponding sub- and quotient graphs



$\subset G, G'$, $\qquad G/\{3,4\} =$ 

$= G'/\{3,4\}$

are the same for G as they are for G', hence they yield the same residue

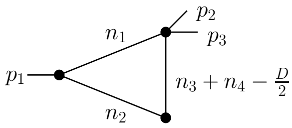
$$\operatorname*{Res}_{\omega(\{3,4\})=0} I_G = P_{\{3,4\}} \cdot I_{G/\{3,4\}} = P_{\{3,4\}} \cdot I_{G'/\{3,4\}} = \operatorname*{Res}_{\omega(\{3,4\})=0} I_{G'}.$$

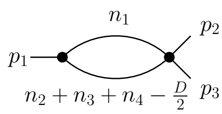
The residue $P_G = \operatorname{Res}_{\omega=0} I_G$ of the overall divergence depends only on $\mathcal{U}$, but not $\mathcal{F}$. Since G and G' differ only in the attachment of the external legs, they have the same $\mathcal{U}_G = \mathcal{U}_{G'}$, and therefore $P_G = P_{G'}$. Hence, the difference $I_G - I_{G'}$ has neither a pole on $\omega = 0$ nor at $\omega(\{3,4\}) = 0$.

5. For internal masses $m_e = 0$, obtain the subleading order ($\propto 1/\varepsilon$) of I_G.

Hint: Compute $I_{G'}$ with the formula for the massless bubble integral in terms of Γ-functions.

Solution: Let $B(n_1, n_2) = \Gamma(D/2 - n_1)\Gamma(D/2 - n_2)\Gamma(n_1 + n_2 - D/2)/[\Gamma(n_1)\Gamma(n_2)\Gamma(D - n_1 - n_2)]$ so that the bubble integral is $(-p^2)^{-\omega} B(n_1, n_2)$. Integrating out the subloop yields

$I_{G'} = B(n_3, n_4) \times$ 

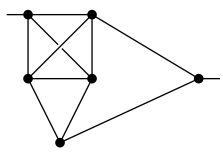
$= B(n_3, n_4) \times$ 

$$= B(n_3, n_4) B(n_1, n_2 + n_3 + n_4 - D/2)(-p_1^2)^{-\omega}$$

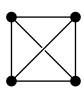
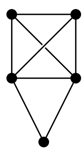
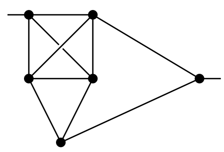
Plugging in $n_e = 1$ and $D = 4 - 2\varepsilon$, this gives the ε-expansion

$$\begin{aligned} I_G(4 - 2\varepsilon, 1, 1, 1, 1, z) = I_{G'} + \mathcal{O}\left(\varepsilon^0\right) &= (-p_1^2)^{-2\varepsilon} \frac{\Gamma(1-\varepsilon)^3 \Gamma(1-2\varepsilon)\Gamma(\varepsilon)\Gamma(2\varepsilon)}{\Gamma(1+\varepsilon)\Gamma(2-2\varepsilon)\Gamma(2-3\varepsilon)} + \mathcal{O}\left(\varepsilon^0\right) \\ &= \frac{1}{2\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{5}{2} - \gamma_E - \log(-p_1^2)\right) + \mathcal{O}\left(\varepsilon^0\right) \end{aligned}$$

6. Compute the leading order in the ε-expansion ($D = 4 - 2\varepsilon$) of the $n_e = 1$ integral



Solution: There is a logarithmic overall divergence $\omega|_{n_e=1} = 5\varepsilon$ and two nested, logarithmic subdivergences

$\delta =$  $\subset \quad \gamma =$  $\subset \quad G =$ 

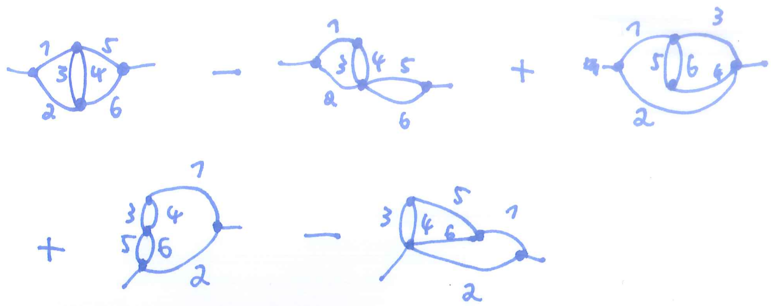
with $\omega(\delta)|_{n_e=1} = 3\varepsilon$ and $\omega(\gamma)|_{n_e=1} = 4\varepsilon$. Hence the leading order is a triple pole $1/[\omega \cdot \omega(\gamma) \cdot \omega(\delta)]|_{n_e=1} = 1/(60\varepsilon^3)$ with coefficient

$$\operatorname*{Res}_{\omega(\delta)=0} \operatorname*{Res}_{\omega(\gamma)=0} \operatorname*{Res}_{\omega=0} I_G = P_\delta \cdot P_{\gamma/\delta} \cdot P_{G/\gamma} = 6\zeta(3) \cdot 1 \cdot 1$$

where δ is the wheel with 3 spokes mentioned in the lecture, and $\gamma/\delta \cong G/\gamma$ are both isomorphic to the 1-loop bubble graph. In conclusion,

