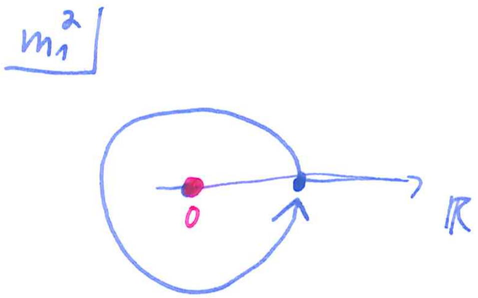
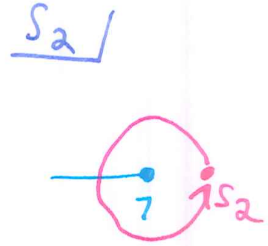
analytic continuation & monodromy

Consider a small loop in m_1^2 around 0:



With $|m_1^2|$ very small

$$\rightsquigarrow \quad s_2 \sim 1 + \frac{m_1^2}{m_2^2 - p^2}$$



Contour deformation:

Integration path:

$$\int \frac{dx}{F} = \underbrace{\int \frac{dx}{F}}_{\text{original } I(z)} + \underbrace{\oint \frac{dx}{F}}_{\text{monodromy / variation / discontinuity}}$$

$$= 2\pi i \cdot \operatorname*{Res}_{x=s_1} \frac{dx}{F}$$

$\Rightarrow$ multi-valuedness

$\Rightarrow$ variations are computed by integrals over deformed / different contours

$\Big\}$ Picard–Lefschetz theory of vanishing cycles ...

$\Rightarrow$ some variations can be computed by "Cutkosky rules"

Computing Landau varieties in higher dimension

- Landau equations in momentum – Schwinger – space
- here: use (Schwinger) parameters (no loop momenta)

<u>Idea:</u> Repeat the 1-dimensional analysis, step by step, for each integration variable.

- Start with: Landau variety of the integrand:

$$S := \{ \mathcal{U} = 0 \} \cup \{ F = 0 \}$$

<u>Def.</u> Given a set of polynomials $\{ f_i(x) \}$. Consider:

(*)
- $f_i(0)$ = constant coefficient (endpoint singularity $x \to 0$)
- leading coeff. of f_i's (endpoint singularities $x \to \infty$)

- $D f_i$ (pinch between two zeros of f_i)
- $[f_i, f_j]_x$ (pinches between zeros of f_i & a zero of f_j)

"resultant"

<u>Ex.</u> If $f_i = ax + b$, then

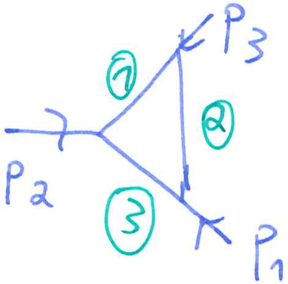
$$[f_i, f_j] = a^{\deg_x f_j} \cdot f_j \big|_{x = -b/a}$$

Denote by S_x the irreducible factors of $(*)$.

<u>Thm.</u> The Landau variety of $I(z)$ is a subset of

$$((\underbrace{S_{x_1}}_{\text{singularities of } \int_0^\infty x_1^{n_1 - 1} dx_1 \frac{1}{\mathcal{U}^{D/2-\omega} \mathcal{F}^{\omega}}})_{x_2}) \dots .$$

Ex.



$D=4$

$n_e = 1$

$m_e = 0$

$h(x) = x_3$

$$\mathcal{U} = x_1 + x_2 + 1$$

$$\mathcal{F} = p_3^2 x_1 x_2 + p_2^2 x_1 + p_1^2 x_2$$

$$\Rightarrow \quad I(z) = \int_0^\infty \int_0^\infty \frac{dx_1\, dx_2}{\mathcal{U} \cdot \mathcal{F}}$$

Landau variety of the integrand:

$$S = \{ x_1 + x_2 + 1 = 0 \} \cup \{ p_3^2 x_1 x_2 + p_2^2 x_1 + p_1^2 x_2 = 0 \}$$

1. Reduce x_1:

- const. part $\quad x_2 + 1 \qquad p_1^2 x_2$
- lead. coeff. $\quad 1 \qquad p_3^2 x_2 + p_2^2$
- discriminants: $\quad / \qquad /$ $\quad$ ($\mathcal{U}$ & $\mathcal{F}$ linear in x_1)
- resultant: $\quad (p_3^2 x_2 + p_2^2)(x_2+1) - p_1^2 x_2 =: \Delta$

$\quad \quad ||$

$\quad \mathcal{F}|_{x_1 = -x_2 - 1}$

$$\Rightarrow \quad S_{x_1} = \{x_2 = 0\} \cup \{p_1^2 = 0\} \cup \{x_2 + 1 = 0\} \cup \{p_3^2 x_2 + p_2^2 = 0\}$$
$$\cup \{\Delta = 0\}$$

$\Rightarrow$ these are the singularities of the fct. $I_1(x_2, z) := \int_0^\infty \frac{dx_1}{UF}$

Next step: Reduce x_2:

- constant parts: p_1^2 , (1) , p_2^2 ,
- leading coeffs: 1, p_3^2
- discriminants: $D\Delta = (p_2^2 + p_3^2 - p_1^2)^2 - 4p_2^2 p_3^2$
- resultants: $p_3^2 - p_2^2$

$$\Rightarrow L \subseteq (S_{x_1})_{x_2} = \{p_1^2 = 0\} \cup \{p_2^2 = 0\} \cup \{p_3^2 = 0\} \cup \{D\Delta = 0\}$$
$$\cup \{p_3^2 = p_2^2\}$$

"polynomial reduction"

Ex. Do the reductions in reverse order: $\Rightarrow (S_{X_2})_{X_1}$ does not contain $\{p_3^2 = p_2^2\}$

$$\Rightarrow \quad L \subseteq (S_{x_1})_{x_2} \cap (S_{x_2})_{x_1} = \underbrace{\{p_1^2=0\} \cup \{p_2^2=0\} \cup \{p_3^2=0\} \cup \{D\Delta=0\}}_{=L}$$