# Exercises (MITP Amplitude Games)

Erik Panzer

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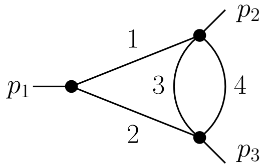
## 1 Power counting and factorization

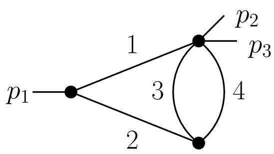
Consider the Feynman integral $I_G(D, n, z)$ of the graph

$G =$ 


1. Compute the graph polynomials $\mathcal{U}$ and $\mathcal{F}$.
2. Determine the two singular hyperplanes that contain the point $(D, n) = (4, 1, 1, 1, 1)$.
3. Show that $\mathcal{U}$ and $\mathcal{F}$ factorize to leading order on the subdivergence, and onclude that the leading order of the $\varepsilon$-expansion is

$$I_G(4 - 2\varepsilon, 1, 1, 1, 1, z) = \frac{1}{2\varepsilon^2} + \mathcal{O}\left(\varepsilon^{-1}\right).$$

4. Show that $I_G - I_{G'}$ is finite at $(D, n) = (4, 1, 1, 1, 1)$, where $G'$ is the graph

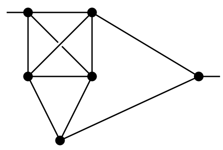
$G' =$ 


*Hint: Compute both residues.*

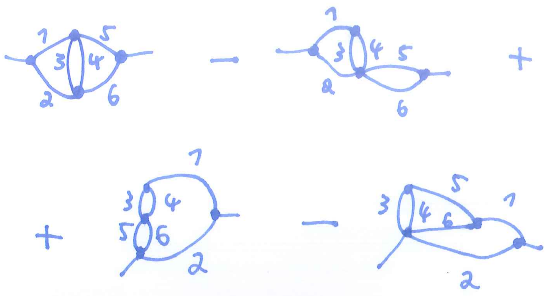
5. For internal masses $m_e = 0$, obtain the subleading order ($\propto 1/\varepsilon$) of $I_G$.

*Hint: Compute $I_{G'}$ with the formula for the massless bubble integral in terms of $\Gamma$-functions.*

6. Compute the leading order in the $\varepsilon$-expansion ($D = 4 - 2\varepsilon$) of the $n_e = 1$ integral



7. Compute all subdivergences near $n_e \to 1$ and $D \to 4$ of the following 5 massless Feynman integrals individually, and deduce also which divergences are left over in the indicated linear combination:



Find a single additional graph with sign such that its addition renders the entire linear combination free of subdivergences.

## 2 Schwinger parameters and graph polynomials

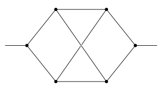
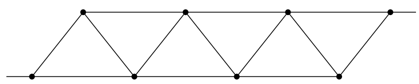
1. Starting from the Schwinger parameter representation

$$I(D,n,z) = \left( \prod_{e=1}^{N} \int_0^\infty \frac{x_e^{n_e-1}\mathrm{d}x_e}{\Gamma(n_e)} \right) \frac{e^{-\mathcal{F}/\mathcal{U}}}{\mathcal{U}^{D/2}},$$

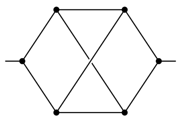
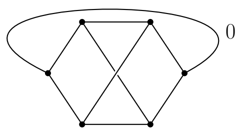
prove the projective and the Lee-Pomeransky representations.

*Hint: Multiply with* $1 = \int_0^\infty \delta(\rho - h(x))\mathrm{d}\rho$ *and change variables* $x_e \to \rho x_e$.

2. Compute the number of spanning trees of the following graphs: *Hint: Use* $\mathcal{U} = \det A$.

$X =$  $Z_6 =$ 

3. Consider a graph $G$ with two external legs, external momentum $p^2 = -1$, and vanishing internal masses $m_e = 0$. Let $V$ denote the "vacuum" graph obtained by gluing the external legs into a new edge "0", for example

$G =$  $\Rightarrow$ $V =$ 

Show that:

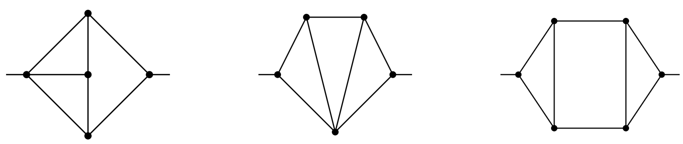
a) $\omega(V) = \omega(G) + n_0 - D/2$,

b) $\mathcal{U}_V = x_0 \mathcal{U}_G + \mathcal{F}_G$, *Hint: trees and 2-forests.*

c) $I_G = \Gamma(D/2) \cdot P_V$ where

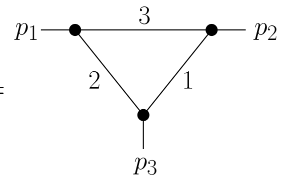
$$P_V := \operatorname*{Res}_{\omega(V)=0} I_V = \left( \prod_{e=0}^{N} \int_0^\infty \frac{x_e^{n_e-1}\mathrm{d}x_e}{\Gamma(n_e)} \right) \frac{\delta(1-h(x))}{\mathcal{U}_V^{D/2}}.$$

d) Conclude that in $D = 4$ dimensions with indices $n_e = 1$, the Feynman integrals of the following graphs all coincide:



# 3 Analytic continuation

Consider the following graph with $m_1^2 = m_2^2 = p_1^2 = p_2^2 = m$ and $m_3 = 0$:

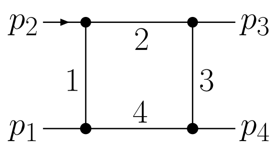
$G =$ 


1. Show that $\mathcal{F}_{\{1,2\}} = 0$ for the tree subgraph with edges $\{1,2\} = G - \{3\}$. Deduce, via the infrared factorization formula, that $\mathcal{F}_G$ must be independent of $x_3$.
2. Confirm by computing $\mathcal{F}_G$ explicitly.
3. Draw the Newton polytope of $\mathcal{U} + \mathcal{F}$. Read off the 5 facets.
4. Describe the convergence domain in $(D, n_1, n_2, n_3)$ by inequalities, and find all finite integrals in $D = 6$ dimensions with integer $n_e$.
5. Set $D = 4 - 2\varepsilon$ and all $n_e = 1$. In the Lee-Pomeransky representation, insert $1 = \int_0^\infty \delta(\rho - x_1)\mathrm{d}\rho$, rescale $x_e \to \rho^{\sigma_e} x_e$ for $\rho = (-1, -1, -2)$, and factor out the lowest powers of $\rho$ to make the infrared divergence explicit.
6. Integrate by parts in $\rho$ and give a convergent integral formula (without $\rho$) for each coefficient in the $\varepsilon$-expansion.
7. Compute the leading order (coefficient of $1/\varepsilon$) and relate it to a bubble integral.
8. Explain where the divergence comes from in momentum space.

# 4 Polynomial reduction

Compute the Landau varieties of:

1. The massless box integral ($m_e^2 = p_i^2 = 0$).




2. The triangle integral for generic $p_1^2, p_2^2, p_3^2$ as in the lecture, but with an internal mass $m_3 \neq 0$ (still $m_1 = m_2 = 0$).