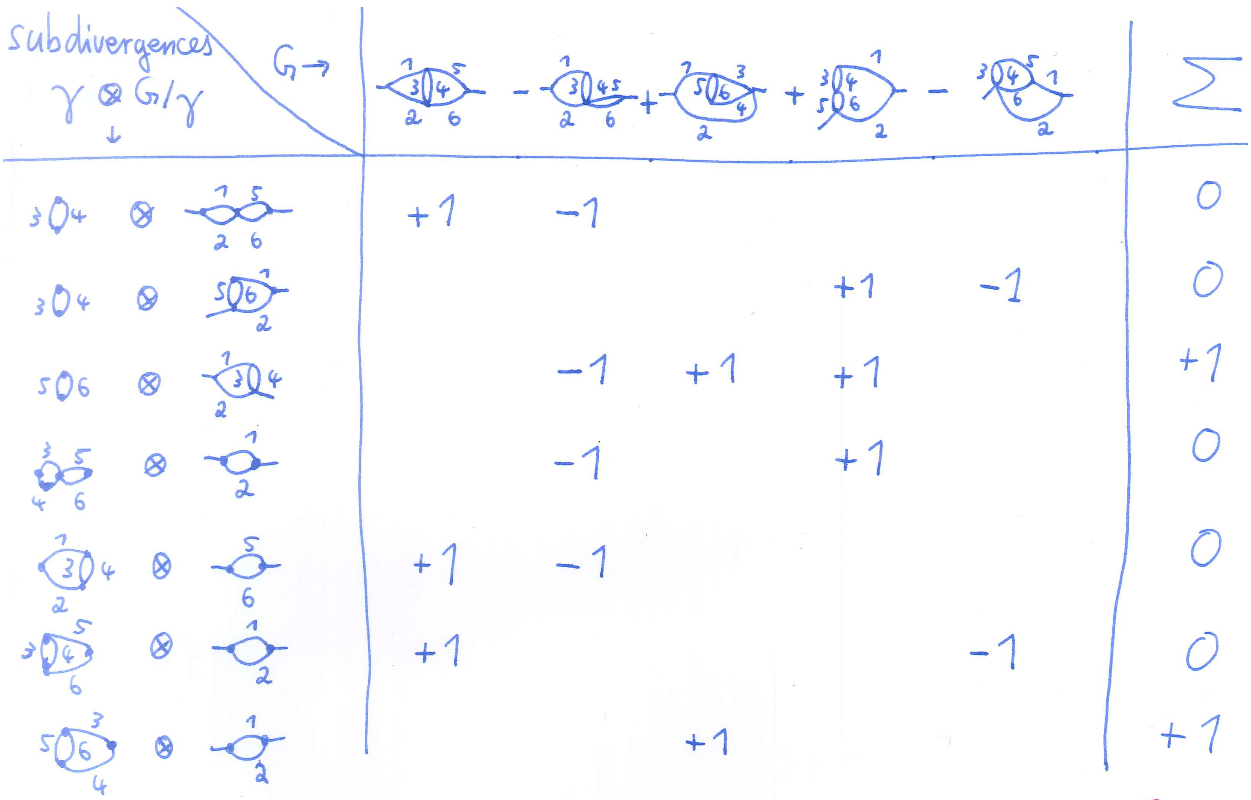
$$I_G(4-2\varepsilon, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, z) = \frac{\zeta(3)}{10\varepsilon^3} + \mathcal{O}\left(\varepsilon^{-2}\right).$$

7. Compute all subdivergences near $n_e \to 1$ and $D \to 4$ of the following 5 massless Feynman integrals individually, and deduce also which divergences are left over in the indicated linear combination:

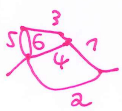


Find a single additional graph with sign such that its addition renders the entire linear combination free of subdivergences.

Solution:



⇒ adding −



cancels

2 Schwinger parameters and graph polynomials

1. Starting from the Schwinger parameter representation

$$I(D,n,z) = \left(\prod_{e=1}^{N} \int_0^\infty \frac{x_e^{n_e-1}\mathrm{d}x_e}{\Gamma(n_e)}\right) \frac{e^{-\mathcal{F}/\mathcal{U}}}{\mathcal{U}^{D/2}},$$

prove the projective and the Lee-Pomeransky representations.

Hint: Multiply with $1 = \int_0^\infty \delta(\rho - h(x))\mathrm{d}\rho$ *and change variables* $x_e \to \rho x_e$.

Solution: Introducing ρ and rescaling x_e as suggested, the polynomials $\mathcal{U}$ and $\mathcal{F}$ get multiplied by ρ^L and ρ^{L+1}, respectively. From $x_e^{n_e-1}\mathrm{d}x_e$ we also get a factor ρ^{n_e} for every edge. Since $h(x)$ is homoeneous, the delta distribution becomes $\delta(\rho - \rho h(x)) = \delta(1-h(x))/\rho$ after the rescaling $x_e \to \rho x_e$. The integral over ρ is a gamma function,

$$I(D,n,z) = \left(\prod_{e=1}^{N} \int_0^\infty \frac{x_e^{n_e-1}\mathrm{d}x_e}{\Gamma(n_e)}\right) \frac{\delta(1-h(x))}{\mathcal{U}^{D/2}} \underbrace{\int_0^\infty \rho^{\omega-1} e^{-\rho\mathcal{F}/\mathcal{U}}\mathrm{d}\rho}_{\Gamma(\omega)(\mathcal{U}/\mathcal{F})^\omega}$$

which proves the projective representation of the Feynman integral. Applying the same procedure to the Lee-Pomeransky integral, we get

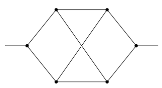
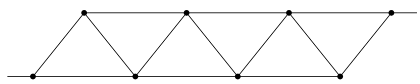
$$\left(\prod_{e=1}^{N} \int_0^\infty \frac{x_e^{n_e-1}\mathrm{d}x_e}{\Gamma(n_e)}\right)(\mathcal{U}+\mathcal{F})^{-D/2} = \left(\prod_{e=1}^{N} \int_0^\infty \frac{x_e^{n_e-1}\mathrm{d}x_e}{\Gamma(n_e)}\right)\delta(1-h(x))\int_0^\infty \rho^{\omega-1}(\mathcal{U}+\rho\mathcal{F})^{-D/2}\mathrm{d}\rho$$

The substitution $\rho \to \rho \cdot \mathcal{U}/\mathcal{F}$ gives

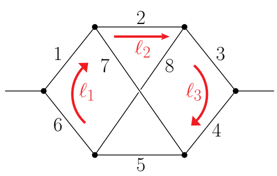
$$\int_0^\infty \rho^{\omega-1}(\mathcal{U}+\rho\mathcal{F})^{-D/2}\mathrm{d}\rho = \left(\frac{\mathcal{U}}{\mathcal{F}}\right)^\omega \mathcal{U}^{-D/2}\int_0^\infty \frac{\rho^{\omega-1}}{(1+\rho)^{D/2}}\mathrm{d}\rho = \frac{\beta(\omega, D/2-\omega)}{\mathcal{U}^{D/2-\omega}\mathcal{F}^\omega},$$

Euler's beta function. In terms of gamma functions, $\beta(\omega, D/2-\omega) = \Gamma(\omega)\Gamma(D/2-\omega)/\Gamma(D/2)$, hence with the prefactor $\Gamma(D/2)/\Gamma(D/2-\omega)$ of the Lee-Pomeransky representation, only $\Gamma(\omega)$ remains. We have thus arrived again at the projective representation of the Feynman integral.

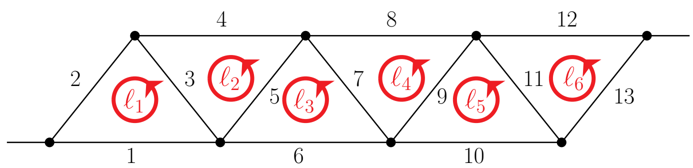
2. Compute the number of spanning trees of the following graphs: *Hint: Use* $\mathcal{U} = \det A$.

$X =$  $Z_6 =$ 

Solution: Consider the loop momentum flows and edge labels as indicated in



and



.

The loop momentum through edges 7,8,5 of X is then $\ell_1 - \ell_2$, $\ell_2 - \ell_3$ and $\ell_1 - \ell_2 + \ell_3$, respectively. This gives the matrix

$$A_X = \begin{pmatrix} x_1 + x_5 + x_6 + x_7 & -x_5 - x_7 & x_5 \\ -x_5 - x_7 & x_2 + x_5 + x_7 + x_8 & -x_5 - x_8 \\ x_5 & -x_5 - x_8 & x_3 + x_4 + x_5 + x_8 \end{pmatrix}.$$

The number of spanning trees is equal to $\mathcal{U} = \det A$ evaluated at $x_e = 1$ for every edge e,

$$\mathcal{U}_X|_{x_e=1} = \det \begin{pmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{pmatrix} = 36.$$

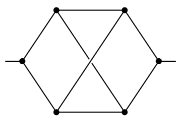
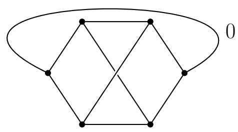
Note that the diagonal entries $A_{ii} = \sum_e x_e$ are the sums over all edges such that ℓ_i flows through e. The off-diagonal terms $A_{ij} = \sum_e x_e - \sum_f x_f$ are sums over edges such that both loop momenta ℓ_i and ℓ_j flow through the edge, with sign $+1$ if they flow in the same direction (e) and sign -1 if they flow through the edge in opposite direction (f). Hence we easily get

$$A_{Z_6} = \begin{pmatrix} x_1+x_2+x_3 & -x_3 & 0 & 0 & 0 & 0 \\ -x_3 & x_3+x_4+x_5 & -x_5 & 0 & 0 & 0 \\ 0 & -x_5 & x_5+x_6+x_7 & -x_7 & 0 & 0 \\ 0 & 0 & -x_7 & x_7+x_8+x_9 & -x_9 & 0 \\ 0 & 0 & 0 & -x_9 & x_9+x_{10}+x_{11} & -x_{11} \\ 0 & 0 & 0 & 0 & -x_{11} & x_{11}+x_{12}+x_{13} \end{pmatrix}$$

and the number of spanning trees of Z_6 is

$$\det \begin{pmatrix} 3 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & -1 & 3 \end{pmatrix} = 377.$$

3. Consider a graph G with two external legs, external momentum $p^2 = -1$, and vanishing internal masses $m_e = 0$. Let V denote the "vacuum" graph obtained by gluing the external legs into a new edge "0", for example

$G =$  $\Rightarrow$ $V =$ 

Show that:

a) $\omega(V) = \omega(G) + n_0 - D/2$,

Solution: V has one additional edge ($\Rightarrow +n_0$) and one additional loop ($\Rightarrow -D/2$).

b) $\mathcal{U}_V = x_0 \mathcal{U}_G + \mathcal{F}_G$, *Hint: trees and 2-forests.*

Solution: Every spanning tree T of V either:

- does *not* contain 0, in which case T is a spanning tree of V; or
- *does* contain 0, in which case $F := T \setminus \{0\} = T_1 \sqcup T_2$ is a 2-forest of G with one external leg in T_1 and the other external leg in T_2, hence a 2-forest that contributes to $\mathcal{F}_G$.

c) $I_G = \Gamma(D/2) \cdot P_V$ where

$$P_V := \operatorname*{Res}_{\omega(V)=0} I_V = \left(\prod_{e=0}^{N} \int_0^\infty \frac{x_e^{n_e-1} \mathrm{d}x_e}{\Gamma(n_e)} \right) \frac{\delta(1-h(x))}{\mathcal{U}_V^{D/2}}.$$

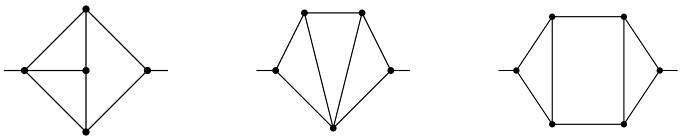
Solution: The integral over x_0 is (compare with the Lee-Pomeransky integral in part 1.)

$$\frac{1}{\Gamma(n_0)} \int_0^\infty \frac{x_0^{n_0-1} \mathrm{d}x_0}{(x_0 \mathcal{U}_G + \mathcal{F}_G)^{D/2}} = \frac{\beta(n_0, D/2-n_0)}{\Gamma(n_0) \mathcal{F}_G^{D/2}} \left(\frac{\mathcal{F}_G}{\mathcal{U}_G} \right)^{n_0} = \frac{\Gamma(D/2-n_0)}{\Gamma(D/2)} \frac{1}{\mathcal{U}_G^{n_0} \mathcal{F}_G^{D/2-n_0}}.$$

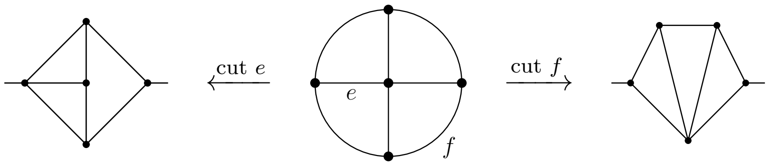
In the integral for P_V, we have $\omega(V)=0$, hence $D/2-n_0=\omega(G)$. The above thus becomes

$$P_V=\frac{\Gamma(\omega(G))}{\Gamma(D/2)}\left(\prod_{e=0}^{N}\int_0^\infty\frac{x_e^{n_e-1}\mathrm{d}x_e}{\Gamma(n_e)}\right)\frac{\delta(1-h(x))}{\mathcal{U}_G^{D/2-\omega(G)}\mathcal{F}_G^{\omega(G)}}=\frac{1}{\Gamma(D/2)}\cdot I_G.$$

d) Conclude that in $D=4$ dimensions with indices $n_e=1$, the Feynman integrals of the following graphs all coincide:

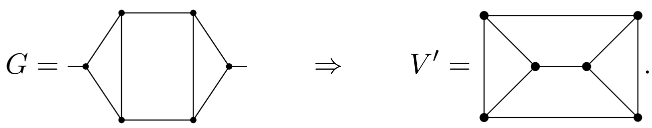


Solution: The left and centre graphs glue to the same graph V, the wheel with 4 spokes:



Also the indices agree, because $n_0=1$ in both cases. Hence both of these propagator integrals are equal to the period of V with unit indices everywhere. (FYI: it is $P_V=20\zeta(5)$.)

The third propagator glues into a different graph:

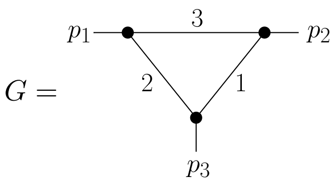


But the central, glued edge has index $n_0=D/2-\omega(G)=0$ so that it can be contracted, therefore $P_{V'}|_{n_0=0}=P_V$ because $V'/0\cong V$.

Remark: This is called the "glue-and-cut" symmetry.

3 Analytic continuation

Consider the following graph with $m_1^2=m_2^2=p_1^2=p_2^2=m^2$ and $m_3=0$:



1. Show that $\mathcal{F}_{\{1,2\}}=0$ for the tree subgraph with edges $\{1,2\}=G-\{3\}$. Deduce, via the infrared factorization formula, that $\mathcal{F}_G$ must be independent of x_3.

   Solution: The graph polynomials for the tree are $\mathcal{U}_{\{1,2\}}=1$ and

   $$\mathcal{F}_{\{1,2\}}=(m^2x_1+m^2x_2)\mathcal{U}_{\{1,2\}}-p_1^2x_2-p_2^2x_1=x_1(m^2-p_2^2)+x_2(m^2-p_1^2)=0.$$

   Under the scaling $(x_1,x_2)\to(\rho x_1,\rho x_2)$ of the tree edges, the IR-factorization formula gives

   $$\mathcal{F}_G=\rho^1\mathcal{U}_{G/\{1,2\}}\mathcal{F}_{\{1,2\}}+\mathcal{O}\left(\rho^2\right)=\mathcal{O}\left(\rho^2\right)$$

   hence every term in $\mathcal{F}_G$ is of degree ≥ 2 in the variables (x_1,x_2). But we know that $\mathcal{F}_G$ is homogeneous of degree 2 in all variables (because G has 1 loop); hence $\mathcal{F}_G$ cannot have any x_3.

2. Confirm by computing $\mathcal{F}_G$ explicitly.

   Solution:

$$\begin{aligned}\mathcal{F}_G &= m^2(x_1+x_2)(x_1+x_2+x_3) - p_1^2 x_2 x_3 - p_2^2 x_1 x_3 - p_3^2 x_1 x_2 \\ &= m^2(x_1+x_2)^2 - p_3^2 x_1 x_2 + x_3 \big(\underbrace{m^2 x_1 + m^2 x_2 - p_1^2 x_2 - p_2^2 x_1}_{=\mathcal{F}_{\{1,2\}}=0}\big) \\ &= m^2(x_1+x_2)^2 - p_3^2 x_1 x_2.\end{aligned}$$

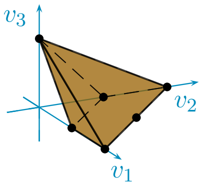
3. Draw the Newton polytope of $\mathcal{U}+\mathcal{F}$. Read off the 5 facets.

   Solution: The Lee-Pomeransky polynomial

$$\mathcal{U}_G + \mathcal{F}_G = x_1 + x_2 + x_3 + x_1^2 m^2 + x_2^2 m^2 + x_1 x_2 \left(2m^2 - p_3^2\right)$$

   has 6 different monomials. We read off the Newton polytope

$$\mathrm{NP} = \operatorname{conv}\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\2\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \right\} =$$



4. Describe the convergence domain in (D, n_1, n_2, n_3) by inequalities, and find all finite integrals in $D=6$ dimensions with integer n_e.

   Solution: The Newton polytope is a pyramid with apex $(0,0,1)$ over a quadrilateral in the (x_1,x_2)-plane. This pyramid has 5 supporting hyperplanes (facets) that we can read off easily:

$$\begin{gathered} v_1 \geq 0, \qquad v_2 \geq 0, \qquad v_3 \geq 0, \\ v_1+v_2+v_3 \geq 1, \qquad v_1+v_2+2v_3 \leq 2. \end{gathered}$$

   For example, $v_3=0$ is the plane containing the base quadrilateral; and $v_1+v_2+2v_3=2$ is the supporting hyperplane that contains the three vertices $(2,0,0)$, $(0,2,0)$ and $(0,0,1)$.

   The convergence region is $\mathrm{Re}(n) \in \operatorname{interior}(D/2 \cdot \mathrm{NP})$, hence the conditions are

$$\begin{gathered} \mathrm{Re}(n_1) > 0, \qquad \mathrm{Re}(n_2) > 0, \qquad \mathrm{Re}(n_3) > 0, \\ \mathrm{Re}(n_1+n_2+n_3) > \mathrm{Re}(D/2), \qquad \mathrm{Re}(n_1+n_2+2n_3) < \mathrm{Re}(D). \end{gathered}$$

   For $D=6$, the only integer solutions are $n=(2,1,1)$ and $n=(1,2,1)$.

5. Set $D=4-2\varepsilon$ and all $n_e=1$. In the Lee-Pomeransky representation, insert $1=\int_0^\infty \delta(\rho - x_1^{-1})\mathrm{d}\rho$, rescale $x_e \to \rho^{\sigma_e} x_e$ for $\sigma=(-1,-1,-2)$, and factor out the lowest powers of ρ to make the infrared divergence explicit.

   Solution: Under this scaling, we have

$$\begin{gathered} \mathrm{d}x_1 \to \rho^{-1}\mathrm{d}x_1, \qquad \mathrm{d}x_2 \to \rho^{-1}\mathrm{d}x_2, \qquad \mathrm{d}x_3 \to \rho^{-2}\mathrm{d}x_3, \\ \mathcal{U} \to \rho^{-2}(x_3 + \rho x_1 + \rho x_2), \qquad \mathcal{F} \to \rho^{-2}\mathcal{F}, \qquad \delta(\rho - x_1^{-1}) \to \rho^{-1}\delta(1-x_1). \end{gathered}$$

   With $D/2 = 2-\varepsilon$ and $\omega = 1+\varepsilon$, the Lee-Pomeransky integral thus becomes

$$I = \frac{\Gamma(2-\varepsilon)}{\Gamma(1-2\varepsilon)} \int_0^\infty \mathrm{d}x_1 \int_0^\infty \mathrm{d}x_2 \int_0^\infty \mathrm{d}x_3\, \delta(1-x_1) \int_0^\infty \frac{\rho^{-2\varepsilon-1}\mathrm{d}\rho}{(x_3+\mathcal{F}+\rho x_1 + \rho x_2)^{2-\varepsilon}}.$$

   The integral over ρ is divergent at the lower boundary, unless $\omega(\sigma) = -2\varepsilon > 0$.

6. Integrate by parts in ρ and give a convergent integral formula (without ρ) for each coefficient in the ε-expansion.

   Solution: The integration by parts in ρ gives

$$\int_0^\infty \rho^{-2\varepsilon-1}\mathrm{d}\rho\,(x_3+\mathcal{F}+\rho x_1+\rho x_2)^{\varepsilon-2} = \frac{\varepsilon-2}{2\varepsilon}\int_0^\infty \frac{(x_1+x_2)\rho^{-2\varepsilon}\mathrm{d}\rho}{(x_3+\mathcal{F}+\rho x_1+\rho x_2)^{3-\varepsilon}}$$

   Inverting the scaling, $x_1 \to x_1\rho$, $x_2 \to x_2\rho$ and $x_3 \to x_3\rho^2$ to return to the original Schwinger parameters and to get rid of ρ, the resulting integral representation is

$$I = -\frac{\Gamma(3-\varepsilon)}{2\varepsilon\Gamma(1-2\varepsilon)}\int_0^\infty \mathrm{d}x_1\int_0^\infty \mathrm{d}x_2\int_0^\infty \mathrm{d}x_3 \frac{x_1+x_2}{(\mathcal{U}+\mathcal{F})^{3-\varepsilon}}.$$

   The divergence at $\varepsilon=0$ is explicit in the prefactor, but (in contrast to the original Lee-Pomeransky representation) the integral over the Schwinger parameters is holomorphic also at $\varepsilon=0$. Hence we can expand under the integral:

$$I = -\frac{\Gamma(3-\varepsilon)}{2\Gamma(1-2\varepsilon)}\sum_{k\geq 0}\frac{\varepsilon^{k-1}}{k!}\int_0^\infty \mathrm{d}x_1\int_0^\infty \mathrm{d}x_2\int_0^\infty \mathrm{d}x_3 \frac{x_1+x_2}{(\mathcal{U}+\mathcal{F})^{3}}\log^k(\mathcal{U}+\mathcal{F}).$$

7. Compute the leading order (coefficient of $1/\varepsilon$) and relate it to a bubble integral.

   Solution: For the leading order, the ρ-integral in 6. simplifies to

$$\frac{1}{2\varepsilon}\int_0^\infty \rho^{-2\varepsilon}\mathrm{d}\rho\,\frac{\partial}{\partial\rho}\frac{1}{(x_3+\mathcal{F}+\rho x_1+\rho x_2)^{2-\varepsilon}} = -\frac{1}{2\varepsilon}\frac{1}{(x_3+\mathcal{F})^2}+\mathcal{O}\left(\varepsilon^0\right)$$

   and the subsequent x_3-integral is straightforward, leaving:

$$I = -\frac{1}{2\varepsilon}\int_0^\infty \mathrm{d}x_1\int_0^\infty \mathrm{d}x_2 \frac{\delta(1-x_1)}{\mathcal{F}}+\mathcal{O}\left(\varepsilon^0\right).$$

   Up to the prefactor, this is identical to the bubble integral of $G/3$ in $D=2$ dimensions.

8. Explain where the divergence comes from in momentum space.

   Solution: Let ℓ denote the momentum flowing through edge 3. Then

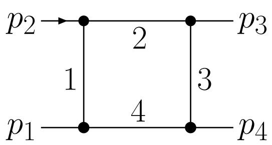
$$\begin{aligned} I(D,1,1,1,z) &= \int_{\mathbb{R}^{1,D-1}}\frac{\mathrm{d}^D\ell}{i\pi^{D/2}}\frac{1}{-\ell^2}\frac{1}{m^2-(\ell+p_1)^2}\frac{1}{m^2-(\ell-p_2)^2} \\ &= \int_{\mathbb{R}^{1,D-1}}\frac{\mathrm{d}^D\ell}{i\pi^{D/2}}\frac{1}{-\ell^2}\frac{1}{\ell^2+2\ell p_1}\frac{1}{\ell^2-2\ell p_2} \end{aligned}$$

   For $\ell \to 0$, the integrand grows with $\|\ell\|^{-4}$, while the volume element in $D=4$ scales as $\|\ell\|^3\,\mathrm{d}\,\|\ell\|$. This shows a logarithmic divergence at $\ell \to 0$.

4 Polynomial reduction

Compute the Landau varieties of:

1. The massless box integral ($m_e^2 = p_i^2 = 0$).



Solution: Start with the singularities of the integrand, in the projective representation with $\delta(1-x_4)$: Set $s=-(p_1+p_2)^2$ and $t=-(p_1+p_4)^2$, then

$$S=\{\mathcal{U}|_{x_4=1},\mathcal{F}|_{x_4=1}\}=\{1+x_1+x_2+x_3, sx_2+tx_1x_3\}\,.$$

Reduction of x_1:

$$S_1=\{\underbrace{1+x_2+x_3,s,x_2}_{x_1\to 0},\underbrace{t,x_3}_{x_1\to\infty},\underbrace{sx_2-tx_3(1+x_2+x_3)}_{\text{resultant}}\}$$

Reduction of x_2:

$$S_{1,2}=\{s,t,x_3,1+x_3,s-tx_3\}$$

Reduction of x_3:

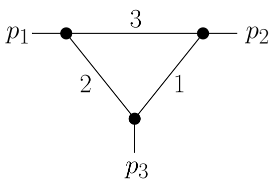
$$S_{1,2,3}=\{s,t,s+t\}$$

Hence the Landau variety $L\subseteq S_{1,2,3}$ has at most 3 components. Clearly $\{s=0\}$ and $\{t=0\}$ are necessary, since the special cases $n=(0,1,0,1)$ and $n=(1,0,1,0)$ correspond to bubble integrals $\propto s^{-\varepsilon}$ and $\propto t^{-\varepsilon}$, respectively, which have singularities at $s=0$ and $t=0$. To see that singularities at $s+t=0$ also appear, consider for example the finite box integral in $D=6$:

$$\begin{aligned}
I(6,1,1,1,1,s,t)&=\int_0^\infty\int_0^\infty\int_0^\infty\frac{\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3}{(sx_2+tx_1x_3)(1+x_1+x_2+x_3)^2},\quad\text{set } x_2\to x_2x_3\\
&=\int_0^\infty\int_0^\infty\int_0^\infty\frac{\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3}{(sx_2+tx_1)(1+x_1+x_3(1+x_2))^2}\\
&=\int_0^\infty\int_0^\infty\frac{\mathrm{d}x_1\mathrm{d}x_2}{(sx_2+tx_1)(1+x_1)(1+x_2)}=\int_0^\infty\frac{\mathrm{d}x_1}{(1+x_1)(tx_1-s)}\log\frac{tx_1}{s}\\
&=\frac{\pi^2+\log^2(s/t)}{2(s+t)}
\end{aligned}$$

Taking two variations around $s=0$, this becomes $(2i\pi)^2/(s+t)$, which clearly has a pole at $s+t=0$. In conclusion, $L=S_{1,2,3}=\{s,t,s+t\}$.

2. The triangle integral for generic p_1^2,p_2^2,p_3^2 as in the lecture, but with an internal mass $m_3\neq 0$ (still $m_1=m_2=0$).



Solution: In the projective representation with $x_3=1$, the singularities of the integrand are

$$S=\Big\{\underbrace{1+x_1+x_2}_{\mathcal{U}|_{x_3=1}},\underbrace{-p_1^2x_2-p_2^2x_1-p_3^2x_1x_2+m_3^2(1+x_1+x_2)}_{\mathcal{F}|_{x_3=1}}\Big\}$$

Reduction of x_1:

$$S_1=\Big\{\underbrace{1+x_2,m_3^2(1+x_2)-p_1^2x_2}_{x_1\to 0},\underbrace{m_3^2-p_2^2-p_3^2x_2}_{x_1\to\infty},\underbrace{(1+x_2)(p_2^2+p_3^2x_2)-p_1^2x_2}_{\text{resultant }[\mathcal{U},\mathcal{F}]}\Big\}$$

Reduction of x_2:

$$S_{1,2}=\Big\{\underbrace{m_3^2,m_3^2-p_2^2,p_2^2}_{x_2\to 0},\underbrace{m_3^2-p_1^2,p_3^2}_{x_2\to\infty},\underbrace{p_1^2,m_3^2+p_3^2-p_2^2,(m_3^2-p_1^2)(m_3^2-p_2^2)-m_3^2p_3^2,\Delta}_{\text{resultants}}\Big\}$$

where $\Delta = p_1^4 + p_2^4 + p_3^4 - 2p_1^2p_2^2 - 2p_1^2p_3^2 - 2p_2^2p_3^2$ denotes the discriminant that also appeared in the massless case. By symmetry (flipping $x_1 \leftrightarrow x_2$ and $p_1 \leftrightarrow p_2$), note

$$S_{2,1} = S_{1,2}\big|_{p_1 \leftrightarrow p_2} = \Big(S_{1,2} \setminus \{m_3^2 + p_3^2 - p_2^2\}\Big) \cup \{m_3^2 + p_3^2 - p_1^2\}.$$

Hence the component $m_3^2 + p_3^2 - p_2^2 \in S_{1,2}$ is spurious, and we get the upper bound

$$L \subseteq S_{1,2} \cap S_{2,1} = \Big\{p_1^2, p_2^2, p_3^2, \Delta, m_3^2, m_3^2 - p_1^2, m_3^2 - p_2^2, (m_3^2 - p_1^2)(m_3^2 - p_2^2) + m_3^2p_3^2\Big\}.$$

In fact, all these singularities indeed appear, hence $L = S_{1,2} \cap S_{2,1}$.

Remark: The following is not explicitly asked for in the question, and for information only.

For example, setting $n_1 = 0$, the contracted graph is a bubble with one massless and one massive propagator. In $D = 4$ with the massive propagator squared ($n_3 = 2$), this bubble integral is

$$I(4, 0, 1, 2, z) = \int_0^\infty \frac{x_3 \mathrm{d}x_3}{(1 + x_3)(m_3^2x_3(1 + x_3) - p_1^2x_3)} = \frac{1}{-p_1^2} \log \frac{m_3^2 - p_1^2}{m_3^2}.$$

This exhibits singularities at $p_1^2 = 0$, $m_3^2 = 0$ and $m_3^2 - p_1^2 = 0$. By symmetry, the bubble with $n = (1, 0, 2)$ also gives singularities at $p_2^2 = 0$ and $m_3^2 - p_2^2 = 0$. The massless bubble $n = (1, 1, 0)$ is proportional to a power of p_3^2. In summary, the bubble quotients imply the lower bound

$$L \supseteq \Big\{p_1^2, p_2^2, p_3^2, m_3^2, m_3^2 - p_1^2, m_3^2 - p_2^2\Big\}.$$

The component $(m_3^2 - p_1^2)(m_3^2 - p_2^2) - m_3^2p_3^2 = 0$ is the leading Landau singularity of the triangle. For example, it appears in

$$I(4, 1, 1, 2, z) = \int_0^\infty \int_0^\infty \frac{\mathrm{d}x_1\mathrm{d}x_2}{\mathcal{F}^2|_{x_3=1}} = \frac{1}{(m_3^2 - p_1^2)(m_3^2 - p_2^2) + m_3^2p_3^2} \log \frac{(m_3^2 - p_1^2)(m_3^2 - p_2^2)}{-p_3^2m_3^2}.$$

Finally, the fact that $\Delta \in L$ is in some sense the most complicated. It is an example of a "singularity of the second type" [1]. As the calculation above shows, Δ arises as the discriminant of the resultant $[\mathcal{U}, \mathcal{F}]$. In particular, it is absent in the integral $I(4, 1, 1, 2, z)$ above because that only depends on $\mathcal{F}$. One can see Δ explicitly for example as the denominator of

$$I(4, 1, 1, 1, z) = \int_0^\infty \int_0^\infty \frac{\mathrm{d}x_1\mathrm{d}x_2}{\mathcal{U}\mathcal{F}|_{x_3=1}} = \frac{1}{\sqrt{\Delta}} \times \{\text{sum of several dilogarithms}\}.$$

References

[1] D. B. Fairlie, P. V. Landshoff, J. Nuttall and J. C. Polkinghorne, *Singularities of the second type*, *Journal of Mathematical Physics* **3** (1962), no. 4 pp. 594–602.